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# On a functional equation related to diversity

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**Abstract**

The general solution of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} k\left(p_i\right)\sum_{j=1}^{m} q_j^{\beta},$$

where $f$, $k$ are real valued mappings each having the domain $I = [0,1]$; $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ being fixed integers; $0 < \beta \in \mathbb{R}, \beta \neq 1$ have been obtained. The relevance of its general solution to the diversity index has been discussed.

*Keywords:* Additive mapping, sum form functional equation, the entropies of degree $\beta$, index of diversity

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## 1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers; $\mathbb{R}$ denotes the set of real numbers and $I$ denotes the closed unit interval $[0,1]$. For $n \in \mathbb{N}$, $\Gamma_n = \left\{(p_1,\ldots,p_n);\ p_i \geq 0,\ i = 1,\ldots,n;\ \sum_{i=1}^{n} p_i = 1\right\}$ denotes the set of all finite $n$-component complete discrete probability distributions.

While studying some problems in statistical thermodynamics, Chaundy and Mcleod [2] came across the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F\left(p_i q_j\right) = \sum_{i=1}^{n} F\left(p_i\right) + \sum_{j=1}^{m} F(q_j), \tag{1.1}$$

where $F$ is real valued mapping with domain $I$; $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$. They [2] proved that if (1.1) holds for all $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$; $n,\ m \geq 2$ and $F : I \to \mathbb{R}$ is assumed to be a continuous mapping, then for all $x \in I$, $F$ is of the form

$$F(x) = -cx\log_2 x \qquad (\text{where } c \in \mathbb{R} \text{ and } 0\log_2 0 := 0). \tag{1.2}$$

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Interestingly, the solution (1.2) was identical to the expression of Shannon entropies [22] defined as:

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \qquad (\text{with } 0 \log_2 0 := 0), \tag{1.3}$$

where $H_n : \Gamma_n \to \mathbb{R}$, $n \in \mathbb{N}$ and $(p_1, \ldots, p_n) \in \Gamma_n$.

Recently, a Pexiderized form of (1.1), containing two unknown mappings was studied by Nath and Singh [17]. Building upon this, another Pexiderization of (1.1), containing three unknown mappings has also been discussed by Nath and Singh [18]; Singh and Grover [27].

Havrda and Charvát [5] generalized the notion of Shannon entropies given by (1.3) by introducing entropies of degree $\beta$. Losonczi and Maksa [10] were first to address the functional equation that characterized the Havrda and Charvát [5] entropies by studying multiplicative type equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f\left(q_j\right), \tag{1.4}$$

where $f : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $n \geq 3, m \geq 3$ being fixed integers. They [10] not only obtained the solutions of (1.4) but also introduced the phenomenon of *general solutions for sum form functional equations*. For further study of this notion we can refer [8] and [11] to the readers.

Following the idea of Losonczi and Maksa [10], Nath and Singh have reviewed Pexiderized forms of (1.4), containing two and three unknown mappings in [13] and [14]. Moreover, with the aim to analyze more such functional equations, Nath and Singh [15] came across the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta}, \tag{1.5}$$

where $f : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $\beta$ is a fixed positive real power different from 1 satisfying the conventions $0^\beta := 0$, $1^\beta := 1$. They obtained its general solutions for $n \geq 3, m \geq 3$ being fixed integers. In the recent past, Singh and Grover [26] have investigated the stability of equation (1.5) for $n \geq 3, m \geq 3$ being fixed integers.

Objective of this paper is to find the general solutions of the Pexiderized form of functional equation (1.5), that is

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} k(p_i) \sum_{j=1}^{m} q_j^{\beta}, \tag{A}$$

where $f : I \to \mathbb{R}$, $k : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3, m \geq 2$ being integers and $\beta$ is a fixed positive real power different from 1 satisfying the conventions $0^\beta := 0$, $1^\beta := 1$.

Apparently functional equation (A) being a Pexiderized form of (1.4) is useful in characterizing the Havrda and Charvát [5] entropies. It is worth a mention that general solutions of (A) has been discussed by Nath and Singh [16] but for $n \geq 3$, $m \geq 3$ being fixed integers. Moreover, recently few sum form functional equations were considered by Nath and Singh [12, 19]; Singh and Dass [23]; Singh and Grover [24, 25] who obtained their general solutions for fixed integers $n \geq 3$, $m \geq 3$ or $n \geq 3$, $m \geq 2$. However, functional equation (A) for $n \geq 3$, $m \geq 2$ being fixed integers seem to have been missed.

This paper is divided into four sections. In next section 2, we mention some definitions and preliminary results which will be used in the subsequent sections. In section 3, the general solution of functional equation (A) is obtained for the fixed integers $n \geq 3$, $m \geq 2$. In section 4, we discuss the relevance of functional equation (A) from the perspective of diversity index.

## 2. Some Preliminary Results

This section provides some known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on $I$ or on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if the equation $a(x + y) = a(x) + a(y)$ holds for all $(x, y) \in \Delta$. Analogously, a mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if the equation $A(x + y) = A(x) + A(y)$ holds for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. Daróczy and Losonczi [3] established an interesting relation between these two additive mappings. They proved that there exists a unique additive extension of the additive mapping $a : I \to \mathbb{R}$ to the set of real numbers.

**Result 2.1** ([10]). *Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geq 3$ a fixed integer and $c$ a real constant. Then there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = a(p) - \frac{1}{n}a(1) + \frac{c}{n}$ for all $p \in I$.*

**Result 2.2** ([9]). *Let $m \geq 2$ be fixed integer; $\beta$ be fixed positive real power different from 1 satisfying the conventions $0^\beta := 0$, $1^\beta := 1$ and $F : I \to \mathbb{R}$ be real valued mapping which satisfy the functional equation*

$$\sum_{j=1}^{m} F(pq_j) = F(p) \sum_{j=1}^{m} q_j^\beta \tag{2.1}$$

*for all $p \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$. If $F(0) = 0$, then any general solution of* (2.1) *is of the form $F(p) = F(1)p^\beta$ for all $p \in I$.*

## 3. The General Solution of the Functional Equation (A)

The main result of this section is the following theorem:

**Theorem 3.1.** *Let $n \geq 3$, $m \geq 2$ be fixed integers; $\beta$ be fixed positive real power different from 1 satisfying the conventions $0^\beta := 0$, $1^\beta := 1$ and $f : I \to \mathbb{R}$, $k : I \to \mathbb{R}$. The pair $(f, k)$ satisfies* (A) *if and only if there exist the additive mappings $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ such that*

$$\left.\begin{array}{ll} \text{(i)} & f(p) = cp^\beta + a_1(p) - \frac{1}{nm}a_1(1), \\ \text{(ii)} & k(p) = cp^\beta + a_2(p) - \frac{1}{n}a_2(1). \end{array}\right\} \tag{3.1}$$

*Proof.* Let us put $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (A). We obtain

$$\sum_{i=1}^{n} \{f(p_i) - k(p_i)\} = -n(m-1)f(0). \tag{3.2}$$

By Result 2.1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) - k(p) = a(p) - \frac{1}{n}a(1) - (m-1)f(0) \tag{3.3}$$

for all $p \in I$. The substitution $p = 0$ in (3.3) along with the fact $a(0) = 0$, it follows that

$$a(1) = n[k(0) - mf(0)]. \tag{3.4}$$

From (3.3) and (3.4), after performing necessary calculations, we obtain

$$k(p) = f(p) - a(p) - f(0) + k(0) \tag{3.5}$$

for all $p \in I$. From (A), (3.4) and (3.5), it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) - \left[\sum_{i=1}^{n} f(p_i) + n(m-1)f(0)\right] \sum_{j=1}^{m} q_j^\beta = 0 \tag{3.6}$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 2$ being fixed integers. By Result 2.1, there exists a mapping $A : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable such that

$$\sum_{j=1}^{m} f\left(pq_j\right) - [f(p) + n(m-1)f(0)p] \sum_{j=1}^{m} q_j^{\beta} = A(p; q_1, \dots, q_m) - \frac{1}{n} A(1; q_1, \dots, q_m) \tag{3.7}$$

for all $p \in I$ and $(q_1, \dots, q_m) \in \Gamma_m$. On substitution $p = 0$ in (3.7) and using the fact that $A(0; q_1, \dots, q_m) = 0$, we get

$$A(1; q_1, \dots, q_m) = -nmf(0) + nf(0) \sum_{j=1}^{m} q_j^{\beta} \tag{3.8}$$

for all $(q_1, \dots, q_m) \in \Gamma_m$. From (3.7) and (3.8), after performing necessary calculations, it follows that

$$\sum_{j=1}^{m} f\left(pq_j\right) - [f(p) + n(m-1)f(0)p - f(0)] \sum_{j=1}^{m} q_j^{\beta} - mf(0) = A(p; q_1, \dots, q_m). \tag{3.9}$$

Let $x \in I$ and $(r_1, \dots, r_m) \in \Gamma_m$ be arbitrary probability distribution. Replacing $p$ by $xr_t$, $t = 1, \dots, m$ consecutively in (3.9); summing the resulting $m$ equations so obtained and using the additivity of the mapping $A : \mathbb{R} \times \Gamma_m \to \mathbb{R}$ in the first variable, we obtain

$$\sum_{t=1}^{m} \sum_{j=1}^{m} f\left(xr_t q_j\right) - \sum_{t=1}^{m} f(xr_t) \sum_{j=1}^{m} q_j^{\beta} - n(m-1)f(0)x \sum_{j=1}^{m} q_j^{\beta} + mf(0) \sum_{j=1}^{m} q_j^{\beta} - m^2 f(0) = A(x; q_1, \dots, q_m) \tag{3.10}$$

for all $x \in I$, $(q_1, \dots, q_m) \in \Gamma_m$ and $(r_1, \dots, r_m) \in \Gamma_m$. Now for $p = x$ and $q_1 = r_1, \dots, q_m = r_m$, functional equation (3.9) gives

$$\sum_{t=1}^{m} f(xr_t) = [f(x) + n(m-1)f(0)x - f(0)] \sum_{t=1}^{m} r_t^{\beta} + mf(0) + A(x; r_1, \dots, r_m). \tag{3.11}$$

With the aid of (3.11), functional equation (3.10) can be written as

$$\begin{aligned} &\sum_{t=1}^{m} \sum_{j=1}^{m} f\left(xr_t q_j\right) - [f(x) + n(m-1)f(0)x - f(0)] \sum_{t=1}^{m} r_t^{\beta} \sum_{j=1}^{m} q_j^{\beta} - m^2 f(0) \\ &= A(x; q_1, \dots, q_m) + [A(x; r_1, \dots, r_m) + n(m-1)f(0)x] \sum_{j=1}^{m} q_j^{\beta}. \end{aligned}$$

Apparently, the left hand side of the above equation is commutative in $r_t$ and $q_j$, $t = 1, \dots, m;\ j = 1, \dots, m$ (Aczél [1]). Thus its right hand side should also be commutative. As a result, we get

$$[A(x; q_1, \dots, q_m) + n(m-1)f(0)x] \left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] = [A(x; r_1, \dots, r_m) + n(m-1)f(0)x] \left[1 - \sum_{j=1}^{m} q_j^{\beta}\right]. \tag{3.12}$$

Now, we assert that for fixed positive real power $\beta \neq 1$

$$1 - \sum_{t=1}^{m} r_t^{\beta} \not\equiv 0 \tag{3.13}$$

for all $(r_1, \dots, r_m) \in \Gamma_m$; $m \geq 2$ being fixed integer. To the contrary suppose $1 - \sum_{t=1}^{m} r_t^{\beta} = 0$ for all $(r_1, \dots, r_m) \in \Gamma_m$. On choosing a probability distribution $\left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right) \in \Gamma_m$, we have $\left(\frac{1}{2}\right)^{\beta} = \frac{1}{2}$. This implies $\beta = 1$, contradicting our presumption that $\beta \neq 1$. Thus our supposition is wrong and so (3.13) follows proving our assertion. Now substituting $r_t = r_t^*, t = 1, \ldots, m$ in (3.13) and using it in (3.12), we obtain

$$A(x; q_1, \ldots, q_m) = a_0(x)\left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] - n(m-1)f(0)x\,, \tag{3.14}$$

where $a_0 : \mathbb{R} \to \mathbb{R}$ defined as

$$a_0(x) = \left[1 - \sum_{t=1}^{m} r_t^{*\beta}\right]^{-1} [A(x; r_1^*, \ldots, r_m^*) + n(m-1)f(0)x]$$

is an additive mapping. Moreover from (3.8), it follows that

$$a_0(1) = -nf(0). \tag{3.15}$$

From (3.9) and (3.14) the functional equation

$$\sum_{j=1}^{m}\left\{f\left(pq_j\right) - a_0(pq_j) + n(m-1)f(0)pq_j - f(0)\right\} - \{f(p) - a_0(p) + n(m-1)f(0)p - f(0)\}\sum_{j=1}^{m} q_j^{\beta} = 0$$

follows. This yields the equation (2.1) where $F : I \to \mathbb{R}$ is defined as

$$F(x) = f(x) - a_0(x) + n(m-1)f(0)x - f(0) \tag{3.16}$$

for all $x \in I$. Clearly $F(0) = 0$, thus by Result 2.2, the mapping $F : I \to \mathbb{R}$ is of the form $F(p) = F(1)p^{\beta}$ for all $p \in I$. We get $F(1) = f(1) + (nm-1)f(0)$ from (3.15) and (3.16). On taking $c := f(1) + (nm-1)f(0)$, the solution (3.1) of functional equation (A) is attained from (3.16), (3.4) and (3.5), where the additive mappings $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ are defined as $a_1(x) = a_0(x) - n(m-1)f(0)x$ with $a_1(1) = -nmf(0)$ and $a_2(x) = a_1(x) - a(x)$ with $a_2(1) = -nk(0)$. This completes the proof. □

## 4. Comments

In this section we establish a connect of sum form functional equations which are emerging from Information Theory to a quantitative measure known as Index of Diversity.

For $n \in \mathbb{N}$, $(p_1, \ldots, p_n) \in \Gamma_n$, Havrda and Charvát [5] defined the entropies $H_n^{\beta} : I \to \mathbb{R}$ of degree $\beta$ as:

$$H_n^{\beta}(p_1, \ldots, p_n) = (1 - 2^{1-\beta})^{-1}\left[1 - \sum_{i=1}^{n} p_i^{\beta}\right], \tag{4.1}$$

where $\beta$ is fixed positive real power different from 1, such that $0^{\beta} := 0$ and $1^{\beta} := 1$.

A Diversity Index is a nonnegative real valued mapping defined on a probability distribution. It indicates the differences within the given sample space. Over the years, it has evolved with a multidisciplinary approach resulting in numerous definitions depending upon the discipline wherein it is conceived. We would prefer the references [5], [6] and [12] for the readers to get familiar with the phenomenon of diversity and its various fields of research. Also, few significant contributions that have recently appeared in this area are by Rajaram and Castellani [20]; Ginebra and Puig [4].

For $n \in \mathbb{N}$, $(p_1, \ldots, p_n) \in \Gamma_n$, Hill [5] defined Diversity Number $N_a : \Gamma_n \to \mathbb{R}$, as:

$$N_a(p_1, \ldots, p_n) = \left(\sum_{i=1}^{n} p_i^{a}\right)^{1/1-a}, \tag{4.2}$$

where '$a$' is referred as the order of diversity; '$n$' as the richness and the term within the parentheses as the basic sum. The general expression (4.2) also referred as the Effective Number or Hill number [7].

In addition to this, Hill [6] reflected upon the case when $a = 1$, where he proved that expression (4.2) is undefined for $a = 1$. However, its limit as '$a$' approaches to 1 exists and is related to Shannon entropies given by (1.3). Further, Rao [21], Tuomisto [28], Jost [7] and Hill [6] have elaborated on these diversity indices for different values of '$a$'. Indeed, Tuomisto further elaborated on the cases $a < 0$, $a > 0$ and logically justified that '$a$' must be restricted to nonnegative values (see p. 5, [28]).

Here, from (4.1) and (4.2), it follows that

$$[N_a(p_1, \ldots, p_n)]^{1-a} = 1 - (1 - 2^{1-a}) H_n^a(p_1, \ldots, p_n). \tag{4.3}$$

This gives that $(1 - a)^{th}$ power of Hill number of order '$a$' is equal to the entropy of order '$a$' corresponding to the same value of '$a$'. It needs to be remarked that (4.3) plays a key role in connecting three branches: Functional Equations, Information Theory and Index of Diversity demonstrated below.

We first demonstrate the connect of sum form functional equation (A) with diversity indices. For this we compute the summands (3.1)(i) and (3.1)(ii) of (A) and making use of (4.2), we obtain

$$\sum_{i=1}^{n} f(p_i) = c[N_\beta(p_1, \ldots, p_n)]^{1-\beta} + n(1 - m) f(0) \tag{4.4}$$

and

$$\sum_{i=1}^{n} k(p_i) = c[N_\beta(p_1, \ldots, p_n)]^{1-\beta}. \tag{4.5}$$

Clearly, the solution (3.1) of functional equation (A) is connected to Hill number of order $\beta$, if $c \neq 0$. Moreover if $c = 0$, then the summands $\sum_{i=1}^{n} f(p_i)$ and $\sum_{i=1}^{n} k(p_i)$ do not represent any diversity indices. Thus, the case $c = 0$ is not of much importance. Hence it can be concluded that functional equation (A) is related to Index of Diversity. Also, with the aid of (4.3) the connect of sum form functional equation (A) with entropies of degree $\beta$ follows from (4.4) and (4.5), thus validating the fact that it is useful in characterising the entropies of degree $\beta$. Summarizing this section we say, (A) is emerging from Information Theory and connected to Index of Diversity.

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