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Some Classes of Finite Sums Related to the Generalized Harmonic Functions and Special Numbers and Polynomials

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Abstract

The aim of this paper is to give some new classes of finite sums involving the numbers $y(m,\lambda)$, the generalized harmonic functions, special numbers and polynomials, the Dedekind sums, and other combinatorial sum. Reciprocity laws for these sums are proven. Some applications of these reciprocity laws are presented. With aid of the reciprocity law of the Dedekind sums, formulas for many new finite sums are obtained. Relations among these new classes of finite sums, partial sum of the generalized harmonic functions, the Riemann zeta function, the Hurwitz zeta function, hypergeometric series, polylogarithms, digamma functions, polygmamma functions, and special numbers and polynomials and other combinatorial sums are given. Moreover, some formulas for the partial sum of the generalized harmonic functions and special numbers and polynomials are given. Finally, coments and observations on the results of this paper are given.

Keywords: Generating functions, Special numbers and polynomials, Finite sum, Generalized harmonic functions, Riemann zeta function, Hurwitz zeta function, Hypergeometric series, Polylogarithms, Digamma functions, Polygmamma functions, Dedekind sums, Hardy sums

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1. Introduction

Finite sums involving special numbers and polynomials have been used to investigate and study many areas of mathematics and other sciences. Many formulas and relations for these sums have been investigated by many methods involving the generating functions, special functions, integral methods, and series.

The main motivation of this paper is to define many families of finite sums including the numbers $y(m,\lambda)$, the generalized harmonic functions, special numbers and polynomials, the Dedekind sums, and other combinatorial sum. Reciprocity laws for these sums are proven.

We use the following standard notations, formulas, and definitions:

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of natural numbers, the ring of integer numbers and the field of complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1,-2,-3,\ldots\}$, and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$. We also tacitly suppose that for $z \in \mathbb{C}$, $\log z$ denotes the principal branch of the many-valued function $\mathrm{Im}(\log z)$ with the imaginary part $\log z$ constrained by

$$-\pi < \mathrm{Im}(\log z) \leq \pi.$$

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Here $\log e = 1$ will be considered throughout this paper.

$$0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N}. \end{cases}$$

The Pochhammer's symbol involving the rising factorial polynomials, is given by

$$(\lambda)_v = \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)} = \lambda(\lambda + 1)\cdots(\lambda + v - 1)$$

and

$$(\lambda)_0 = 1$$

for $\lambda \neq 1$, where $v \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\Gamma(\lambda)$ denotes the gamma function.

$$\binom{\lambda}{v} = \frac{\lambda(\lambda - 1)\cdots(\lambda - v + 1)}{v!} = \frac{(\lambda)^{\underline{v}}}{v!} \ (v \in \mathbb{N}, \lambda \in \mathbb{C})$$

and

$$\binom{\lambda}{0} = 1.$$

Observe that

$$(-\lambda)_v = (-1)^v (\lambda)^{\underline{v}}$$

(*cf.* [41–53]; and references therein).

The Stirling numbers of the second kind are also given by the following generating function including falling factorial:

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k \tag{1.1}$$

(*cf.* [1–7], [13–19], [26–53] and see the references cited therein).

The Bernoulli numbers, B_n, are defined by means of the following generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

(*cf.* [1–7], [13–19], [26–53] and see the references cited therein).

The harmonic numbers, denoted by $f(n)$ in Euler's works and also in generally represented with the symbol h_n, H_n and $H(n)$, are defined by means of the following generating function:

$$\sum_{n=1}^{\infty} H_n z^n = \frac{\log(1 - z)}{z - 1}, \tag{1.2}$$

where $|z| < 1$ (*cf.* [1, 5, 9, 15, 16, 18, 19, 21, 34, 36], [46–48], [51, 52]).

The Harmonic numbers are also given by the following integral representation:

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x} dx \tag{1.3}$$

and

$$H_n = -n \int_0^1 x^{n-1} \log(1 - x) dx = -n \int_0^1 (1 - x)^{n-1} \log(x) dx, \tag{1.4}$$

where $H_0 = 0$ (*cf.* [9, 36, 46, 47]).

The generalized harmonic functions $H_m^{(s)}(w)$ are given by

$$H_m^{(s)}(w) = \sum_{n=1}^{m} \frac{1}{(n+w)^s},$$

$(m \in \mathbb{N};\ s \in \mathbb{C};\ w \in \mathbb{C}\backslash\mathbb{Z}^-; \mathbb{Z}^- = \{-1,-2,-3,\ldots\})$ so that, obviously, $H_0^{(0)}(0) = 1$, and also for $w = 0$, we have the generalized harmonic numbers:

$$H_m^{(s)} = H_m^{(s)}(0) = \sum_{n=1}^{m} \frac{1}{n^s},$$

(*cf.* [1, 5, 9, 15, 16, 18, 19, 21, 34, 36], [46–48], [51, 52]).

For $s = 1$, we have the generalized harmonic numbers or also known as the generalized harmonic functions are defined by the following formula:

$$H_m(w) = \sum_{n=1}^{m} \frac{1}{n+w},$$

where $H_0(x) = 0$, w is an indeterminate and $m \in \mathbb{N}$ (*cf.* [1, 5, 9], [15–19], [21, 34, 36], [46–48], [51, 52]).

It is clear that

$$H_m = H_m^{(1)}(0).$$

(*cf.* [1, 5, 9], [15–19], [21, 34, 36], [46–48], [51, 52]).

The polylogarithm function is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{1.5}$$

The definition of $Li_s(z)$ is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$. This definition can be extended to $|z| \geq 1$ by the process of analytic continuation.

For $\mathrm{Re}(s) > 1$, and $z = 1$, we have

$$Li_s(1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(*cf.* [51]). The special case $s = 1$ involves the ordinary natural logarithm, that is

$$Li_1(z) = -\log(1-z)$$

and

$$Li_1\left(\frac{1}{2}\right) = \log(2)$$

while the special cases $s = 2$ and $s = 3$ are called the dilogarithm (also referred to as Spence's function) and trilogarithm respectively. For $s = m$ with $|z| \leq 1$; $m \in \mathbb{N}\backslash\{1\}$, one has

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$$

(*cf.* [51]).

It is time to give brief summary of this paper as follows:

In subsection of Section 1, we study some properties of the interpolation function for the numbers $y_{8,n}(\lambda; a)$. To give some explicit formulas for the numbers $y(n, \lambda)$, we make use of the generating function of the numbers $y_{8,n}(\lambda; a)$, and some explicit formulas for the harmonic numbers.

In Section 2, we give a new family of finite sums involving generalized harmonic functions, generalized harmonic numbers and the Hurwitz zeta function, and digamma function.

In Section 3, we give some formulas for families finite sums involving Dedekind type sums and the numbers $y(n, \lambda)$. We also give some interesting reciprocity laws for these sums.

1.1. Interpolation function for the numbers $y_{8,n}(\lambda;a)$

The numbers $y_{8,n}(\lambda;a)$ and the polynomials $y_{8,n}(x,\lambda;a)$ are respectively defined by means of the following generating functions:

$$K_1(t;a,\lambda)=\frac{\log(\lambda+a^t)}{a^t+\lambda-1}=\sum_{n=0}^{\infty}y_{8,n}(\lambda;a)\frac{t^n}{n!} \tag{1.6}$$

and

$$\begin{aligned} K_2(t,x;a,\lambda) &= a^{xt}K_1(t;a,\lambda) \\ &= \sum_{n=0}^{\infty}y_{8,n}(x,\lambda;a)\frac{t^n}{n!} \end{aligned} \tag{1.7}$$

(*cf.* [44, Eq. (4.1)-Eq. (4.2), p. 47]).

For functions $K_1(t;a,\lambda)$ and $K_2(t,x;a,\lambda)$, when $\lambda\neq 0$, we assume that

$$\left|\frac{a^t}{\lambda}\right|<1$$

and

$$\left|t\log a+\log\left(\frac{1}{\lambda-1}\right)\right|<\pi.$$

For $\lambda=0$, by using (1.6), we have

$$\frac{t\log a}{e^{t\log a}-1}=\sum_{n=0}^{\infty}y_{8,n}(0;a)\frac{t^n}{n!}.$$

Combining the above function with (1), we get

$$y_{8,n}(0;a)=(\log a)^n B_n$$

(*cf.* [44, p. 48]).

We have recently defined many different Peters and Boole type combinatorial numbers and polynomials. We gave some notations for these numbers and polynomials. For instance, in order to distinguish them from each other, these polynomials are labeled by the following symbols:

$y_{j,n}(x;\lambda,q)$, $j=1,2,\ldots,9$, and also $Y_n(x;\lambda)$. Therefore, for the numbers $y_{9,n}(\lambda;a)$ the number 9 is only used for index representation of these polynomials.

For $x=0$, we have

$$y_{8,n}(0,\lambda;a)=y_{8,n}(\lambda;a)$$

(*cf.* [44, p. 48]).

From the previous equation, the following relations are derived:

$$y_{8,0}(\lambda;1)=\frac{\log(\lambda+1)}{\lambda} \tag{1.8}$$

and for $n\geq 1$,

$$y_{8,n}(\lambda;1)=0$$

(*cf.* [44, pp. 48-49]).

Interpolation function for the number $y_{8,k}(\lambda;a)$ are given by the following definition.

Definition 1.1. (*cf.* [44, p. 52]) Let $a\geq 1$. For $\lambda\in\mathbb{C}\setminus\{0,1\}$ ($|\frac{1}{\lambda-1}|<1$) and $s\in\mathbb{C}$, a unification of zeta type function $Z_1(s;a,\lambda)$ is defined by

$$\mathcal{Z}_1(s;a,\lambda)=\frac{(\log\lambda)}{(\log a)^s}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s(\lambda-1)^{n+1}}+\frac{1}{(\log a)^s}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}}\right)\frac{1}{(n+1)^s} \tag{1.9}$$

where $\lambda\in\mathbb{C}\setminus\{0,1\}$ ($|\frac{1}{\lambda-1}|<1$; $\mathrm{Re}(s)>1$).

Substituting $a = e$ into (1.9), we have

$$\mathcal{Z}_1(s; e, \lambda) = (\log \lambda) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s (\lambda - 1)^{n+1}} + \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)\lambda^{j+1} (\lambda - 1)^{n+1-j} (n+1)^s}$$

(*cf.* [44, p. 52]).

When $\lambda = 2$ and $\mathrm{Re}(s) > 1$, the above relation reduces to the following interesting formula:

$$\mathcal{Z}_1(s; e, 2) = (\log 2) \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s} \sum_{j=0}^{n} \frac{1}{(j+1)2^{j+1}} \tag{1.10}$$

(*cf.* [44, Eq. (4.13), p. 52]).

The function $Z_1(s; a, \lambda)$ is analytic continuation, except $s = 1$ and $\lambda = 2$ in whole complex plane.

Theorem 1.2. (*cf.* [44, p. 52]). *Let $\lambda \in \mathbb{C}$ ($|\frac{1}{\lambda}| < 1$) and $k \in \mathbb{N}$. Then we have*

$$\mathcal{Z}_1(-k; a, \lambda) = y_{8,k}(\lambda; a). \tag{1.11}$$

Combining (1.10) with (1.14), we have

$$\mathcal{Z}_1(s; e, 2) = \log 2\zeta(s) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s} \left(H_{n+1} - 2^{-n-1} \sum_{v=0}^{n+1} \binom{n+1}{v} H_v \right).$$

Substituting $s = -k$, $k \in \mathbb{N}$, into the above equation, and using (1.11) and with the help of the following well-known formula involving analytic continuation of the Riemann zeta function:

$$\zeta(-k) = -\frac{B_k}{k}$$

(*cf.* [50, 51]), we arrive at the following theorem:

Theorem 1.3. *Let $k \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{\infty} (-1)^n (n+1)^k \left(H_{n+1} - 2^{-n-1} \sum_{v=0}^{n+1} \binom{n+1}{v} H_v \right) = y_{8,k}(2; e) + \frac{(\log 2) B_k}{k}.$$

The numbers $y(n, \lambda)$, which are arised from the equation (1.9), are defined by the following relation:

$$y(n, \lambda) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)\lambda^{j+1} (\lambda - 1)^{n+1-j}} \tag{1.12}$$

(*cf.* [44, Eq. (6.1), p. 57]).

Thus, for $\lambda \neq 1$, the function $\mathcal{Z}_1(s; a, \lambda)$ is modified as follows:

$$Li_s\left(\frac{1}{1-\lambda}\right) = \frac{(\lambda - 1)(\log a)^s \mathcal{Z}_1(s; a, \lambda)}{\log \lambda} - \frac{1}{\log \lambda} \sum_{n=0}^{\infty} \frac{y(n, \lambda)}{(n+1)^s}.$$

Substituting $\lambda = 2$ into (1.12), we get

$$y(n) := y(n, 2) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)2^{j+1}} \tag{1.13}$$

(*cf.* [44, Eq. (6.2), p. 57]).

In [44, p. 57], we gave the following open problems, involving the numbers $y(n)$ and the numbers $y(n, \lambda)$:

1. One of the first questions that comes to mind what is generating function for the numbers $y(n)$ and the numbers $y(n,\lambda)$.
2. Is there other special number families related to the numbers $y(n)$?
3. What are the combinational applications of the numbers $y(n)$?
4. Can we find a special arithmetic function representing this family of numbers?

We can partially solve some of the above open question (*cf.* [42, 44–46]). For example, since

$$y(n)=(-1)^n\left(\sum_{j=1}^{n}\frac{1}{j2^j}+\frac{1}{(n+1)2^{n+1}}\right),$$

with the aid of the following known finite sum involving the harmonic sum:

$$\sum_{j=1}^{n}\frac{1}{j2^j}=H_n-2^{-n}\sum_{k=0}^{n}\binom{n}{k}H_k \tag{1.14}$$

(*cf.* [16, Eq. (1.25)]), the equation (1.13) reduces to the following formula:

$$y(n)=(-1)^nH_n+\frac{(-1)^n}{(n+1)2^{n+1}}+(-1)^{n+1}2^{-n}\sum_{k=0}^{n}\binom{n}{k}H_k.$$

For other solutions, see (*cf.* [44–46]).

By combining (1.13) with (1.14), we get the following finite sum:

$$\sum_{j=1}^{n}\frac{1}{j2^j}=(-1)^ny(n)-\frac{1}{(n+1)2^{n+1}}. \tag{1.15}$$

For $\lambda=1$, we gave the following interesting and novel finite sum:

$$\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}=2\,(n+1)\,y(n,-1) \tag{1.16}$$

(*cf.* [44, Eq. (6.5), p. 58]).

We now present the other examples for some known finite sums:

For $n\in\mathbb{N}$, by applying $\frac{d^n}{dt^n}$ to the function

$$\left(e^t+1\right)^k$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^2e^{tk},$$

the following well-known finite sums associated with binomial coefficients are given:

$$\frac{d^n}{dt^n}\left\{\left(e^t+1\right)^k\right\}|_{t=0}=\sum_{j=1}^{k}\binom{k}{j}j^n \tag{1.17}$$

and

$$\frac{d^m}{dt^m}\left\{\sum_{k=0}^{n}\binom{n}{k}^2e^{tk}\right\}|_{t=0}=\sum_{k=1}^{n}\binom{n}{k}^2k^m \tag{1.18}$$

(*cf.* [23, 24, 41, 43, 53]).

By the aid of the hypergeometric function, various interesting finite sums have been defined. For instance, we [43] defined the following finite sums for the combinatorial numbers $y_6(n,k;\lambda,p)$ involving finite sums of powers of binomial coefficients:

$$F_{y_6}(t,n;\lambda,p) = \frac{1}{n!}\, {}_pF_{p-1}\left[\begin{array}{c} -n,-n,...,-n \\ 1,1,...,1 \end{array} ;(-1)^p \lambda e^t\right] = \sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}, \tag{1.19}$$

where $n,p \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) and ${}_pF_q$ denotes the well-known generalized hypergeometric function which is defined by

$${}_pF_q\left[\begin{array}{c} \alpha_1,...,\alpha_p \\ \beta_1,...,\beta_q \end{array} ;z\right] = \sum_{m=0}^{\infty}\left(\frac{\prod_{j=1}^{p}\left(\alpha_j\right)^{\overline{m}}}{\prod_{j=1}^{q}\left(\beta_j\right)^{\overline{m}}}\right)\frac{z^m}{m!},$$

where the above series converges for all z if $p < q+1$, and for $|z| < 1$ if $p = q+1$. Assuming that all parameters have general values, real or complex, except for the β_j, $j = 1,2,...,q$ none of which is equal to zero or a negative integer and also

$$(\lambda)^{\overline{v}} = \prod_{j=0}^{v-1}(\lambda + j),$$

and $(\lambda)^{\overline{0}} = 1$ for $\lambda \neq 1$, where $v \in \mathbb{N}$, $\lambda \in \mathbb{C}$. For the generalized hypergeometric function and their applications, it is also recommended to refer to the following resource (*cf.* [27], [51]).

By using the equation (1.19), we have

$$y_6(m,n;\lambda,p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p k^m \lambda^k, \tag{1.20}$$

where $n,m,p \in \mathbb{N}_0$ and also

$$y_6(m,n;\lambda,p) = \frac{\partial^m}{\partial t^m}\left\{F_{y_6}(t,m,n;\lambda,p)\right\}|_{t=0} \tag{1.21}$$

where $m \in \mathbb{N}$ (*cf.* [43]).

Putting $\lambda = 1$ in (1.20), we have

$$n!y_6(m,n;1,p) = M_{m,p}(n) = \sum_{k=0}^{n}\binom{n}{k}^p k^m$$

(*cf.* [24], [31, Eq. (5.1.1), p. 159], [43]).

With the aid of finite sums, we now calculate the following integral in terms of the numbers $y(n)$

$$\int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n}dx.$$

The above integral is given by the of the following known formula

$$\int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n}dx = \frac{(2^n+1)\log 2 + H_n}{2^{n+1}} - \frac{1}{2}\sum_{j=1}^{n}\frac{1}{j2^j} - \frac{\pi}{2^{n+2}}\sum_{j=0}^{n-1}\frac{\binom{2j}{j}}{2^j} + \frac{1}{2^{n+1}}\sum_{j=1}^{n-1}\frac{\binom{2j}{j}}{2^j}\sum_{k=1}^{j}\frac{2^k}{k\binom{2k}{k}}$$

(*cf.* [52, Eq. (1.60), p. 22]).

Combining the above integral formula with (1.15), we get

$$\int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n}dx = \frac{(2^n+1)\log 2 + H_n}{2^{n+1}} + \frac{(-1)^{n+1}y(n)}{2} + \frac{1}{(n+1)2^{n+2}} - \pi\sum_{j=0}^{n-1}\frac{\binom{2j}{j}}{2^{j+n+2}} + \sum_{j=1}^{n-1}\frac{\binom{2j}{j}}{2^{j+n+1}}\sum_{k=1}^{j}\frac{2^k}{k\binom{2k}{k}},$$

where it is assumed that

$$\sum_{j=1}^{-1}\frac{\binom{2j}{j}}{2^j}\sum_{k=1}^{j}\frac{2^k}{k\binom{2k}{k}}=0. \tag{1.22}$$

By combining the above equation with the following well-known explicit formula for the Catalan numbers:

$$C_j=\frac{1}{j+1}\binom{2j}{j}; \quad j\in\mathbb{N}_0,$$

we arrive at the modification of Theorem 3.4 in [45] as follows:

Theorem 1.4. *Let $n\in\mathbb{N}_0$. Let* (1.22) *be satisfied. Then we have*

$$\begin{aligned}\int_0^1\frac{x^{2n}}{(1+x)(1+x^2)^n}dx &= \frac{(2^n+1)\log 2+H_n}{2^{n+1}}+\frac{(-1)^{n+1}}{2}y(n)+\frac{1}{(n+1)2^{n+2}}\\ &\quad -\pi\sum_{j=0}^{n-1}\frac{(j+1)C_j}{2^{j+n+2}}+\sum_{j=1}^{n-1}\frac{(j+1)C_j}{2^{n+j+1}}\sum_{k=1}^{j}\frac{2^k}{k(k+1)C_k}.\end{aligned} \tag{1.23}$$

With the aid of (1.23), some special values for the following integral

$$\int_0^1\frac{x^{2n}}{(1+x)(1+x^2)^n}dx$$

are given as follows:

Substituting $n=0$ into (1.23), we have

$$\int_0^1\frac{1}{1+x}dx=\log 2.$$

Putting $n=1$ in (1.23), we have

$$\begin{aligned}\int_0^1\frac{x^2}{(1+x)(1+x^2)}dx &= \frac{3\log 2+H_1}{4}+\frac{y(1)}{2}+\frac{1}{16}-\frac{\pi}{8}C_0\\ &= \frac{3}{4}\log 2-\frac{\pi}{8}.\end{aligned}$$

Substituting $n=2$ into (1.23), we have

$$\begin{aligned}\int_0^1\frac{x^4}{(1+x)(1+x^2)^2}dx &= \frac{5\log 2+H_2}{8}-\frac{1}{2}y(2)+\frac{1}{48}-\pi\sum_{j=0}^{1}\frac{(j+1)C_j}{2^{j+4}}+\sum_{j=1}^{1}\frac{(j+1)C_j}{2^{3+j}}\sum_{k=1}^{j}\frac{2^k}{k(k+1)C_k}\\ &= \frac{5}{4}\log 2-\frac{\pi}{8}.\end{aligned}$$

2. A new family of finite sums involving generalized harmonic functions

In this section, we investigate a family of finite sum involving generalized harmonic functions. We give some properties of this sum.

We set the following finite sum:

$$\mathfrak{s}_{m,s}(a_1,a_2,\ldots,a_{v-1};a_v,v)=\sum_{j=1}^{v}\sum_{k=1}^{m}\frac{1}{\left(a_j+k\right)^s} \tag{2.1}$$

$\Big(v, m \in \mathbb{N};\ s \in \mathbb{C};\ a_j \in \mathbb{C}\backslash\mathbb{Z}^-; j \in \{1, 2, 3, \ldots, v\}\Big)$.

Substituting $a_1 = a_2 = \cdots = a_{v-1} = a_v = 0$, and $v = 1$ into (2.1), we have

$$\mathfrak{s}_{m,s}(0, 0, \ldots, 0; 0, 1) = H_m^{(s)}$$

(*cf.* [1, 5, 9, 15, 16, 18, 19, 21, 34, 36], [46–48], [51, 52]).

Some special values of the sum $\mathfrak{s}_{m,s}(a_1, a_2, \ldots, a_{v-1}; a_v, v)$ are given as follows:

Substituting $a_1 = z - 1$ and $a_2 = \cdots = a_{v-1} = a_v = 0$ into (2.1), we have

$$\mathfrak{s}_{m,s}(z - 1, 0, \ldots, 0; 0, v) = vH_m^{(s)}(z - 1)$$

(*cf.* [1, 5, 9, 15, 16, 18, 19, 21, 34, 36], [46–48], [51, 52]).

Putting $v = 1$ in equation (2.1), for $m \in \mathbb{N}$; $\kappa \in \mathbb{C}$; $z \in \mathbb{C}\backslash\mathbb{Z}_0^-$, we have the following well-known formula for the generalized harmonic numbers $H_m^{(\kappa)}(z)$, which are also denoted by $H_m(z; \kappa)$:

$$\mathfrak{s}_{m,\kappa}(z - 1, 1) = \sum_{k=1}^{m} \frac{1}{(z + k - 1)^{\kappa}} = H_m^{(\kappa)}(z), \tag{2.2}$$

which, for $s = \kappa$, is recorded by Rassias and Srivastava [34, Eq. (1.11)].

Substituting $a_1 = a_2 = \cdots = a_{v-1} = a_v = 0$ into (2.1), we obtain

$$\mathfrak{s}_{m,s}(0, 0, \ldots, 0; 0, v) = vH_m^{(s)},$$

for $v = 1$, we have

$$\mathfrak{s}_{m,s}(0) = H_m^{(s)},$$

for $v = 1$, we also have

$$\mathfrak{s}_{m,1}(0) = H_m$$

(*cf.* [1, 5, 9, 15, 16, 18, 19, 21, 34, 36], [46–48], [51, 52]).

Substituting $a_j = b_j - 1$; $j \in \{1, 2, 3, \ldots, v\}$ into (2.1), we obtain

$$\begin{aligned}
\mathfrak{s}_{m,d+1}(b_1 - 1, b_2 - 1, \ldots, b_{v-1} - 1; b_v - 1, v) &= \sum_{j=1}^{v}\sum_{k=1}^{m} \frac{1}{\left(b_j + k - 1\right)^{d+1}} \\
&= \sum_{j=1}^{v} H_m^{(d+1)}(b_j - 1) \\
&= \sum_{j=1}^{v} \Big(\zeta(d + 1, b_j) - \zeta(d + 1, b_j + m)\Big).
\end{aligned}$$

The sum $\mathfrak{s}_{m,s}(a_1, a_2, \ldots, a_{v-1}; a_v, v)$ is related to the generalized harmonic functions $H_m^{(s)}(z)$. That is

$$\mathfrak{s}_{m,s}(a_1, a_2, \ldots, a_{v-1}; a_v) = \sum_{j=1}^{v} H_m^{(s)}(a_j). \tag{2.3}$$

The finite sum, given in (2.1), satisfies the following *Symmetric Property*:

$$\mathfrak{s}_{m,s}(a_2, a_3, \ldots, a_v; a_1, v) = \mathfrak{s}_{m,s}(a_1, a_3, \ldots, a_v; a_2, v) = \cdots = \mathfrak{s}_{m,s}(a_1, a_2, \ldots, a_{v-1}; a_v, v).$$

For instance, $v = 2$, we have

$$\mathfrak{s}_{m,s}(a_2; a_1, 2) = \mathfrak{s}_{m,s}(a_1; a_2, 2).$$

Theorem 2.1 (Reciprocity law)**.** *Reciprocity law for $s_{m,s}(a_1, a_2, \ldots, a_{v-1}; a_v, v)$ is given by*

$$\begin{aligned}
& \mathfrak{s}_{m,d+1}(a_2-1, a_3-1, \ldots, a_v-1; a_1-1, v) + \mathfrak{s}_{m,d+1}(a_1-1, a-1, \ldots, a_v-1; a_2-1, v) \\
& + \cdots + \mathfrak{s}_{m,d+1}(a_1-1, a_2-1, \ldots, a_{v-1}-1; a_v-1, v) \\
= & \sum_{j=1}^{v} \frac{\psi^{(d)}(a_j-1+m) - \psi^{(d)}(a_j-1)}{(-1)^d d!}.
\end{aligned}$$

Proof. In order to give proof of the assertion of theorem, the following well-known formula involving the polygamma functions $\psi^{(d)}(s)$ ($d \in \mathbb{N}$), the function $H_m^{(s)}(z)$ (*cf.* [51, Eq. (20), p. 22]; see also [16, Eq. (1.8)], [34])

$$\psi^{(d)}(s+m) - \psi^{(d)}(s) = (-1)^d d! H_m^{(d+1)}(s-1) \tag{2.4}$$

where $m, d \in \mathbb{N}_0$. Combining the equation (2.4) with the equation (2.1) and (2.3), for $s = d+1$ and $a_j = b_j - 1$; $j \in \{1, 2, 3, \ldots, v\}$, we obtain

$$\begin{aligned}
\mathfrak{s}_{m,d+1}(b_1-1, b_2-1, \ldots, b_{v-1}-1; b_v-1, v) &= \sum_{j=1}^{v} \sum_{k=1}^{m} \frac{1}{\left(b_j+k-1\right)^{d+1}} \\
&= \sum_{j=1}^{v} H_m^{(d+1)}(b_j-1) \\
&= \sum_{j=1}^{v} \frac{\psi^{(d)}(b_j-1+m) - \psi^{(d)}(b_j-1)}{(-1)^d d!}.
\end{aligned}$$

Therefore

$$\mathfrak{s}_{m,d+1}(b_1-1, b_2-1, \ldots, b_{v-1}-1; b_v-1) = \sum_{j=1}^{v} \frac{\psi^{(d)}(b_j-1+m) - \psi^{(d)}(b_j-1)}{(-1)^d d!}. \tag{2.5}$$

By using symmetric property of the sum:

$$\mathfrak{s}_{m,d+1}(b_1-1, b_2-1, \ldots, b_{v-1}-1; b_v-1, v),$$

we get

$$\begin{aligned}
& \mathfrak{s}_{m,d+1}(b_2-1, b_3-1, \ldots, b_v-1; b_1-1, v) + \mathfrak{s}_{m,d+1}(b_1-1, b_3-1, \ldots, b_v-1; b_2-1, v) \\
& + \cdots + \mathfrak{s}_{m,d+1}(b_1-1, b_2-1, \ldots, b_{v-1}-1; b_v-1, v) \\
= & v \mathfrak{s}_{m,d+1}(b_1-1, b_2-1, \ldots, b_{v-1}-1; b_v-1, v).
\end{aligned}$$

By using the above equation in conjunction with the equation (2.5), we obtain

$$\begin{aligned}
& \mathfrak{s}_{m,d+1}(b_2-1, b_3-1, \ldots, b_v-1; b_1-1, v) + \mathfrak{s}_{m,d+1}(b_1-1, b_3-1, \ldots, b_v-1; b_2-1, v) \\
& + \cdots + \mathfrak{s}_{m,d+1}(b_1-1, b_2-1, \ldots, b_{v-1}-1; b_v-1, v) \\
= & \sum_{j=1}^{v} \frac{\psi^{(d)}(b_j-1+m) - \psi^{(d)}(b_j-1)}{(-1)^d d!}.
\end{aligned}$$

Proof of theorem is completed. □

A relation between the sum $\mathfrak{s}_{m,1}(a_1, a_2, \ldots, a_{v-1}; a_v, v)$ and generalized harmonic functions of order b is given by

$$\mathfrak{s}_{m,b}(a_1, a_2, \ldots, a_{v-1}; a_v, v) = \sum_{j=1}^{v} H_m^{(b)}(a_j)$$

and for $b = 1$, we also have

$$\mathfrak{s}_{m,1}(a_1, a_2, \ldots, a_{v-1}; a_v, v) = \sum_{j=1}^{v} H_m(a_j).$$

For $v = 3$ and $b = 1$, we have known relation between the sum $\mathfrak{s}_m(a_1, a_2; a_3, 3)$ and the Gauss's hypergeometric series is given by

$$\mathfrak{s}_{m,1}(a_1 - 1, a_2 - 1; 0, 3) - 3H_m = \sum_{j=1}^{m} \frac{1}{a_1 - 1 + j} + \sum_{j=1}^{m} \frac{1}{a_1 - 1 + j} - 2\sum_{j=1}^{m} \frac{1}{j}$$

or

$$\mathfrak{s}_{m,1}(a_1 - 1, a_2 - 1; 0, 3) - 3H_m = H_m(a_1 - 1) + H_m(a_2 - 1) - 2H_m.$$

Observe that, the right-hand side of the above equation derive from the following equation, which was proved by Beukers [8, p. 25]:

$$\log z \, {}_2F_1\left[\begin{matrix} a_1, a_2 \\ 1 \end{matrix} ; z\right] + \sum_{m=1}^{\infty} \frac{(a_1)_m (a_2)_m}{(m!)^2} z^m \left(H_m(a_1 - 1) + H_m(a_2 - 1) - 2H_m\right).$$

A relation between the sum $\mathfrak{s}_m(a_1, a_2, \ldots, a_{v-1}; 0, v)$ and the numbers $y\left(n, \frac{1}{2}\right)$ is given by

$$\mathfrak{s}_{m,1}(a_1, a_2, \ldots, a_{v-1}; 0, v) = \sum_{j=1}^{v-1} H_m(a_j) + 2^{n+2}\left(H_{\left[\frac{m}{2}\right]} + \frac{(-1)^{m+1}}{m+1}\right) - 2^{-m-2} y\left(m, \frac{1}{2}\right),$$

where $[x]$ denotes the integer part of x and $y\left(m, \frac{1}{2}\right)$ is related to the following relation, which was proved in [46, Theorem 3]:

$$y\left(m, \frac{1}{2}\right) = 2^{m+2}\left(\frac{(-1)^{m+1}}{m+1} + H_{\left[\frac{m}{2}\right]} - H_m\right); \qquad m \in \mathbb{N}_0. \tag{2.6}$$

In order to give explicit formula for

$$\sum_{j=1}^{v} \frac{1}{\left(a_j\right)_m} \frac{d}{d\lambda} \left\{\left(a_j\right)_m\right\},$$

we need the following known relations:

$$\frac{d}{d\lambda} \{(\lambda)_v\} = (\lambda)_v \left(\psi(\lambda + v) - \psi(\lambda)\right) \tag{2.7}$$

and

$$\psi(s + m) - \psi(s) = H_m(s - 1) \tag{2.8}$$

(*cf.* [15, Eq. (1.18)]). By combining the equation (2.7) with the equation (2.1), for $d = 0$ and $j \in \{1, 2, ..., v\}$, we get

$$\sum_{j=1}^{v} \frac{1}{\left(a_j\right)_m} \frac{d}{d\lambda} \{(\lambda)_m\} \big|_{\lambda = a_j} = \mathfrak{s}_{m,1}(a_1 - 1, a_2 - 1, \ldots, a_{v-1} - 1; a_v - 1, v),$$

which yields the assertion of the following theorem:

Theorem 2.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{s}_{m,1}(a_1 - 1, a_2 - 1, \ldots, a_{v-1} - 1; a_v - 1, v) = \sum_{j=1}^{v} \frac{1}{\left(a_j\right)_m} \frac{d}{d\lambda} \{(\lambda)_m\} \big|_{\lambda = a_j}.$$

Combining the following well-known formula:

$$\frac{d}{d\lambda} \{(\lambda)_m\} \big|_{\lambda = -m} = -m! H_m,$$

where $m \in \mathbb{N}_0$ (*cf.* [15, Eq. (1.19)]) with the equation (2.6), we also arrive at the following theorem:

Theorem 2.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{d}{d\lambda}\{(\lambda)_m\}|_{\lambda=-m} = m!\,(\lambda)_m\left(2^{-n-2}y\left(m,\frac{1}{2}\right)-H_{\left[\frac{m}{2}\right]}+\frac{(-1)^m}{m+1}\right).$$

Assuming $z,a,b,c \in \mathbb{C}$ with $|z|<1;\ |z|=0$ when $\operatorname{Re}\{c-a-b\}>0;\ c \notin \mathbb{Z}_0^-$, in [34, Eq. (1.5)-Eq. (1.7)], Rassias and Srivastava gave the following partial derivative formulas for the Gauss's hypergeometric series:

$$\frac{\partial}{\partial a}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}=\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\left(\psi(m+a)-\psi(a)\right)z^m, \tag{2.9}$$

$$\frac{\partial}{\partial b}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}=\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\left(\psi(m+b)-\psi(b)\right)z^m, \tag{2.10}$$

and

$$\frac{\partial}{\partial c}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}=\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\left(\psi(m+c)-\psi(c)\right)z^m. \tag{2.11}$$

Combining (2.8) and (2.2) respectively with the equations from (2.9) to (2.11), we have the following results:

$$\frac{\partial}{\partial a}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}=\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\mathfrak{s}_{m,1}(a-1,1)z^m, \tag{2.12}$$

$$\frac{\partial}{\partial b}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}=\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\mathfrak{s}_{m,1}(b-1,1)z^m, \tag{2.13}$$

and

$$\frac{\partial}{\partial c}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}=\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\mathfrak{s}_{m,1}(c-1,1)z^m. \tag{2.14}$$

Adding both sides of the equations from (2.12) to (2.14), we get the following theorem:

Theorem 2.4. *Let $z,a,b,c \in \mathbb{C}$ with $|z|<1;\ |z|=0$ when $\operatorname{Re}\{c-a-b\}>0;\ c \notin \mathbb{Z}_0^-$. Then we have*

$$\begin{aligned}&\frac{\partial}{\partial a}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}+\frac{\partial}{\partial b}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}+\frac{\partial}{\partial c}\left\{\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}z^m\right\}\\ =\ &\sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m\,m!}\left(\mathfrak{s}_{m,1}(a-1,1)+\mathfrak{s}_{m,1}(b-1,1)+\mathfrak{s}_{m,1}(c-1,1)\right)z^m.\end{aligned}$$

3. Finite sums derived from Dedekind type sums and the numbers $y(n,\lambda)$

The Dedekind eta function $\eta(\tau)$, introduced by Dedekind in 1877, plays a central role in elliptic modular functions, in number theory, in analysis, in combinatorics, in q-series, in theory of the Weierstrass elliptic functions, in modular forms, in Kronecker limit formula, and in theory of cryptography, in other areas. For $\tau \in \mathbb{H}=\{\tau=u+iv \in \mathbb{C}: v>0, v \in \mathbb{R}\}$, denotes the upper helf-plane, the function $\eta(\tau)$ is defined by the following relation:

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{m=1}^{\infty}\left(1-e^{2\pi i m\tau}\right)$$

(*cf.* [4, 32]). For $\tau \in \mathbb{H}$, and $|\ x\ |< 1$, the product $\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$ converges absolutely and is nonzero. Since the convergence is uniform on compact subsets of $\mathbb{H}$, the function $\eta(\tau)$ is analytic on $\mathbb{H}$.

The Dedekind sums are $s(d,c)$ derived from the behavior of the function $\log \eta(z)$ under the modular group $\Gamma(1)$, which is given by

$$\Gamma(1)=\left\{A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc=1, a,b,c,d\in\mathbb{Z}\right\},$$

where $Az=\frac{az+b}{cz+d}$. That is

$$\log \eta(Az)=\log \eta(z)+\frac{\pi i(a+d)}{12c}-\pi i\left(s(d,c)-\frac{1}{4}\right)+\frac{1}{2}\log(cz+d), \tag{3.1}$$

where $z\in\mathbb{H}$ and the Dedekind sum $s(d,c)$ is given by

$$s(d,c)=\sum_{j \bmod c}\left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right), \tag{3.2}$$

where $d,c\in\mathbb{Z}$ and $c>0$, $(d,c)=1$ and

$$\begin{aligned}((x)) &= \begin{cases} x-[x]-\frac{1}{2}, & x\notin\mathbb{Z} \\ 0, & x\in\mathbb{Z}\end{cases} \\ &= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}\end{aligned}$$

(*cf.* [2, 4, 7, 29, 32, 33, 40]).

By using the definition of $((x))$ together with the equation (3.2), we have the following known result:

$$s(d,c)=\frac{(c-1)(2d(2c-1)-3c)}{12c}-\frac{1}{c}\sum_{j=0}^{c-1} j\left[\frac{dj}{c}\right].$$

The sum $s(d,c)$ has the following reciprocity law:

$$s(d,c)+s(c,d)=-\frac{1}{4}+\frac{1}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{1}{dc}\right), \tag{3.3}$$

where d and c are coprime positive integers, that is $(c,d)=1$ (*cf.* [32]).

We set

$$h(c,d)=s(d,c)+s(c,d),$$

$$f(c,d)=-\frac{1}{4}+\frac{d}{12c}+\frac{c}{12d}+\frac{1}{12dc},$$

and

$$v(c,d)=\frac{(8d^2-9d+2)c^2+3d(2-3d)c+2d^2}{12cd}.$$

In order to give our main results, we now briefly explain the notations written under the sigma summation symbol by examples:

The following sum runs over coprime integers:

$$\sum_{\substack{c=1 \\ (c,d)=1}}^{m} h(c,d).$$

For example,

$$\sum_{\substack{c=1 \\ (c,2)=1}}^{10} h(c,d)=h(1,2)+h(3,2)+h(5,2)+h(7,2)+h(9,2).$$

Similarly, the following sum runs over non-coprime integers:

$$\sum_{\substack{c=1 \\ (c,d)>1}}^{m} f(c,d).$$

For example,

$$\sum_{\substack{c=1 \\ (c,2)>1}}^{10} h(c,d) = h(2,2) + h(4,2) + h(6,2) + h(8,2) + h(10,2).$$

If we sum both sides of (3.3) from $c = 1$ to m, we get

$$\sum_{\substack{c=1 \\ (c,d)=1}}^{m} h(c,d) + \sum_{\substack{c=1 \\ (c,d)>1}}^{m} f(c,d) = \frac{m(m+1) - 6dm + 2\left(d^2+1\right) H_m}{24d}, \tag{3.4}$$

where $m, d \in \mathbb{N}$.

If we sum both sides of (3.3) from $d = 1$ to m, we obtain

$$\sum_{\substack{d=1 \\ (d,c)=1}}^{m} h(c,d) + \sum_{\substack{d=1 \\ (d,c)>1}}^{m} f(c,d) = \frac{m(m+1) - 6cm + 2\left(c^2+1\right) H_m}{24c}, \tag{3.5}$$

where $m, c \in \mathbb{N}$.

By combining (3.4) with (2.6), we obtain

$$\begin{aligned} y\left(m, \frac{1}{2}\right) &= 2^{m+2}\left(H_{\left[\frac{m}{2}\right]} + \frac{(-1)^{m+1}}{m+1} - \frac{6dm - m(m+1)}{d^2+1}\right) \\ &\quad - \frac{3d2^{m+4}}{d^2+1}\left(\sum_{\substack{d=1 \\ (d,c)=1}}^{m} h(c,d) + \sum_{\substack{d=1 \\ (d,c)>1}}^{m} f(c,d)\right). \end{aligned}$$

After some calculations, we arrive at the following theorem:

Theorem 3.1. *Let c and d be coprime positive integers. Let $m \in \mathbb{N}$. Then we have*

$$\begin{aligned} y\left(m, \frac{1}{2}\right) &= 2^{m+2}\left(H_{\left[\frac{m}{2}\right]} + \frac{(-1)^{m+1}}{m+1} + \frac{m(m+1) - 6dm}{d^2+1}\right) \\ &\quad - 2^{m+4}\frac{3d}{d^2+1}\left(\sum_{\substack{d=1 \\ (d,c)=1}}^{m} h(c,d) + \sum_{\substack{d=1 \\ (d,c)>1}}^{m} f(c,d)\right). \end{aligned} \tag{3.6}$$

Combining (3.4) with (3.5), we obtain the following theorem:

Theorem 3.2. *Let c and d be coprime positive integers. Let $m \in \mathbb{N}$. Then we have*

$$\sum_{\substack{c=1\\(c,d)=1}}^{m} h(c,d) + \sum_{\substack{d=1\\(d,c)=1}}^{m} h(c,d) + \sum_{\substack{c=1\\(c,d)>1}}^{m} f(c,d) + \sum_{\substack{d=1\\(d,c)>1}}^{m} f(c,d) \tag{3.7}$$
$$= \frac{m(m+1)(d+c) - 12cdm + 2(c+d)(cd+1)H_m}{24cd}.$$

By using the equation (3.4), we get

$$-\sum_{\substack{c=1\\(c,d)>1}}^{m} v(c,d) + \sum_{\substack{c=1\\(c,d)>1}}^{m} f(c,d) - \sum_{\substack{c=1\\(c,d)=1}}^{m} \left(\frac{1}{c}\sum_{j=0}^{c-1} j\left[\frac{dj}{c}\right] + \frac{1}{d}\sum_{j=0}^{d-1} j\left[\frac{cj}{d}\right]\right)$$
$$= \frac{m(m+1)(-8d^2+9d-1) - 18dm(1-d)c + 2(1-d^2)H_m}{24d}.$$

By using the above equation, we arrive at the following theorem:

Theorem 3.3 (Reciprocity law for the function $[x]$)**.** *Let $d, m \in \mathbb{N}$. Then we have*

$$\sum_{\substack{c=1\\(c,d)=1}}^{m} \left(\frac{1}{c}\sum_{j=0}^{c-1} j\left[\frac{dj}{c}\right] + \frac{1}{d}\sum_{j=0}^{d-1} j\left[\frac{cj}{d}\right]\right) \tag{3.8}$$
$$= \sum_{\substack{c=1\\(c,d)>1}}^{m} (f(c,d) - v(c,d)) + \frac{m(m+1)(8d^2-9d+1) + 18dm(1-d)c + 2(d^2-1)H_m}{24d}.$$

By using (3.8), we also obtain

$$\sum_{\substack{c=1\\(c,d)=1}}^{m} \frac{1}{12cd}\left(12d\sum_{j=0}^{c-1} j\left[\frac{dj}{c}\right] + 12c\sum_{j=0}^{d-1} j\left[\frac{cj}{d}\right]\right)$$
$$= \sum_{\substack{c=1\\(c,d)>1}}^{m} (f(c,d) - v(c,d)) + \frac{m(m+1)(8d^2-9d+1) + 18dm(1-d)c + 2(d^2-1)H_m}{24d}.$$

Combining the above equation with the following well-known reciprocity law for the function $[x]$ (*cf.* [32, 38]):

$$12d\sum_{j=0}^{c-1} j\left[\frac{dj}{c}\right] + 12c\sum_{j=0}^{d-1} j\left[\frac{cj}{d}\right] = (c-1)(d-1)(8dc-c-d-1),$$

and after some elementary calculations, we arrive at the following theorem:

Theorem 3.4. *Let $d, m \in \mathbb{N}$. Then we have*

$$\sum_{\substack{c=1\\(c,d)=1}}^{m} \frac{(c-1)(8dc-c-d-1)}{c} = \frac{12d}{d-1} \sum_{\substack{c=1\\(c,d)>1}}^{m} (f(c,d)-v(c,d)) + \frac{m(m+1)(8d^2-9d+1)+18dm(1-d)c+2(d^2-1)H_m}{2(d-1)}.$$

Mathematica Code 1. The following code returns the values of the finite sum at the left-hand side of the Theorem 3.4.

```
TableForm[ Table[ Sum[ If[CoprimeQ[c, d] == True, ((c − 1)*(8*d*c − c − d − 1))/c, 0], {c, 1, m}], {m, 1, 10}, {d, 1, 5}],
    TableHeadings −> {{"m=1", "m=2", "m=3", "m=4", "m=5", "m=6", "m=7", "m=8", "m=9", "m=10"}, {"d=1", "d=2", "d=3", "d=4", "d=5"}}]
```

By Mathematica Code 1, we present Table 1 involving some values of the finite sum at the left-hand side of the Theorem 3.4 when $d = \{1, 2, 3, 4, 5\}$ and $m = \{1, 2, \dots, 10\}$.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
$m=1$	0	0	0	0	0
$m=2$	6	0	21	0	36
$m=3$	$\frac{56}{3}$	28	21	$\frac{176}{3}$	110
$m=4$	$\frac{229}{6}$	28	87	$\frac{176}{3}$	$\frac{445}{2}$
$m=5$	$\frac{1937}{30}$	$\frac{428}{5}$	$\frac{879}{5}$	$\frac{536}{3}$	$\frac{445}{2}$
$m=6$	$\frac{979}{10}$	$\frac{428}{5}$	$\frac{879}{5}$	$\frac{536}{3}$	$\frac{825}{2}$
$m=7$	$\frac{9673}{70}$	$\frac{6056}{35}$	$\frac{10863}{35}$	$\frac{7568}{21}$	$\frac{8979}{14}$
$m=8$	$\frac{25961}{140}$	$\frac{6056}{35}$	$\frac{32751}{70}$	$\frac{7568}{21}$	$\frac{25455}{28}$
$m=9$	$\frac{301969}{1260}$	$\frac{30488}{105}$	$\frac{32751}{70}$	$\frac{38048}{63}$	$\frac{102125}{84}$
$m=10$	$\frac{379081}{1260}$	$\frac{30488}{105}$	$\frac{46989}{70}$	$\frac{38048}{63}$	$\frac{102125}{84}$

Table 1. Some values of the finite sum at the left-hand side of the Theorem 3.4 when $d = \{1, 2, 3, 4, 5\}$ and $m = \{1, 2, \dots, 10\}$.

Remark 3.5. In [12], Cetin et al. gave many reciprocity laws and open questions for the function $[x]$, the Dedekind type sums and the Hardy type sums.

In [39], we defined the following sum $Y(h, k)$:

$$Y(h, k) = 4ks_5(h, k),$$

where h and k are odd integers with $(h, k) = 1$, $s_5(h, k)$ denotes one of the Hardy sums defined by

$$s_5(h, k) = \sum_{j=1}^{k} (-1)^{j+\left[\frac{hj}{k}\right]} \left(\left(\frac{j}{k}\right)\right),$$

where

$$(-1)^{[x]} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1}$$

(*cf.* [12, 22, 25, 30, 40]).

The sum $Y(h,k)$ has the following reciprocity law:

If h and k are positive odd integers with $(h,k)=1$, then we have

$$hY(h,k)+kY(k,h)=2hk-2, \tag{3.9}$$

(*cf.* [39, Eq. (2.1)]).

If we sum both sides of (3.9) from $h=1$ to m, we arrive at the following problem:

Problem 1. What kind of results are obtained when the method applied to the Dedekind sums is also applied to the formula (3.9)?

Using the following known Hardy type sum $C(a_1,a_2,\ldots,a_{v-1};a_v;m)$:

$$C(a_1,a_2,\ldots,a_{v-1};a_v;m)=\sum_{j=1}^{a_v-1} j^m(-1)^{j+\left[\frac{a_1 j}{a_v}\right]+\left[\frac{a_2 j}{a_v}\right]+\cdots+\left[\frac{a_{v-1} j}{a_v}\right]},$$

where $a_1,a_2,\ldots,a_{v-1},a_v$ are pairwise coprime positive integers, $m\in\mathbb{N}_0$ and $v\in\mathbb{N}$ with $v\geq 2$ (*cf.* [12, Definition 1]), we arrive at the following another problem:

Problem 2.

$$\sum_{\substack{k=1\\(k,h)=1}}^{m}\left(4kC(h;k;1)-2k^2C(h;k;0)+2k^2(-1)^{k+h}\right)=?$$

By using the following well-known formula:

$$\sum_{j=1}^{n-1} j^m=\frac{B_{m+1}(n)-B_{m+1}}{m+1}$$

where $B_m(n)$ and B_m denotes respectively the Bernoulli numbers and the Bernoulli polynomials (*cf.* [26, 28, 51]), and also

$$\sum_{j=1}^{n-1}(-1)^j j^m=\frac{(-1)^{n+1}E_m(n)-E_m}{2},$$

where $E_m(n)$ and E_m denotes respectively the Euler numbers and the Euler polynomials, which are defined by means of the following generating functions:

$$\frac{2e^{tn}}{e^t+1}=\sum_{l=0}^{\infty}E_l(n)\frac{t^l}{l!}$$

and

$$\frac{2}{e^t+1}=\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}$$

(*cf.* [26, 28, 51]), how to find relation between the Problem 1 and the Problem 2?

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