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Analytic Solvability of a Class of Symmetric Nonlinear Second Order Differential Equations

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Abstract

In this study, we introduce a solvability of special type of symmetric algebraic differential equations (SADEs) in virtue of geometric function theory by considering a symmetric differential operator. The analytic solutions of the SADEs are considered by utilizing the Caratheodory functions joining the subordination concept. A class of Caratheodory functions involving special functions gives the upper bound solution.

Keywords: Analytic function, Subordination, Univalent function, Open unit disk, Differential equations

2010 MSC: 30C45, 30C55

1. Introduction

The discussion of algebraic differential equations in a complex domain is categorized under the subject of complex ordinary differential equations, that can be interconnected in algebraic relations. There are different directives to explore this discussion attaching with complex domains. These discussions are involved in the higher-order homogeneous linear differential equation [3]. The meropmorphic studies given by suggesting the Painleve analysis can be located in [17]. The univalent symmetric investigation by considering a singular type of Painleve analysis is studied in [10]. The fractional calculus of algebraic differential equations is considered in [4, 9, 11]. The Nevanlinna technique, for ordinary modules, and algebraic differential equations is employed in [14]. The Asymmetrical and systematic certain and exponential solvability is indicated in [12]. Using the operating distinctive functions such as Zeta function can be found in [15]. The geometric solutions are presented in [2]. Finally, the quantum technique and other analytic methods are selected in [1], [5]–[7]. A special case of symmetric algebraic differential equations (SADEs) is considered in view of the geometry investigations of normalized analytic functions in the open unit disk $\cup := \{z \in \mathbb{C} : |z| < 1\}$. The derivatives are considered in view of the following complex symmetric differential operator $\Delta : \cup \to \cup$ of fractional order $\nu \in [0, 1)$ [8] for the class of normalized analytic functions $\chi(z)$, which is denoted by $\wedge$.

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$$
\begin{aligned}
\Delta_{\nu}^{0}\chi(z) &= \chi(z) = z + \sum_{n=2}^{\infty}\chi_n z^n \\
\Delta_{\nu}^{1}\chi(z) &= \nu z\chi'(z) - (1-\nu)z\chi'(-z) = \nu\left(z + \sum_{n=2}^{\infty} n\chi_n z^n\right) - (1-\nu)\left(-z + \sum_{n=2}^{\infty} n(-1)^n\chi_n z^n\right) \\
&= z + \sum_{n=2}^{\infty}[n(\nu - (1-\nu)(-1)^n)]\chi_n z^n \qquad (1.1)\\
\Delta_{\nu}^{2}\chi(z) &= \Delta_{\nu}^{1}[\Delta_{\nu}^{1}\chi(z)] = z + \sum_{n=2}^{\infty}[n(\nu - (1-\nu)(-1)^n)]^2\chi_n z^n, \cdots \\
\Delta_{\nu}^{k}\chi(z) &= \Delta_{\nu}^{1}[\Delta_{\nu}^{k-1}\chi(z)] = z + \sum_{n=2}^{\infty}[n(\nu - (1-\nu)(-1)^n)]^k\chi_n z^n \\
&\qquad \big(\nu \in [0,1],\ z \in \cup,\ \chi \in \wedge\big).
\end{aligned}
$$

Hence, $\Delta_{\nu}^{k}\chi(z) \in \wedge$, whenever, $\chi \in \wedge$. Moreover, we indicate that $\Delta_{\nu}^{k}\chi(z)$ is a convolution operator of two normalized analytic functions as follows

$$
\begin{aligned}
\Delta_{\nu}^{k}\chi(z) &:= \Pi_{\nu}^{k}(z) * \chi(z) \\
&= \left(z + \sum_{n=2}^{\infty}(\Pi_n^{\nu})^k z^n\right) * \left(z + \sum_{n=2}^{\infty}\chi_n z^n\right),
\end{aligned}
$$

where $(\Pi_n^{\nu})^k = [n(\nu - (1-\nu)(-1)^n)]^k$.

Remark 1.1.

- Note that when $\nu = 1$ we have the Sàlàgean differential operator.
- The class $\mathcal{P}$ indicates all functions ρ analytic in $\cup$ and consuming positive real part in $\cup$ with $\rho(0) = 1$. In fact $\varphi \in \mathcal{S}^*$ (the class of starlike analytic functions) if and only if $z\chi'(z)/\chi(z) \in \mathcal{P}$ and $\chi \in \mathcal{C}$ (the class of convex analytic functions) if and only if $1 + z\chi''(z)/\chi'(z) \in \mathcal{P}$.

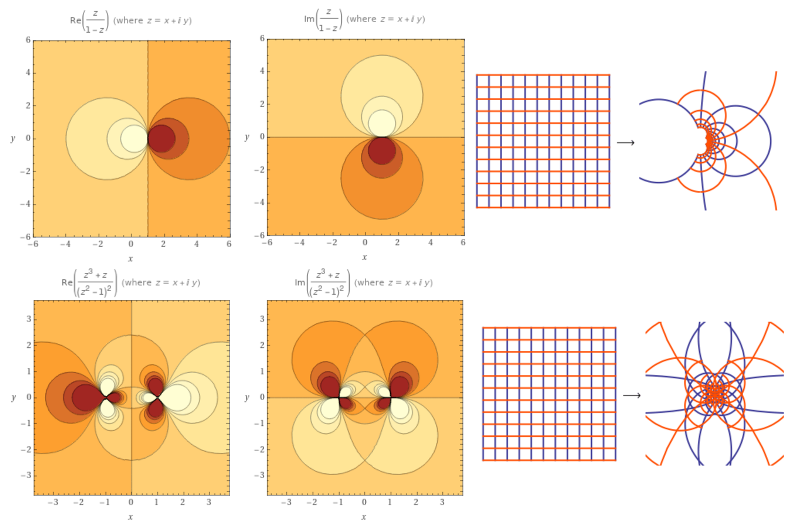

Figure 1. The first roe is the function $\chi(z)$ and the second row its graph under the symmetric operator $\Delta_{0.5}^{k}(\Delta_{\nu}^{k}\chi(z))$.

Example 1.2. Let $\chi(z) = z/(1-z)$ then for $\nu = 0.5$, the $\Delta^k_{0.5}\chi(z) = z + 3z^3 + 5z^5 + O(z^7)$ (see Fig.1). In general, we have $\Delta^k_\nu\chi(z) = z + (4\nu-2)z^2 + 3z^3 + (8\nu-4)z^4 + 5z^5 + (12\nu-6)z^6 + O(z^7)$.

In our discussion, we shall operate $\Delta^k_\nu\chi(z)$ in a special category of algebraic differential equations to define the class of symmetric algebraic differential equations (SADEs). The study based on the Caratheodory functions theory and the subordination concept. This technique indicates the upper bound of the symmetric solution in the open unit disk.

2. Algebraic differential equations

A class of SADEs, which is indicated in [17] and formulated by the structure

$$\varsigma[\chi(z)\chi''(z) + (\chi'(z))^2] + \Upsilon^m_\chi(z) = 0, \quad z \in \mathbb{C}, \tag{2.1}$$

where σ and $\varsigma_\iota \in \mathbb{C}, \iota = 0, ..., m$ are constants such that

$$\Upsilon^m_\chi(z) := \varsigma_m\chi^m(z) + \varsigma_{m-1}\chi^{m-1}(z) + ... + \varsigma_1\chi(z) + \varsigma_0$$

can be generalized by using the conformable differential operator Δ^k_ν.

2.1. Homogeneous case

We deal with the generalization of Eq.(2.1)by using the symmetric operator Δ^k_ν. Our study is to explore geometric properties in $\cup$. Then the solution is majorized by employing special function $\cup$. A rearrangement of Eq.(1.1) yields the homogeneous form when $\varsigma \neq 0$

$$\left(\frac{z\chi''(z)}{\chi'(z)} + \frac{z\chi'(z)}{\chi(z)}\right) \times \left(\frac{z\chi'(z)}{\chi(z)}\right) = 0. \tag{2.2}$$

Consequently, by using (1.1), the generalization type of Eq.(2.2) is indicated as follow:

$$\left(\frac{z\,(\Delta^k_\nu\chi(z))''}{(\Delta^k_\nu\chi(z))'} + \frac{z(\Delta^k_\nu\chi(z))'}{\chi(z)}\right) \quad \times \quad \left(\frac{z(\Delta^k_\nu\chi(z))'}{\chi(z)}\right) = 0. \tag{2.3}$$

We request the following facts.

Definition 2.1.

- The notion subordination denoting by $\chi \prec \psi$ is satisfying the equation $\chi = (\psi(\omega)), |\omega| < |z| < 1$ (see [13]).
- The Ma-Minda classes $S^*(w)$ and $K(w)$ of starlike and convex functions respectively are given by $\left(\frac{z\ell'(z)}{\ell(z)}\right) \prec w(z)$ and $\left(1 + \frac{z\,\ell''(z)}{\ell'(z)}\right) \prec w(z)$, where w has a positive real part in $\cup, w(0) = 1, w'(0) > 1$.
- For $\mathcal{P}(A, B)$ actuality the class of functions

  $$\rho(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \prec \frac{1 + Az}{1 + Bz},$$

  where w is the Schwarz function and $-1 \leq B < A \leq 1$, then $\mathcal{P}(A, B) \subset \mathcal{P}(\frac{1-A}{1-B})$ is the Janowski class.

Our study is indicated by using the above inequality to define the following special class.

Definition 2.2. Let $h(z) = z + \sum_{n=2}^{\infty} h_n z^n$, $z \in \cup$. Then it belongs to the class $\mathbf{M}^\nu(\rho)$ if and only if

$$G(z) := \left(\frac{z\,(\Delta^k_\nu h(z))''}{(\Delta^k_\nu\, h(z))'} + \frac{z(\Delta^k_\nu h(z))'}{\Delta^k_\nu h(z)}\right) \quad \times \quad \left(\frac{z(\Delta^k_\nu h(z))'}{\Delta^k_\nu h(z)}\right) \prec \rho(z). \tag{2.4}$$

$$\Big(z \in \cup,\ \rho(0) = 1,\ \rho'(0) > 1\Big)$$

It is clear that $G(0) = 1$. In the sequel, we shall consider a starlike function with a positive real part:

$$\rho_e(z) = \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + ...$$

as well as a convex univalent function (see [13]-P415)

$$\varrho_e(z) := 1/\rho_e(z) = 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + ... $$

It is well known that, the connections are meeting to the Bernoulli values. Additionally, we have

$$\Re\left(\frac{e^{\xi z} - 1}{\xi z}\right) \geq \frac{1}{2}, \quad 0 < \xi \leq 1.793.... $$

Hence, $\Re\left(\frac{e^{\xi z} - 1}{\xi z}\right) \geq 1/\rho_e(-1) = \frac{1}{2}$.

Our design is generated by the analytic technique of Caratheodory functions which are operated in [8]. In this situation, we establish the necessary conditions of the joining bounds of $\Upsilon_\chi^m(z)$, for transformed values of $m = 0, 1, .., $ consuming a Caratheodory function.

2.2. Non-homogeneous case

In this part, we suppose that $\varsigma = \varsigma_1 = 1$ and $\varsigma_0 = 0$. Then we get

$$h(z)h''(z) + (h'(z))^2 + h(z) = 0, \quad z \in \mathbb{C}. \tag{2.5}$$

Eq. (2.5) can be rearranged as follows:

$$\left(\frac{zh'(z)}{h(z)}\right)\left(\frac{z\,h''(z)}{h'(z)} + \frac{zh'(z)}{h(z)} + \gamma(z)\right) = 0, \quad z \in \cup, \tag{2.6}$$

where

$$\gamma(z) := \frac{z}{h'(z)} = z - 2h_2 z^2 - ... \in \wedge .$$

Consequently, by employing (1.1), the generalization type of Eq.(2.6) is indicated as follow:

$$\left(\frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}\right)\left(\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k h(z))'} + \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)} + g(z)\right) = 0. \tag{2.7}$$

$$\Big(z \in \cup, \quad g(z) := \frac{z}{\left(\Delta_\nu^k h(z)\right)'}\Big).$$

We have the following generalized symmetric class:

Definition 2.3. Let

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \; z \in \cup.$$

Then it belongs to the class $\mathbf{M}_g^\nu(\wp)$ if and only if

$$E(z) := \left(\frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}\right) \quad \times \quad \left(\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k\, h(z))'} + \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)} + g(z)\right) \prec \wp(z). \tag{2.8}$$

$$\Big(z \in \cup, \; \wp(0) = 1, \; \wp'(0) > 1\Big).$$

Obviously, $\Xi(0) = 1$. For example, consider $h(z) = z/(1-z)$, then

- $$\left(\frac{zh'(z)}{h(z)}\right)\left(\frac{z\,h''(z)}{h'(z)} + \frac{zh'(z)}{h(z)}\right) = 1 + 4z + 7z^2 + 10z^3 + 13z^4 + 16z^5 + O(z^6);$$
- $$\left(\frac{zh'(z)}{h(z)}\right)\left(\frac{z\,h''(z)}{h'(z)} + \frac{zh'(z)}{h(z)} + g(z)\right) = 1 + 5z + 6z^2 + 10z^3 + 13z^4 + 16z^5 + O(z^6);$$
- $$\begin{aligned} G(z) &= \Big(1 + (8\nu-5)z + (13-8\nu)z^2 + (32\nu-25)z^3 + (41-32\nu)z^4 + (72\nu-61)z^5 + O(z^6)\Big) \\ &\quad * \Big((8\nu-4)z + (-64\nu^2+64\nu+2)z^2 + (512\nu^3-768\nu^2+264\nu-4)z^3 \\ &\quad + (-4096\nu^4+8192\nu^3-4864\nu^2+768\nu+2)z^4 + 4(8192\nu^5-20480\nu^4+17280\nu^3-5440\nu^2+450\nu-1)z^5 \\ &\quad + (-262144\nu^6+786432\nu^5-860160\nu^4+409600\nu^3-77376\nu^2+3648\nu+2)z^6 + O(z^7)\Big); \end{aligned}$$
- $$\begin{aligned} E(z) &= \Big(1 + (8\nu-5)z + (13-8\nu)z^2 + (32\nu-25)z^3 + (41-32\nu)z^4 + (72\nu-61)z^5 + O(z^6)\Big) \\ &\quad * \Big((8\nu-3)z - 64((\nu-1)\nu)z^2 + (512\nu^3-768\nu^2+264\nu-3)z^3 \\ &\quad + (-4096\nu^4+8192\nu^3-4864\nu^2+768\nu+2)z^4 + 4(8192\nu^5-20480\nu^4+17280\nu^3-5440\nu^2+450\nu-1)z^5 \\ &\quad + (-262144\nu^6+786432\nu^5-860160\nu^4+409600\nu^3-77376\nu^2+3648\nu+2)z^6 + O(z^7)\Big). \end{aligned}$$

2.3. Geometric properties

Some geometric properties are illustrated as follows:

Proposition 2.4. *Consider the functional $G(z)$ such that $p(z) = z(\Delta_\nu^k h(z))'/\Delta_\nu^k h(z)$. Then $h(z)$ is starlike in $\cup$, whenever $\nu \to 1$.*

Proof. Suppose the functional

$$G(z) = \left(\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k h(z))'} + \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}\right) \quad \times \quad \left(\frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}\right).$$

Let $p(z) = z(\Delta_\nu^k h(z))'/\Delta_\nu^k h(z)$, then

$$\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k h(z))'} = \frac{zp'(z)}{p(z)} + p(z) - 1$$

yields that

$$G(z) = zp'(z) + 2p^2(z) - p(z), \quad z \in \cup.$$

By [13]-Example 2.4m, we have $A(z) = 1, B(z) = 2, C(z) = -1$ and $D(z) = 0$ achieving

$$\Re(A(z) + 2B(z)).\Re(A(z) - 2D(z)) = 5 > 0$$

then we have the conclusion

$$\begin{aligned} \Re[A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)] &= \Re[zp'(z) + 2p^2(z) - p(z)] \\ &> 0 \Rightarrow \Re(p(z)) > 0. \end{aligned}$$

Corresponding to the above conclusion and $\nu \to 1$, we have

$$\Re(p(z)) > 0 \Rightarrow \Re\left(\frac{zh'(z)}{h(z)}\right)\left(1 + \frac{z\,h''(z)}{h'(z)}\right) > 0,$$

which indicates that $h(z)$ is starlike. □

Proposition 2.5. *Consider the functional* $E(z)$ *such that* $p(z) = z(\Delta_\nu^k h(z))'/\Delta_\nu^k h(z)$. *If* $\Re(g(z)) \leq 1/2$ *then* χ *is starlike in* $\cup$ *whenever* $\nu \to 1$.

Proof. Consider the function

$$E(z) = \left(\frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}\right)\left(\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k h(z))'} + \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)} + g(z)\right).$$

Then $p(z) = \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}$, implies

$$E(z) = zp'(z) + 2p^2(z) - p(z) + g(z), \quad z \in \cup.$$

Again by [13]-Example 2.4m, we get $A(z) = 1, B(z) = 2, C(z) = -1$ and $D(z) = g(z)$ fulfilling

$$\Re(A(z) + 2B(z)).\Re(A(z) - 2D(z)) = 5\Re(1 - 2g(z)) > 0$$

then we have the conclusion

$$\begin{aligned}\Re[A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)] &= \Re[zp'(z) + 2p^2(z) - p(z) + g(z)] \\ &> 0 \Rightarrow \Re(p(z)) > 0.\end{aligned}$$

Corresponding to the above conclusion and $\nu \to 1$, we have

$$\Re(p(z)) > 0 \Rightarrow \Re\left(\frac{zh'(z)}{h(z)}\right)\left(1 + \frac{zh''(z)}{h'(z)}\right) > 0,$$

which indicates that h is starlike. □

Proposition 2.6 (Integral existence result)**.** *Consider the functional* $G(z) = p(z).q(z)$, *where* $p(z) = z(\Delta_\nu^k h(z))'/\Delta_\nu^k h(z)$ *and*

$$q(z) = \left(\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k h(z))'} + \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)}\right).$$

If the subordination

$$(a_1 + a_2) + \frac{zq'(z)}{q(z)} \prec (a_1 + a_2)\left(\frac{1+z}{1-z}\right) + \frac{2z}{1-z^2}$$

holds and $a_1 + a_2 = b_1 + b_2 > 0$ *then the integral*

$$L(z) := \left(\frac{b_1 + b_2}{z^{(b_2)}p(z)} \int \zeta^{a_1+a_2-1} q(\zeta) d\zeta\right)^{1/(b_1)}$$

has the following facts

- $L(z) \in \wedge$;
- $L(z)/z \neq 0$;
- $\Re\left(b_1 \frac{zL'(z)}{L(z)} + \frac{zq'(z)}{q(z)}\right) > -b_2$.

Proof. Since, $p(0) = 1$ and $q(0) = 1$ with $G(z) = p(z)q(z) \neq 0$ for some $z_0 \in \cup$. Then in view of [13]-Theorem 2.5c, we have the desired conclusion. □

Similarly, we obtain the following result.

Proposition 2.7. *Consider the functional* $E(z) = p(z).q_g(z)$, *where* $p(z) = z(\Delta_\nu^k h(z))'/\Delta_\nu^k h(z)$ *and*

$$q_g(z) = \left(\frac{z\,(\Delta_\nu^k h(z))''}{(\Delta_\nu^k h(z))'} + \frac{z(\Delta_\nu^k h(z))'}{\Delta_\nu^k h(z)} + g(z)\right).$$

If the subordination

$$(a_1 + a_2) + \frac{zq_g'(z)}{q_g(z)} \prec (a_1 + a_2)\left(\frac{1+z}{1-z}\right) + \frac{2z}{1-z^2}$$

holds and $a_1 + a_2 = b_1 + b_2 > 0$ *then the integral*

$$L_g(z) := \left(\frac{b_1 + b_2}{z^{(b_2)} p(z)} \int \zeta^{a_1+a_2-1} q_g(\zeta) d\zeta\right)^{1/(b_1)}$$

has the following facts

- $L_g(z) \in \wedge$;
- $L_g(z)/z \neq 0$;
- $\Re\left(b_1 \frac{zL_g'(z)}{L_g(z)} + \frac{zq_g'(z)}{q_g(z)}\right) > -b_2.$

Proposition 2.8. *Consider the function* $\Pi_\nu^k(z)$, $z \in \cup$. *If*

$$\left|(\Pi_n^\nu)^k\right| \leq \binom{2(-1+\nu)}{n-1}, \quad \nu \in [0,1]$$

and $\chi \in \mathcal{S}^*(\nu)$ *then* $\Delta_\nu^k \chi(z) \in \mathcal{S}^*(\nu)$. *Moreover, there exists a probability measure* τ *on* $\partial\cup$ *such that*

$$\Delta_\nu^k h(z) = \int_{\partial\cup} \frac{z}{(1 - z\eta)^{2-2\nu}} d\tau(\eta).$$

Proof. Consider the function

$$\begin{aligned} f_c(z) &= \frac{z}{(1-z)^{2-2\nu}} \\ &= z + (2-2\nu)z^2 + (\nu-1)(2\nu-3)z^3 - 2/3((\nu-2)(\nu-1)(2\nu-3))z^4 \\ &+ 1/6(\nu-2)(\nu-1)(2\nu-5)(2\nu-3)z^5 - 1/15((\nu-3)(\nu-2)(\nu-1)(2\nu-5)(2\nu-3))z^6 + O(z^7) \\ &= \sum_{n=1}^{\infty} (-1)^{(1+n)} \binom{2(-1+\nu)}{n-1} z^n, \quad z \in \cup. \end{aligned}$$

The function $f_\nu(z) \in \mathcal{S}^*(\nu)$ ([16]). By the assumption, $\Pi_\nu^k(z)$ is majored by $f_\nu(z)$

$$\Pi_\nu^k(z) \ll \frac{z}{(1-z)^{2-2\nu}} \Rightarrow \Pi_\nu^k(z) \prec f_\nu(z),$$

then $\Pi_\nu^k(z) \in \mathcal{S}^*(\nu)$. Consequently $\Delta_\nu^k h(z) \in \mathcal{S}^*(\nu)$. The second part is indicated by [16]-Corollary 2.2. □

3. Results

Here, we illustrate our computational outcomes.

Theorem 3.1. *Let the function $\chi \in \wedge$ fulfilling the inequality*

$$1+\mu\left(\frac{z\,G'(z)}{[G(z)]^{\iota}}\right) \prec \sqrt{z+1}, \quad \iota = 0,1,2,$$

where $G(z) = \left(\frac{z(\Delta_\nu^\iota \chi(z))'}{\chi(z)}\right)\left(\frac{z\,(\Delta_\nu^\iota \chi(z))''}{(\Delta_\nu^\iota \chi(z))'} + \frac{z(\Delta_\nu^\iota \chi(z))'}{\chi(z)}\right)$. *Then*

$$G(z) \prec \rho_e(z) = \frac{z}{e^z - 1}, \; z \in \cup$$

when $\mu \geq \max \mu_\iota$,

- $$\max \mu_0 = \max\left\{2(e-1)(-1+\sqrt{(2)}+\log(2)-\log(1+\sqrt{(2)})), \; \frac{-2((e-1)(\log(2)-1))}{(e-2)}\right\}.$$
- $$\max \mu_1 = \max\left\{\frac{2(-1+\sqrt{(2)}+\log(2)-\log(1+\sqrt{(2)}))}{\log(-e/(1-e))}, \; \frac{2(1-\log(2))}{\log(e-1)}\right\}.$$
- $$\max \mu_2 = \max\left\{2e(-1+\sqrt{(2)}+\log(2)-\log(1+\sqrt{(2)})), \; \frac{2(1-\log(2))}{(e-2)}\right\}.$$

Proof. Part I, when $\iota = 0$, then we have $1+\mu\,(z\,G'(z)) \prec \sqrt{z+1}$.
Define a function $W_\mu : \cup \to \mathbb{C}$ as follows:

$$W_\mu(z) = 1 + \frac{2}{\mu}\left(\sqrt{z+1} - \log(1+\sqrt{z+1}) - 1 + \log(2)\right).$$

Clearly, $W_\mu(z)$ is analytic in $\cup$ satisfying $W_\mu(0) = 1$ and it is a solution of the first order differential equation

$$1+\mu\left(z\,W'_\mu(z)\right) = \sqrt{z+1}, \quad z \in \cup. \tag{3.1}$$

Therefore, this yields $\mathfrak{U}(z) := \mu\left(z\,W'_\mu(z)\right) = \sqrt{z+1} - 1$ is starlike in $\cup$. So for $\mathfrak{W}(z) := \mathfrak{U}(z) + 1$, we round off that

$$\Re\left(\frac{z\,\mathfrak{U}'(z)}{\mathfrak{U}(z)}\right) = \Re\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{U}(z)}\right) > 0.$$

Thus, Miller-Mocanu Lemma (see [13]-P132) indicates that

$$1+\mu\,(z\,G'(z)) \prec 1+\mu z W'_\mu(z) \Rightarrow G(z) \prec W_\mu(z).$$

To end this argument, we must show that $W_\mu(z) \prec \rho_e(z)$. Obviously, we have that $W_\mu(z)$ is rising in $I := (-1,1)$ and fulfilling the following relation

$$W_\mu(-1) \leq W_\mu(1).$$

According to the inequality

$$(e-1)^{-1} \leq \Re(\rho_e(z)) \approx 1 - \frac{\cos(\vartheta)}{2} + \sum_{n=1}^{\infty} \frac{\beta_{2n}\cos(2n\vartheta)}{(2n)!} \leq e(e-1)^{-1},$$

we have

$$(e-1)^{-1} \leq W_\mu(-1) \leq W_\mu(1) \leq e(e-1)^{-1}$$

if and only if μ satisfies

$$\mu \geq \max \mu_0 = \max\left\{2(e-1)(-1+\sqrt{(2)}+\log(2)-\log(1+\sqrt{(2)})),\ \frac{-2((e-1)(\log(2)-1))}{(e-2)}\right\}.$$

Thus, we have

$$G(z) \prec W_\mu(z) \prec \rho_e(z) \Rightarrow G(z) \prec \rho_e(z), \quad z \in \cup.$$

Part II: Suppose that $\iota = 1$ then we get the inequality $1+\mu\left(\frac{zG'(z)}{G(z)}\right) \prec \sqrt{z+1}$.
Define a function $S_\mu : \cup \to \mathbb{C}$ formulating the structure

$$S_\mu(z) = \exp\left(\frac{2}{\mu}\left(\sqrt{z+1}-\log(1+\sqrt{z+1})-1+\log(2)\right)\right).$$

It is clear that, the functional $S_\mu(z)$ is analytic in $\cup$ satisfying $S_\mu(0) = 1$ with

$$1+\mu\left(\frac{zS'_\mu(z)}{S_\mu(z)}\right) = \sqrt{z+1}, \quad z \in \cup. \tag{3.2}$$

By assuming $\mathfrak{U}(z) = \sqrt{z+1}-1$, which is starlike in $\cup$ and $\mathfrak{W}(z) = \mathfrak{U}(z)+1$, we get

$$\Re\left(\frac{z\,\mathfrak{U}'(z)}{\mathfrak{U}(z)}\right) = \Re\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{U}(z)}\right) > 0, \quad z \in \cup.$$

Then again, in virtue of the Miller-Mocanu Lemma, we arrive at

$$1+\mu\left(\frac{zG'(z)}{G(z)}\right) \prec 1+\mu\left(\frac{zS'_\mu(z)}{S_\mu(z)}\right) \Rightarrow G(z) \prec S_\mu(z).$$

Proceeding, we obtain

$$(e-1)^{-1} \leq S_\mu(-1) \leq S_\mu(1) \leq e(e-1)^{-1}$$

if μ gets the upper value

$$\mu \geq \max \mu_1 = \max\left\{\frac{2(\sqrt{2}-1+\log(2)-\log(1+\sqrt{2}))}{\log(-e/(1-e))},\ \frac{2(1-\log(2))}{\log(e-1)}\right\}.$$

As a consequence, we receive the subordination

$$G(z) \prec S_\mu(z) \prec \rho_e(z) \Rightarrow G(z) \prec \rho_e(z), \quad z \in \cup.$$

Part III: Consume that $\iota = 2$, then we obtain the subordination $1+\mu\left(\frac{zG'(z)}{G^2(z)}\right) \prec \sqrt{z+1}$.
Define a function $Q_\mu : \cup \to \mathbb{C}$ formulating the structure

$$Q_\mu(z) = \left(1-\frac{2}{\mu}\left(\sqrt{z+1}-\log(1+\sqrt{z+1})-1+\log(2)\right)\right)^{-1}.$$

Readily, $Q_\mu(z)$ is analytic in $\cup$ with $Q_\mu(0) = 1$ and

$$1+\mu\left(\frac{zQ'_\mu(z)}{Q_\mu(z)}\right) = \sqrt{1+z}, \quad z \in \cup. \tag{3.3}$$

By applying the functions $\mathfrak{U}(z) = \sqrt{z+1} - 1$, which is starlike in $\cup$ and $\mathfrak{W}(z) = \mathfrak{O}(z) + 1$, we receive

$$\Re\left(\frac{z\,\mathfrak{U}'(z)}{\mathfrak{U}(z)}\right) = \Re\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{U}(z)}\right) > 0, \quad z \in \cup.$$

Hence, the Miller-Mocanu Lemma yields

$$1 + \mu\left(\frac{z\,G'(z)}{G^2(z)}\right) \prec 1 + \mu\left(\frac{zQ'_\mu(z)}{Q^2_\mu(z)}\right) \Rightarrow G(z) \prec Q_\mu(z).$$

Accordingly, we have

$$(e-1)^{-1} \le Q_\mu(-1) \le Q_\mu(1) \le e(e-1)^{-1}$$

if μ_2 recognizes the upper and lower bounds

$$\begin{aligned} \mu &\ge \max \mu_2 \\ &= \max\left\{2e(-1 + \sqrt{(2)} + \log(2) - \log(1 + \sqrt{(2)})), \frac{2(1-\log(2))}{(e-2)}\right\}. \end{aligned}$$

This implies the conclusion

$$G(z) \prec Q_\mu(z) \prec \rho_e(z) \Rightarrow G(z) \prec \rho_e(z), \quad z \in \cup.$$

□

Theorem 3.1 can be generalized by using $p(z) \in \mathcal{P}$ as follows.

Theorem 3.2. *Let $p(z) \in \mathcal{P}$. In addition, let*

$$1 + \mu\left(\frac{z\,p'(z)}{[p(z)]^\iota}\right) \prec \sqrt{z+1}, \quad \iota = 0,1,2, \quad \mu \ge 5.9462.$$

Then

$$p(z) \prec \rho_e(z), \quad z \in \cup.$$

We proceed to study the convex univalent function $\varrho_e(z) = \frac{e^z - 1}{z}$, $z \in \cup$.

Theorem 3.3. *For $\chi \in \wedge$ achieves*

$$1 + \upsilon\left(\frac{z\,G'(z)}{[G(z)]^\iota}\right) \prec \sqrt{z+1}, \quad \iota = 0,1,2,$$

where $G(z) = \left(\frac{z(\Delta^\iota_\nu\chi(z))'}{\chi(z)}\right)\left(\frac{z\,(\Delta^\iota_\nu\chi(z))''}{(\Delta^\iota_\nu\chi(z))'} + \frac{z(\Delta^\iota_\nu\chi(z))'}{\chi(z)}\right)$. Then

$$G(z) \prec \varrho_e(z),\ z \in \cup$$

when $\upsilon \ge max\upsilon_k$,

- $$\max \upsilon_0 = \max\left\{\frac{2(-1 + \sqrt{(2)} + \log(2) - \log(1+\sqrt{2}))}{(e-2)}, 2(e-1)(\log(2)-1)\right\}.$$
- $$\max \upsilon_1 = \max\left\{\frac{2(-1 + \sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)}))}{\log(e-1)}, 2\left(\frac{1-\log(2)}{\log(e-1)-1}\right)\right\}.$$
- $$\max \upsilon_2 = \max\left\{\frac{2((e-1)(-1 + \sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)})))}{(e-2)}, 2e(\log(2)-1)\right\}.$$

Proof. The function $\varrho_e(z)$ achieves the real formula

$$(e-1)e^{-1} \leq \Re(\varrho_e(z)) \leq e-1, \quad z \in \cup.$$

According to Theorem 3.1, we have

$$(e-1)e^{-1} \leq W_\upsilon(-1) \leq W_\upsilon(1) \leq e-1$$

with the upper value, for $\iota = 0$, we have

$$\begin{aligned} \upsilon &\geq \max \upsilon_0 \\ &= \max \upsilon_0 = \left\{ \frac{-2(1-\sqrt{(2)} - \log(2) + \log(1+\sqrt{(2)}))}{2-e}, 2(e-1)(\log(2)-1) \right\}. \end{aligned}$$

This leads to the following consequences

$$W_\upsilon(z) \prec \varrho_e(z) \Rightarrow E(z) \prec \varrho_e(z), \quad z \in \cup.$$

Similarly, for the second part $\iota = 1$, we have

$$\begin{aligned} \upsilon &\geq \max \upsilon_1 \\ &= \max \left\{ \frac{2(-1+\sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)}))}{\log(e-1)}, 2\left(\frac{1-\log(2)}{\log(e-1)-1}\right) \right\}. \end{aligned}$$

This implies that

$$S_\upsilon(z) \prec \varrho_e(z) \Rightarrow G(z) \prec \varrho_e(z), \quad z \in \cup.$$

Finally, for the third part $\iota = 2$, we obtain

$$\begin{aligned} \upsilon &\geq \max \upsilon_2 \\ &= \max \left\{ \frac{2((e-1)(-1+\sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)})))}{(e-2)}, 2e(\log(2)-1) \right\}. \end{aligned}$$

This leads to conclusion

$$Q_\upsilon(z) \prec \varrho_e(z) \Rightarrow G(z) \prec \varrho_e(z), \quad z \in \cup.$$

□

Theorem 3.3 can be extended by utilizing $p(z) \in \mathcal{P}$ as follows:

Theorem 3.4. *Let $p(z) \in \mathcal{P}$ satisfying*

$$1 + \mu\left(\frac{z\,p'(z)}{[p(z)]^\iota}\right) \prec \sqrt{z+1}, \quad \iota = 0,1,2, \quad \mu \geq 2.8283.$$

Then

$$p(z) \prec \varrho_e(z) = \frac{e^z - 1}{z}, \; z \in \cup.$$

Note that Theorems 3.1 and 3.3 can be suggested when $\chi \in \mathbf{M}_g^\nu(\wp)$ achieving the inequality

$$1 + \mu\left(\frac{z\,E'(z)}{[E(z)]^\iota}\right) \prec \sqrt{z+1}, \quad \iota = 0,1,2.$$

Next result indicates the upper bound $\dfrac{1+A\,z}{1+B\,z}$, $(-1 \leq B < A \leq 1)$, (bi-linear transformation) which is starlike function with positive real part. Also, one can consider $\chi \in \mathbf{M}_g^\nu(\wp)$ under the functional $E(z)$.

Theorem 3.5. *Let $p(z) \in \mathcal{P}$ indicated the following inequalities*

(a) $1 + \epsilon\,(zp'(z)) \prec \sqrt{z+1}, \epsilon \geq max\{\epsilon_0, \epsilon_1\}$, *where*

$$\epsilon_0 = \frac{2(0.22599B - 0.22599)}{(A-B)}, \quad B+1 \neq 0,\ A-B \neq 0;$$

and

$$\epsilon_1 = \frac{2((B-1)(\log(2)-1))}{(A-B)},\ B-1 \neq 0,\ A-B \neq 0.$$

(b) $1 + \epsilon\left(z\frac{p'(z)}{p(z)}\right) \prec \sqrt{z+1}, \epsilon \geq max\{\epsilon_2, \epsilon_3\}$, *where for* $B-1 \neq 0,\ A-1 \neq 0$, *and*

$$\log(\frac{B+1}{A+1}) \neq 2\pi n i$$

$$\epsilon_2 = \frac{2(i(-1+\sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)})))}{2\pi n - i\log(\frac{B+1}{A+1})},$$

and

$$\epsilon_3 = \frac{-2(i(\log(2)-1))}{2\pi n - i\log(\frac{A-1}{B-1})};\ \ \log(\frac{A-1}{B-1}) \neq 2i\pi n.$$

(c) $1 + \epsilon\left(z\frac{p'(z)}{p^2(z)}\right) \prec \sqrt{z+1}, \epsilon \geq max\{\epsilon_4, \epsilon_5\}$, *where*

$$\epsilon_4 = \frac{2(0.225987A + 0.225987)}{(A-B)} B+1 \neq 0,\ A \neq B;$$

$$\epsilon_5 = \frac{2((A-1)(\log(2)-1))}{(A-B)}, \quad B-1 \neq 0 A \neq B.$$

Then $p(z) \prec \dfrac{1+Az}{1+Bz}, (-1 \leq B < A \leq 1)$.

As an application of Theorem 3.5, one can consider $G(z)$ and $E(z)$. We advance to extant the upper bound result of Eq.(2.4) by the singular function $1 + \sin(z), z \in \cup$, where it is with positive real part.

Theorem 3.6. *Consider one of the following inequalities*

(a) $1 + \tau\,(zG'(z)) \prec \sqrt{z+1}, \tau \geq max\{\tau_0, \tau_1\}$, *where*

$$\tau_0 = 2(-1+\sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)}))\csc(1);$$

and

$$\tau_1 = 2(\log(2)-1)(-\csc(1)).$$

(b) $1 + \tau\left(z\frac{G'(z)}{G(z)}\right) \prec \sqrt{z+1}, \tau \geq max\{\tau_2, \tau_3\}$, *where*

$$\tau_2 = 2\left(\frac{(-1+\sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)}))}{\log(1+\sin(1))}\right),$$

and

$$\tau_3 = \frac{2(\log(2)-1)}{\log(1-\sin(1))}.$$

(c) $1 + \tau\left(z\frac{G'(z)}{G^2(z)}\right) \prec \sqrt{z+1}, \tau \geq max\{\tau_4, \tau_5\}$, *where*

$$\tau_4 = 2(-1+\sqrt{(2)} + \log(2) - \log(1+\sqrt{(2)}))(1+\csc(1));$$

$$\tau_5 = 2(\log(2)-1)(-(\csc(1)-1)).$$

Then $G(z) \prec 1 + \sin(z), z \in \cup$.

Theorem 3.6 can be modified by using $p(z) \in \mathcal{P}$ as follows:

Theorem 3.7. *Let* $p(z) \in \mathcal{P}$ *achieving*

$$1 + \mu\left(\frac{z\,p'(z)}{[p(z)]^{\iota}}\right) \prec \sqrt{z+1}, \quad \iota = 0, 1, 2, \quad \mu \geq 2.5076.$$

Then $p(z) \prec 1 + \sin(z), z \in \cup$.

4. Conclusion

A discussion of symmetric algebraic differential equations (SADEs) of complex variables is presented in statement of geometric function theory by consuming current complex symmetric differential operator. We established a class of normalized functions connecting the construction of SADEs. Assembled by the subordination inequality, we offered the upper bound resolution of a class of Caratheodory functions containing distinct functions. As ongoing determinations in this style, one can study Eq.(2.4) in terminologies of differential operators such as fractional differential and convolution operator in the open unit disk. Alternatively, one can suggest a quantum calculus.

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