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Iterative algorithms for solving nonlinear quasi-variational inequalities

Muhammad Aslam Noor [a], Khalida Inayat Noor [b]

[a]*Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan*
[b]*Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan*

Abstract

In this paper, we consider the quasi-variational inequalities. It is shown that quasi-variational inequalities are equivalent to the implicit fixed point problems. Some new iterative methods for solving quasi-variational inequalities and related optimization problems are suggested by using projection methods, Wiener-Hopf equations and dynamical systems coupled with finite difference technique. Convergence analysis of these methods is investigated under monotonicity. Some special cases are discussed as applications of the main results.

Keywords: Variational inequalities, projection method, Wiener-Hopf equations, dynamical system, convergence, applications

2010 MSC: 49J40, 90C33

1. Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory, which was introduced and considered in early sixties by Stampacchia [46], can be viewed as a novel extension and generalization of the variational principles. It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities. See [1–6], [8–11], [18–28], [30–35], [40–44], [46].

If the set involved in the variational inequality depends upon the solution explicitly or implicitly, then the variational inequalities are called the quasi-variational inequality, introduced by Bensoussan and Lions [3] in the field of impulse control. Noor [19–21] proved that the quasi-variational inequalities are equivalent to the implicit fixed point problem. This equivalent formulation played an important role in developing numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi-variational inequalities. See [3, 4, 9], [19–21], [29, 30, 43] and references therein.

The Wiener-Hopf equations were introduced and studied by Shi [44] and Robinson [43]. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. Noor [24] have proved that quasi-variational inequalities are equivalent to the the Wiener-Hopf equations. This equivalence has been used to study the existence and stability of the solution of variational inequalities. Noor et al. [35] have been shown that the dynamical system can be used to suggest some implicit iterative method for solving

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Email addresses: noormaslam@gmail.com (Muhammad Aslam Noor), khalidan@gmail.com (Khalida Inayat Noor)

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⋆*Corresponding Author:* Muhammad Aslam Noor

variational inequalities. For the applications and numerical methods of the dynamical systems, see [26, 35, 36] and the references therein.

In this paper, we consider the quasi-variational inequalities involving the difference of two monotone operators, which was introduced and studied by Noor [21, 22]. It is known that quasi-variational inequalities are equivalent to fixed point problem. We used this alternative form to suggest and investigate some new implicit and explicit iterative methods for solving nonlinear quasi-variational inequalities. The convergence criteria of the proposed implicit methods is discussed under some mild conditions. For a special case of convex-valued set, we establish that quasi variational inequalities are equivalent to strongly general variational inequalities. Several important special cases are discussed as applications of our results. It is expected the techniques and ideas of this paper may be starting point for further research.

2. Formulations and basic facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis, which are needed in the derivation of the main results.

Definition 2.1. A set Ω in $\mathcal{H}$ is said to be a convex set, if

$$\mu + \lambda(\nu - \mu) \in \Omega, \qquad \forall \mu, \nu \in \Omega, \lambda \in [0, 1].$$

Definition 2.2. A function Φ is said to be a convex function, if

$$\Phi((1-\lambda)\mu + \lambda\nu) \leq (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu), \quad \forall \mu, \nu \in \Omega, \quad \lambda \in [0, 1]. \tag{2.1}$$

For $\lambda = \frac{1}{2}$, the convex function reduces to:

$$\Phi(\frac{\mu+\nu}{2}) \leq \frac{1}{2}\{\Phi(\mu) + \Phi(\nu)\}, \quad \forall \mu, \nu \in \Omega,$$

which is known as the mid-convex (Jensen-convex) function. It is known that, if the function is continuous on the interior of the convex set, then convex function and mid-convex are equivalent.

Convex functions are closely related to the integral inequalities and variational inequalities. These type of inequalities have played crucial part in developing fields such as: numerical analysis, operations research, transportation, financial mathematics, structural analysis, dynamical systems, sensitivity analysis, etc.

It is well known that a function Φ is a continuous convex functions, if and only if, it satisfies the inequality

$$\Phi\Big(\frac{a+b}{2}\Big) \leq \frac{2}{b-a}\int_a^b \Phi(x)dx \leq \frac{\Phi(a)+\Phi(b)}{2}, \quad \forall a, b \in I = [a, b], \tag{2.2}$$

which is known as the Hermite-Hadamard type inequality. Such type of the inequalities provide us with the upper and lower bounds for the mean value integral.

If the convex function Φ is differentiable, then $\mu \in \Omega$ is the minimum of the function Φ, if and only if, $\mu \in \Omega$ satisfies the inequality

$$\langle \Phi'(\mu), \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.3}$$

The inequalities of the type (2.3) are called the variational inequalities, which were introduced and studied by Stampacchia [46]. These facts motivated to consider more general variational inequalities of which (2.3) is a special case. To be more precise, for a given operator $\mathcal{T}$, find $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.4}$$

which is called the variational inequality. Note that, for $\Phi'(\mu) = \mathcal{T}\mu$, problem (2.4) is exactly the problem (2.3). For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [1–6], [8–28], [30–36], [40–46] and the references therein.

In many applications, the convex set Ω depends upon the solution explicitly or implicitly. In such cases, variational inequality is called the quasi variational inequality, that is, find $\mu \in \Omega(\mu)$ and a point-to-set mapping $\Omega : \mu \to \Omega(\mu)$, which associates a closed convex-valued set $\Omega(\mu)$ with any element μ of $\mathcal{H}$, find $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.5}$$

which is known as the quasi variational inequality, introduced by Bensoussan and Lions [3] in the impulse control theory. For the numerical analysis, sensitivity analysis, dynamical systems and other aspects of quasi variational inequalities and related optimization programming problems, see [2–4], [15], [20–22], [24], [31–35], [45] and the references therein.

Noor [20, 21] has considered and studied a class quasi-variational inequality, which include all the above problems as special cases.

For given nonlinear operators $\mathcal{T}$, we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq \langle \mathcal{A}(\mu), \nu - \mu \rangle, \quad \forall \nu \in \Omega(\mu). \tag{2.6}$$

We now mention some very important and interesting problems of the problem (2.6).

(I). Problem (2.6) can be interpreted as variational inequality involving difference of two monotone operators, which is itself a very difficult problem. This problem can be viewed as a problem of finding the the minimum of two difference of convex functions, known DC-problem. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake, see for example, Noor et al. [39].

(II). If $\Omega^*(\mu) = \{\mu \in \mathcal{H} : \langle \mu, \nu \rangle \geq 0, \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathcal{H}$, then problem (2.6) is equivalent to finding $\mu \in \Omega(\mu)$ such that

$$\mathcal{T}\mu - \mathcal{A}(\mu) \in \Omega^*(\mu) \quad \text{and} \quad \langle \mathcal{T}\mu - \mathcal{A}(\mu), \mu \rangle = 0, \tag{2.7}$$

which is known as the strongly nonlinear quasi complementarity problems [20, 21]. Obviously strongly complementarity problems include the complementarity problems, which were introduced by Lemake [11], Cottle [4] and Noor [22] in game theory, management sciences and quadratic programming as special cases.

(III). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$, then problem (2.6) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq \langle \mathcal{A}(\mu), \nu - \mu \rangle, \quad \forall \nu \in \Omega, \tag{2.8}$$

which is known as the strongly nonlinear variational inequalities, introduced and studied by Noor [18, 21, 22].

(IV). Problem (2.8) includes the absolute value equations, which is being investigated extensively in recent years using quite different techniques and ideas. To be more precise, take $\Omega = \mathcal{H}, \mathcal{A}(\mu) = \mathcal{A}|\mu|$, then one easily show that problem (2.8) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\mathcal{T}u - \mathcal{A}|\mu| = b, \tag{2.9}$$

which is called the absolute value equations, where b is a given data. This problem was rediscovered by Mangasarian [13] and Noor et al. [36–38]. Clearly, system of absolute value equations is a very important special case of strongly nonlinear variational inequalities, which were introduced by Noor [18].

(V). If $\langle \mathcal{A}(\mu), \nu \rangle = \mathcal{A}(\mu, \nu)$, then (2.6) collapses to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq \mathcal{A}(\mu, \nu - \mu), \quad \forall \nu \in \Omega(\mu), \tag{2.10}$$

which is called nonlinear quasi hemivariational inequality and appears to be a new one.

If $\Omega(\mu) = \Omega$, then the problem (2.10) becomes the hemivariational inequality, which was introduced by Panagiotopoulos [40] in structural analysis. For the applications, formulation and other aspects of hemivariational inequalities, see [27, 40, 41] and the references therein.

Remark 2.3. It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, \mathcal{A}$, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of variational inequalities, complementarity problems and optimization problems as special cases of the nonlinear quasi-variational inequalities (2.6). This shows that the problem (2.6) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the nonlinear quasi-variational inequalities.

Example 2.4. To convey an idea of the applications of the nonlinear quasi variational inequalities (2.6), we consider the second-order implicit obstacle boundary value problem of finding μ such that

$$\left.\begin{array}{ll} -\mu'' \geq \phi(x,\mu) & \text{on } \Omega_1 = [a,b] \\ \mu \geq \mathcal{M}(\mu) & \text{on } \Omega_1 = [a,b] \\ [-\mu'' - \phi(x,\mu)][\mu - \mathcal{M}(\mu)] = 0 & \text{on } \Omega_1 = [a,b] \\ \mu(a) = 0, \quad \mu(b) = 0. & \end{array}\right\} \tag{2.11}$$

where $\phi(x,\mu)$ is a continuous function and $\mathcal{M}(\mu)$ is the cost (obstacle) function. The prototype encountered is

$$\mathcal{M}(\mu) = \eta + \inf_i \{\mu^i\}. \tag{2.12}$$

In (2.12), η represents the switching cost. It is positive, when the unit is turned on and equal to zero when the unit is turned off. The operator $\mathcal{M}$ provides the coupling between the unknowns $\mu = (\mu^1, \mu^2, \ldots, \mu^i)$, see [12]. We study the problem (2.11) in the framework of strongly nonlinear quasi variational inequality approach. To do so, we first define the set as

$$\Omega(\mu) = \{\nu : \nu \in \mathcal{H}_0^1(\Omega_1) : \nu \geq \mathcal{M}(\mu), \quad \text{on } \Omega_1\},$$

which is a closed convex-valued set in $\mathcal{H}_0^1(\Omega)$, where $\mathcal{H}_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (2.11) is

$$\begin{aligned} \mathcal{I}[\nu] &= -\int_a^b \left(\frac{d^2\nu}{dx^2}\right)\nu dx - 2\int_a^b \{\int_0^\nu \phi(x,\zeta)d\zeta\}dx, \quad \forall \nu \in \Omega(\mu) \\ &= \int_a^b \left(\frac{d\nu}{dx}\right)^2 dx - 2\int_a^b \{\int_0^\nu f(x,\zeta)d\zeta\}dx \\ &= \langle \mathcal{T}\nu, \nu\rangle - 2\Phi(\nu), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \langle \mathcal{T}\mu, \nu\rangle &= -\int_a^b \left(\frac{d^2\mu}{dx^2}\right)(\nu)\, dx = \int_a^b \frac{d\mu}{dx}\frac{d\nu}{dx}dx \\ \Phi(\nu) &= \int_a^b \{\int_0^\nu \phi(x,\zeta)d\zeta\}dx. \end{aligned} \tag{2.14}$$

It has been shown [18] that

$$\langle \Phi'(\mu), \nu\rangle = \int_a^b \phi(x,\zeta)\nu dx.$$

It is clear that the operator $\mathcal{T}$ defined by (2.14) is linear, symmetric and positive. Using the technique of Noor [18], one can show that the minimum of the functional $\mathcal{I}[\nu]$ defined by (2.13) associated with the problem (2.11) on the closed convex-valued multifunction $\Omega(\mu)$ can be characterized by the inequality of type

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq \langle \Phi'(u), \nu - u\rangle, \quad \forall \nu \in \Omega(\mu), \tag{2.15}$$

which is exactly the quasi variational inequality (2.6) with $\langle Au, \nu\rangle = \langle \Phi'(u), \nu\rangle$. For the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities, see [3], [6–21].

We also need the following result, known as the projection Lemma (best approximation) Lemma, which plays a crucial part in establishing the equivalence between the quasi variational inequalities and the fixed point problems. This result is used in the analysing the convergence analysis of the projective implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.5 ([12])**.** *Let $\Pi_{\Omega(\mu)}$ be a closed and convex-valued multifunction in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega(\mu)$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.16}$$

if and only if,

$$\mu = \Pi_{\Omega(\mu)}(z),$$

where $\Pi_{\Omega(\mu)}$ is implicit projection of $\mathcal{H}$ onto the closed convex-valued multifunction $\Omega(\mu)$.

It is well known that the projection operator $\Pi_{\Omega(\mu)}$ is required to satisfy the following assumption, which plays an important part in the derivation of the results.

Assumption 2.6.

$$\|\Pi_{\Omega(\mu)}\omega - \Pi_{\Omega(\nu)}\omega\| \leq \nu\|\mu - \nu\|, \forall \mu, \nu, \omega \in \mathcal{H}. \tag{2.17}$$

Definition 2.7. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:

(i) Strongly monotone, if there exist a constant $\alpha > 0$, such that

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

(ii) Lipschitz continuous, if there exist a constant $\beta > 0$, such that

$$\|\mathcal{T}\mu - \mathcal{T}\nu\| \leq \beta\|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

(iii) Monotone, if

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

(iv) Pseudo monotone, if

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}\nu, \nu - \mu \rangle \geq 0, \quad \forall \mu, \nu \in .$$

Remark 2.8. Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.

3. Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the quasi variational inequalities.

Using Lemma 2.5, one can show that the quasi variational inequalities are equivalent to the fixed point problems.

Lemma 3.1 ([2])**.** *The function $\mu \in \Omega(\mu)$ is a solution of the quasi variational inequality* (2.6)*, if and only if, $\mu \in \Omega(\mu)$ satisfies the relation*

$$\mu = \Pi_{\Omega(\mu)}[\mu - \rho(T\mu - A(\mu))], \tag{3.1}$$

where $\Pi_{\Omega(\mu)}$ is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the problem (2.6) and the problem (3.1) are equivalent. The equivalent fixed point formulation (3.1) is used to suggest the iterative methods for solving the problem (2.6).

Algorithm 3.2. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}[\mu_n - \rho(\mathcal{T}\mu_n - \mathcal{A}(\mu_n))], \quad n = 0, 1, 2, ... \tag{3.2}$$

which is known as the projection method and has been studied extensively.

Algorithm 3.3. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\mu_n - \rho(\mathcal{T}\mu_{n+1} - \mathcal{A}(\mu_{n+1}))], \quad n = 0, 1, 2, ... \tag{3.3}$$

which is known as the implicit projection method and is equivalent to the following two-step method.

Algorithm 3.4. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho(\mathcal{T}\mu_n - \mathcal{A}(\mu_n))] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\mu_n - \rho(T\omega_n - A(\omega_n))], \quad n = 0, 1, 2, ... \end{aligned}$$

For the convergence of the implicit method, see Noor [7].

Algorithm 3.5. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\mu_{n+1} - \rho(\mathcal{T}\mu_{n+1} - \mathcal{A}(\mu_{n+1}))], \quad n = 0, 1, 2, ... \tag{3.4}$$

which is known as the modified projection method and is equivalent to the iterative method.

Algorithm 3.6. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho(\mathcal{T}u_n - \mathcal{A}(\mu_n))] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\omega_n - \rho(\mathcal{T}\omega_n - \mathcal{A}(\omega_n))], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.5).

We can rewrite the equation (3.1) as:

$$\mu = \Pi_{\Omega(\mu)}[\frac{\mu + \mu}{2} - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)]. \tag{3.5}$$

This fixed point formulation was used to suggest the following implicit method.

Algorithm 3.7 ([32]). For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\frac{\mu_n + \mu_{n+1}}{2} - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1})], \quad n = 0, 1, 2, ... \tag{3.6}$$

Noor et al. [32] used the predictor-corrector technique to suggest the following inertial iterative method for solving the problem (2.6).

Algorithm 3.8. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\frac{\omega_n + \mu_n}{2} - \rho\mathcal{T}\omega_n + \rho\mathcal{A}(\omega_n)], \quad \lambda \in [0, 1], \quad n = 0, 1, 2, ... \end{aligned}$$

From equation (3.1), we have

$$\mu = \Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}(\frac{\mu + \mu}{2}) + \rho\mathcal{A}(\frac{\mu + \mu}{2})]. \tag{3.7}$$

This fixed point formulation (3.7) is used to suggest the implicit method for solving the problem (2.6) as

Algorithm 3.9. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\mu_n - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2}) + \rho\mathcal{A}(\frac{\mu_n + \mu_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.8}$$

We can use the predictor-corrector technique to rewrite Algorithm 3.9 as:

Algorithm 3.10. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)], \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\mu_n - \rho\mathcal{T}(\frac{\mu_n + \omega_n}{2}) + \rho\mathcal{A}(\frac{\mu_n + \omega_n}{2})], \quad n = 0, 1, 2, ... \end{aligned}$$

is known as the mid-point implicit method for solving the problem (2.6).

We again use the above fixed formulation to suggest the following implicit iterative method.

Algorithm 3.11. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\mu_{n+1} - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2}) + \rho\mathcal{A}(\frac{\mu_n + \mu_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.9}$$

Using the predictor-corrector technique, Algorithm 3.10 can be written as:

Algorithm 3.12. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)], \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\omega_n - \rho\mathcal{T}(\frac{\mu_n + \omega_n}{2}) + \rho\mathcal{A}(\frac{\mu_n + \omega_n}{2})], \quad n = 0, 1, 2, ... \end{aligned}$$

which appears to be new one.

It is obvious that Algorithm 3.4 and Algorithm 3.5 have been suggested using different variant of the fixed point formulations (3.1). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the problem (2.6) and related optimization problems, which is the main motivation of this paper.

One can rewrite (3.1) as

$$\mu = \Pi_{\Omega(\mu)}[\frac{\mu + \mu}{2} - \rho\mathcal{T}(\frac{\mu + \mu}{2}) + \rho\mathcal{A}(\frac{\mu + \mu}{2})]. \tag{3.10}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.6).

Algorithm 3.13. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\frac{\mu_n + \mu_{n+1}}{2} - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2}) + \rho\mathcal{A}(\frac{\mu_n + \mu_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.11}$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.5 as the predictor and Algorithm 3.13 as corrector. Thus, we obtain a new two-step method for solving the the problem (2.6).

Algorithm 3.14. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\Big[\Big(\frac{\omega_n + \mu_n}{2}\Big) - \rho\mathcal{T}\Big(\frac{\omega_n + \mu_n}{2}\Big) + \rho\mathcal{A}\Big(\frac{\omega_n + \mu_n}{2}\Big)\Big], \quad n = 0, 1, 2, ... \end{aligned}$$

which is a new predictor-corrector two-step method.

For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \Pi_{\Omega(u)}(1 - \xi)\mu + \xi\mu - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)]. \tag{3.12}$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the the problem (2.6).

Algorithm 3.15. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_n)}[(1 - \xi)\mu_n + \xi\mu_{n-1} - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)], \quad n = 0, 1, 2, ...$$

It is noted that Algorithm 3.15 is equivalent to the following two-step method.

Algorithm 3.16. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[\omega_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(u_n)], \quad n = 0,1,2,.... \end{aligned}$$

Algorithm 3.16 is known as the inertial projection method, which is mainly due to Noor [25] and Noor et al. [34, 35].

Using this idea, we can suggest the following iterative methods for solving quasi variational inequalities.

Algorithm 3.17. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[\omega_n - \rho\mathcal{T}\omega_n + \rho\mathcal{A}(\omega_n)], \quad n = 0,1,2,.... \end{aligned}$$

Algorithm 3.18. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\alpha)u_n + \alpha u_{n-1} \\ u_{n+1} &= P_{K(y_n)}[y_n - \rho T y_n + \rho A(y_n)], \quad n = 0,1,2,.... \end{aligned}$$

Using the technique of Noor et al. [31] and Jabeen et al. [9], one can investigate the convergence analysis of these inertial projection methods.

4. Wiener-Hopf Equations Technique

In this section, we discuss the Wiener-Hopf equations associated with the quasi variational inequalities. It is worth mentioning that the Wiener-Hopf equations associated with variational inequalities were introduced and studied by Shi [44] and Ronbinson [43] independently using different techniques. Noor [24] proved that the quasi variational inequalities are equivalent to the implicit Wiener-Hopf equations to study the sensitivity analysis.

Let $\mathcal{T}$ be an operator and $_{\Omega(\mu)} = \mathcal{I} - \Pi_{\Omega(\mu)}$, where $\mathcal{I}$ is the identity operator and $\Pi_{\Omega(\mu)}$ is the projection operator.

We consider the problem of finding $z \in \mathcal{H}$ such that

$$\mathcal{T}\Pi_{\Omega(\mu)}z + \rho^{-1}\mathcal{R}_{\Omega(\mu)}z = \mathcal{A}(\Pi_{\Omega(\mu)}z). \tag{4.1}$$

The equations of the type (4.1) are called the implicit Wiener-Hopf equations, which play part in the developments of iterative methods, sensitivity analysis and other aspects of the variational inequalities, see [24–26], [34, 35] and references therein.

Lemma 4.1. *The element $\mu \in \Omega(\mu)$ is a solution of the quasi variational inequality* (2.6)*, if and only if, $z \in \mathcal{H}$ satisfies the resolvent equation* (4.1)*, where*

$$\mu = \Pi_{\Omega(\mu)}z, \tag{4.2}$$

$$z = \mu - \rho\mathcal{T}u + \rho\mathcal{A}(\mu), \tag{4.3}$$

where $\rho > 0$ is a constant.

From Lemma 4.1, it follows that the problems inequalities (2.6) and the problems (4.1) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the strongly quasi variational inequalities and related optimization problems.

We use the Wiener-Hopf equations (4.1) to suggest some new iterative methods for solving the quasi variational inequalities. From (4.2) and (4.3),

$$\begin{aligned} z &= \Pi_{\Omega(\mu)}z - \rho\mathcal{T}\Pi_{\Omega(\mu)}z + \rho\mathcal{A}(\Pi_{\Omega(\mu)}z) \\ &= \Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)] - \rho\mathcal{T}\Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)]. \end{aligned}$$

Thus, we have

$$u = \rho T u - \rho A(u) + [P_{K(u)}[u - \rho T u + \rho A(u)] - \rho T P_{K(u)}[u - \rho T u + \rho A(u)].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n \Pi_{\Omega(\mu)}\{\Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}u + \rho\mathcal{A}(\mu)] + \rho\mathcal{T}\mu - \rho\mathcal{A}(\mu) - \rho\mathcal{T}\Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)]\} \\ &= (1-\alpha_n)\mu + \alpha_n \Pi_{\Omega(\mu)}\{\omega - \rho\mathcal{T}\omega + \rho\mathcal{T}\mu - \rho\mathcal{A}(\mu)\}, \end{aligned} \tag{4.4}$$

where

$$\omega = \Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)]. \tag{4.5}$$

Using (4.4) and (4.5), we can suggest the following new predictor-corrector method for solving the quasi variational inequalities.

Algorithm 4.2. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)] \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n \Pi_{(\omega_n)}\Big\{\omega_n - \rho\mathcal{T}\omega_n - \rho\mathcal{A}(\mu_n) + \rho\mathcal{T}\mu_n\Big\}. \end{aligned}$$

If $\alpha_n = 1$, then Algorithm 4.2 reduces to

Algorithm 4.3. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu)] \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[\omega_n - \rho\mathcal{T}\omega_n + \rho\mathcal{T}\mu_n - \rho\mathcal{A}(\mu_n)], \end{aligned}$$

which appears to be a new one.

In a similar way, we can suggest and analyse the predictor-corrector method for solving the quasi variational inequalities (2.6), which only involve only one projection.

Algorithm 4.4. For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \mu_n - \xi(\mu_n - \mu_{n-1}) \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\omega_n - \rho\mathcal{T}\omega_n + \rho\mathcal{T}\mu_n - \rho\mathcal{A}(\mu_n)]. \end{aligned}$$

One can study the convergence of the Algorithm 4.4 using the technique of Jabeen et al. [9].

Remark 4.5. We have only given some glimpse of the technique of the Wiener-Hopf equations for solving the quasi variational inequalities. One can explore the applications of the Wiener-Hopf equations in developing efficient numerical methods for variational inequalities and related nonlinear optimization problems.

5. Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving quasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [7]. The dynamical system is a first order initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It have been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We consider some iterative methods for solving the quasi variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operator.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \mu - \Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu + \rho\mathcal{A}(\mu)]. \tag{5.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem (2.6), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{5.2}$$

We now consider a dynamical system associated with the quasi variational inequalities. Using the equivalent formulation (3.1), we suggest a class of project dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}u + \rho\mathcal{A}(\mu)] - \mu\}, \quad \mu(t_0) = \alpha, \tag{5.3}$$

where λ is a parameter. The system of type (5.3) is called the project dynamical system associated with the problem (2.6). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (5.3) can be studied.

We use the project dynamical system (5.3) to suggest some iterative for solving the quasi variational inequalities (2.6). These methods can be viewed in the sense of Koperlevich [14] and Noor [26] involving the double projection.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (5.3) becomes

$$\frac{d\mu}{dt} + \mu = \Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}u + \rho\mathcal{A}(\mu)], \quad \mu(t_0) = \alpha. \tag{5.4}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (5.4), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_{n+1} = \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1})], \tag{5.5}$$

where h is the step size.

Now, we can suggest the following implicit iterative method for solving the problem (2.6).

Algorithm 5.1. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\Big[\mu_n - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1}) - \frac{\mu_{n+1} - \mu_n}{h}\Big],$$

This is an implicit method, which is quite different from the implicit method of [4]. Algorithm 5.1 is equivalent to the following two-step method.

Algorithm 5.2. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}
\omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)] \\
\mu_{n+1} &= \Pi_{\Omega(\omega_n)}[\mu_n - \rho\mathcal{T}\omega_n + \rho\mathcal{A}(\omega_n) - \frac{\omega_n - \mu_n}{h}],
\end{aligned}$$

Discretizing (5), we now suggest an other implicit iterative method for solving (2.6).

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\mu_{n+1} - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1})], \tag{5.6}$$

where h is the step size.

This formulation enables us to suggest the two-step iterative method.

Algorithm 5.3. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned}
\omega_n &= \Pi_{\omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)] \\
\mu_{n+1} &= \Pi_{\Omega(\omega_n)}\Big[\omega_n - \rho\mathcal{T}\omega_n + \rho\mathcal{A}(\omega_n) - \frac{\omega_n - \mu_n}{h}\Big], \quad n = 0, 1, 2, \ldots.
\end{aligned}$$

Again using the project dynamical systems, we can suggested some iterative methods for solving the quasi variational inequality problems and related optimization problems.

Algorithm 5.4. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}\Big[\frac{(h+1)\mu_n - \mu_{n+1}}{h} - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)\Big], \quad n = 0, 1, 2, \ldots .$$

or equivalenly

Algorithm 5.5. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\Big[\frac{(h+1)\mu_n - \omega_n}{h} - \rho\mathcal{T}u_n + \rho\mathcal{A}(\mu_n)\Big], \quad n = 0, 1, 2, \ldots . \end{aligned}$$

Discretizing (5.4), we have

$$\frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = \Pi_{\Omega(\mu_{n_1})}[\mu_n - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1})], \tag{5.7}$$

where h is the step size.

This helps us to suggest the following implicit iterative method for solving the problem (2.6).

Algorithm 5.6. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)] \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n)}\Big[\frac{(h+1)\mu_n - \omega_n}{h} - \rho\mathcal{T}\mu_n + \rho\mathcal{A}(\mu_n)\Big], \quad n = 0, 1, 2, \ldots . \end{aligned}$$

Discretizing (5.4), we propose another implicit iterative method.

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_n = \Pi_{\Omega(\mu_{n+1})}[\mu_n - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1})],$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (2.6).

Algorithm 5.7. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \Pi_{\Omega(\mu_{n+1})}[\mu_n - \rho\mathcal{T}\mu_{n+1} + \rho\mathcal{A}(\mu_{n+1})], \quad n = 0, 1, 2, 3, ...$$

Algorithm 5.7 is an implicit iterative method in the sense of Koperlevich and can be written as:

Algorithm 5.8. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\langle \rho(\mathcal{T}\mu_{n+1} - \mathcal{A}(\mu_{n+1})) + \mu_{n+1} - \mu_n, \nu - \mu_{n+1}\rangle \geq 0. \tag{5.8}$$

We study the convergence analysis of Algorithm 5.8, which is the main motivation of our next result.

Theorem 5.9. *Let $\mu \in \Omega(\mu)$ be a solution of* (2.6) *and μ_n be the approximate solution obtained from (5.8). If the operators $\mathcal{T}$ and $-\mathcal{A}$ are monotone, then,*

$$\|\mu - \mu_{n+1}\|^2 \leq \|\mu - \mu_n\|^2 - \|\mu_n - \mu_{n+1}\|^2. \tag{5.9}$$

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (2.6). Then

$$\langle \mathcal{T}\mu - A(\mu), \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

implies that

$$\langle \mathcal{T}\nu - \mathcal{A}\nu), \nu - \mu\rangle \geq 0, \tag{5.10}$$

since $\mathcal{T}$ and $-\mathcal{A}$ are monotone operators.

Taking $\nu = \mu_{n+1}$ in (5.10), we have

$$\langle \mathcal{T}\mu_{n+1} - \mathcal{A}(\mu_{n+1}), \mu_{n+1} - \mu \rangle \geq 0. \tag{5.11}$$

Now taking $\nu = \mu$ in (5.16), we obtain

$$\langle \rho(\mathcal{T}\mu_{n+1} - \mathcal{A}(\mu_{n+1})) + \mu_{n+1} - \mu_n, \mu - \mu_{n+1} \rangle \geq 0. \tag{5.12}$$

From (5.11) and (5.12), we have

$$\langle \mu_{n+1} - \mu_n, \mu - \mu_{n+1} \rangle \geq 0, \tag{5.13}$$

from which, using the inequality $2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in \mathcal{H}$, we obtain

$$\|\mu - \mu_{n+1}\|^2 \leq \|\mu - \mu_n\|^2 - \|\mu_n - \mu_{n+1}\|^2,$$

which is the required result (5.9). □

Theorem 5.10. *Let μ_{n+1} be the approximate solution obtained from Algorithm* 5.8 *and $\mu \in \Omega(\mu)$ be a solution of* (2.6)*. If the operators $\mathcal{T}$ and $-\mathcal{A}$ are monotone operators, then*

$$\lim_{n\to\infty} \mu_n = \mu.$$

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (2.6). Then, from (5.9), it follows that that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{\mu_n\}$ is bounded. Also from (5.9), we have

$$\sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\|^2 \leq \|\mu_0 - \mu\|^2,$$

which implies that

$$\lim_{n\to\infty} \mu_n = \mu. \tag{5.14}$$

Let $\hat{\mu}$ be the cluster point of $\{\mu_n\}$ and the subsequence $\{\mu_{n_j}\}$ of the sequence $\{\mu_n\}$ converge to $\hat{\mu} \in \mathcal{H}$. Replacing μ_n by μ_{n_j} in (5.16) and taking the limit $n_j \to \infty$ and using (5.14), we have

$$\langle \mathcal{T}\hat{\mu} - \mathcal{A}(\hat{\mu}), \nu - \hat{\mu} \rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

which implies that $\hat{\mu}$ solves the problem (2.6) and

$$\|\mu_{n+1} - \mu_n\|^2 \leq \|\mu_n - \hat{\mu}\|^2.$$

Thus it follows from the above inequality that the sequence $\{\mu_n\}$ has exactly one cluster point $\hat{\mu}$ and $\lim_{n\to\infty} \mu_n = \hat{\mu}$. □

Using (5.4), we have

$$\frac{d\mu}{dt} + \mu = \Pi_{\Omega((1-\alpha)\mu+\alpha\mu)}[(1-\alpha)\mu + \alpha\mu - \rho\mathcal{T}((1-\alpha)\mu + \alpha\mu) + \rho\mathcal{A}((1-\alpha)\mu + \alpha\mu))], \tag{5.15}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (5.15 and taking $h = 1$, we have

$$\mu_{n+1} = \Pi_{\Omega((1-\alpha)\mu_n+\alpha\mu_{n-1})}[(1-\alpha)\mu_n + \alpha\mu_{n-1} - \rho\mathcal{T}((1-\alpha)\mu_n + \alpha\mu_{n-1}) + \rho\mathcal{A}((1-\alpha)\mu_n + \alpha\mu_{n-1})],$$

which is an inertial type iterative method for solving the quasi variational inequality (2.6). Using the predictor-corrector techniques, we have

Algorithm 5.11. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative schemes

$$\begin{aligned} \omega_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ \mu_{n+1} &= \Pi_{\Omega(\omega_n}[\omega_n - \rho\mathcal{T}(\omega_n) + \rho\mathcal{A}(\omega_n)], \end{aligned}$$

which is known as the inertial two-step iterative method.

Remark 5.12. For appropriate and suitable choice of the operators $\mathcal{T}, \mathcal{A}$, convex-valued set, parameter α and the spaces, one can propose a wide class of implicit, explicit and inertial type methods for solving nonlinear quasi variational inequalities and related nonlinear optimization problems. Using the techniques and ideas of Noor et al. [36], one can discuss the convergence analysis of the proposed methods.

6. Applications

In this section, we show that the quasi variational inequalities are equivalent to the general variational inequalities, see Noor [23].

In many applications, the convex-valued multifunction $\Omega(\mu)$ is of the form:

$$\Omega(\mu) = \eta(\mu) + \Omega, \tag{6.1}$$

where Ω is a convex set and η is a point-to-point mapping.

Let $\mu \in \Omega(\mu)$ be a solution of the problem (2.6). Then from Lemma 3.1, it follows that $\mu \in \Omega(\mu)$ such that

$$\mu = \Pi_{\Omega(\mu)}[\mu - \rho(\mathcal{T}\mu - \mathcal{A}(\mu))]. \tag{6.2}$$

Combining (6.1) and (6.2), we obtain

$$\begin{aligned} \mu &= \Pi_{\Omega(\eta(\mu)+\Omega)}[\mu - \rho(\mathcal{T}\mu - \mathcal{A}(\mu))] \\ &= \eta(\mu) + \Pi_{\Omega}[\mu - \eta(\mu) - \rho(\mathcal{T}u - \mathcal{A}(\mu))]. \end{aligned}$$

This implies that

$$g(\mu) = \Pi_{\Omega}[g(\mu) - \rho(\mathcal{T}\mu - \mathcal{A}(\mu))].$$

which is equivalent to finding $\mu \in \mathcal{H} : g(\mu) \in \Omega$ such that

$$\langle(\mathcal{T}\mu - \mathcal{A}(\mu)), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \mathcal{H} : g(\nu) \in \Omega. \tag{6.3}$$

The inequality of the type (6.3) is called the general variational inequality, investigated by Noor [23]. It have shown that odd-order and nonsymmetric obstacle boundary value problems can be studied in the general variational inequalities. For more details, see [26, 34, 35]. Thus all the results proved for quasi variational inequalities continue to hold for strongly general variational inequalities (6.3) with suitable modifications and adjustment. Despite the research activates, very few results are available. The development of efficient implementable numerical methods requires further efforts.

7. Conclusion

In this paper, we have used the equivalence between the quasi variational inequalities and fixed point formulation to suggest some new iterative methods for solving the variational inequalities. These new methods include extragradient method, modified double projection methods and inertial type are suggested using the techniques of projection method, Wiener-Hopf equations and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. It is an open problem to compare these proposed methods with other methods. We have shown that the nonlinear quasi variational inequalities are equivalent to the strongly general variational inequalities under suitable conditions of the convex-valued set. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of variational inequalities and related problems.

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