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Quasi-Hadamard product of certain classes with respect to symmetric points connected with $q-$Salagean operator

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Abstract

The foremost determination of this paper is to attain results associated with the quasi-Hadamard product of certain starlike and convex functions with respect to symmetric points related with q-Salagean operator.

Keywords: Quasi-Hadamard product, symmetric points, q-Salagean operator

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1. Introduction

Let $\mathbb{A}$ denote the class of analytic functions of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in \Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \tag{1.1}$$

and S be the subclass of $\mathbb{A}$, which are univalent functions.

Srivastava [17] presented and inspired about brief expository outline of the classical q-analysis versus the self-styled (p, q)-analysis with an obviously terminated additional parameter p (see Srivastava and Karlsson [18, pp. 350–351], Srivastava [15, 16]).

We recall some basic definitions and concepts details of the q-calculus (see [1, 9, 10]).

Definition 1.1. Let $0 < q < 1$, the q-number $[\gamma]_q$ and $[\gamma]_q!$ are defined by

$$[\gamma]_q := \begin{cases} \frac{1-q^{\gamma}}{1-q} & \gamma \in \mathbb{C} \\ 1 + \sum\limits_{j=1}^{n-1} q^j & \gamma = n \in \mathbb{N} \end{cases} \tag{1.2}$$

and

$$[\gamma]_q! := \begin{cases} 1 & \gamma = 0 \\ \prod\limits_{k=1}^{n} [k]_q & \gamma = n \in \mathbb{N}. \end{cases}$$

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The q-derivative of a function $h(z)$ is $\mathfrak{D}_q h(z)$ defined as follows:

Definition 1.2. The *q-derivative operator* for h is defined by

$$\mathfrak{D}_q\, h(z) := \begin{cases} \frac{h(qz)-h(z)}{z(q-1)} & z \neq 0 \\ h'(0) & z = 0, \end{cases} \tag{1.3}$$

provided that $h'(0)$ exists.

We note that

$$\lim_{q\to 1^-} \mathfrak{D}_q\, h(z) := \lim_{q\to 1^-} \frac{h(qz)-h(z)}{z(q-1)} = h'(z).$$

From (1.1) and (1.3), we have

$$\mathfrak{D}_q\, h(z) := 1 + \sum_{k=2}^{\infty} [k]_q\, a_k z^{k-1},\ z \neq 0, \tag{1.4}$$

where $[k]_q$ is defined by (1.2).
Now, using the q-derivative operator $\mathfrak{D}_q$, Govindaraj and Sivasubramanian [6] (see Govindaraj et al. [7], also El-Deeb and El-Matary [5, with $p = 1$]) defined the operator $\mathfrak{D}_q^n : \mathbb{A} \to \mathbb{A}$ as follows:

$$\mathfrak{D}_q^0\, h(z) = h(z),$$

$$\begin{aligned} \mathfrak{D}_q^1 h(z) &= z\mathfrak{D}_q h(z) = z + \sum_{k=2}^{\infty} [k]_q\, a_k z^k \\ \mathfrak{D}_q^2 h(z) &= z\mathfrak{D}_q\left(\mathfrak{D}_q^1 h(z)\right) = z + \sum_{k=2}^{\infty} \left([k]_q\right)^2 a_k z^k \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.5) \\ \mathfrak{D}_q^n h(z) &= z\mathfrak{D}_q\left(\mathfrak{D}_q^{n-1} h(z)\right) = z + \sum_{k=2}^{\infty} \left([k]_q\right)^n a_k z^k \qquad\qquad (1.6) \\ &\left(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ 0 < q < 1\right). \end{aligned}$$

By specializing the parameter q, we obtain the following operators:

$$\lim_{q\to 1^-} \mathfrak{D}_q^n h(z) = \mathfrak{D}^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \tag{1.7}$$

where $\mathfrak{D}^n$ is called Salagean opeator (see Salagean [14]).

In [13], Sakaguchi familiarized the class $\mathfrak{S}_s^*$ of functions $h(z) \in S$ and satiating the succeeding condition:

$$\Re\left\{\frac{zh'(z)}{h(z)-h(-z)}\right\} > 0 \ \ (z \in \Delta).$$

These functions are called starlike with respect to symmetric points.

In [19], Sudharsan et al. introduced the class $\mathfrak{S}_s^*(\vartheta, \varsigma)$ of functions $h(z) \in S$ and satiating the succeeding condition

$$\left|\frac{zh'(z)}{h(z)-h(-z)} - 1\right| < \varsigma\left|\vartheta\frac{zh'(z)}{h(z)-h(-z)} + 1\right|$$

for some $0 \leq \vartheta \leq 1,\ 0 < \varsigma \leq 1$ and $z \in \Delta$.

Let $\mathfrak{T}$ denote the subclass of S consisting of functions of the form:

$$h(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \ (a_1 > 0,\ a_k \geq 0). \tag{1.8}$$

Let the functions of the forms

$$h_i(z) = a_{1,i} z - \sum_{k=2}^{\infty} a_{k,i} z^k \ (a_{1,i} > 0,\ a_{k,i} \geq 0), \tag{1.9}$$

$$g(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \ (b_1 > 0,\ b_k \geq 0), \tag{1.10}$$

and

$$g_j(z) = b_{1,j} z - \sum_{k=2}^{\infty} b_{k,j} z^k \ (b_{1,j} > 0,\ b_{k,j} \geq 0), \tag{1.11}$$

be regular and univalent in the unit disk Δ.
Let us define quasi-Hadamard product of functions $h(z)$ and $g(z)$ as follows:

$$(h * g)(z) = a_1 b_1 z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * h)(z). \tag{1.12}$$

We can define quasi-Hadamard product of more than two functions as follows:

$$(h_1 * h_2 * ... * h_m)(z) = \prod_{i=1}^{m} a_{1,i} z - \sum_{k=2}^{\infty} \prod_{i=1}^{m} a_{k,i} z^k, \tag{1.13}$$

where the functions h_i $(i = 1, 2, ..., m)$ are given by (1.9).

In [8] Halim et al. defined the class $\mathfrak{S}_s^* \mathfrak{T}(\vartheta, \varsigma)$ as follows:

Definition 1.3 ([8]). A function $h(z)$ be given by (1.8). Then $h(z)$ is in the class $\mathfrak{S}_s^* \mathfrak{T}(\vartheta, \varsigma)$ if it satisfies the following condition:

$$\left| \frac{zh'(z)}{h(z) - h(-z)} - 1 \right| < \varsigma \left| \vartheta \frac{zh'(z)}{h(z) - h(-z)} + 1 \right|, \tag{1.14}$$

for some $0 \leq \vartheta \leq 1,\ 0 < \varsigma \leq 1$ and $z \in \Delta$.

In [3, with $n = 1$] Aouf et al. defined the class $\mathfrak{C}_s \mathfrak{T}(\vartheta, \varsigma)$ as follows:

Definition 1.4 ([3], with $n = 1$). A function $h(z)$ be given by (1.8). Then $h(z)$ is in the class $\mathfrak{C}_s \mathfrak{T}(\vartheta, \varsigma)$ if it satisfies the following condition:

$$\left| \frac{\left(zh'(z)\right)'}{h'(z) + h'(-z)} - 1 \right| < \varsigma \left| \vartheta \frac{\left(zh'(z)\right)'}{h'(z) + h'(-z)} + 1 \right|. \tag{1.15}$$

For some $0 \leq \vartheta \leq 1,\ 0 < \varsigma \leq 1$ and $z \in \Delta$. From (1.14) and (1.15) it follows that

$$h(z) \in \mathfrak{C}_s \mathfrak{T}(\vartheta, \varsigma) \Leftrightarrow zh'(z) \in \mathfrak{S}_s^* \mathfrak{T}(\vartheta, \varsigma). \tag{1.16}$$

In [3, with $n = 1$], Aouf et al. defined the class $\mathfrak{S}^* \mathfrak{T}_{s,n}(\vartheta, \varsigma)$ as follows:

Definition 1.5 ([3], with $n = 1$). A function $h(z)$ be defined by (1.8). Then $h(z)$ is in the class $\mathfrak{S}^* \mathfrak{T}_{s,n}(\vartheta, \varsigma)$ if it satisfies the following condition:

$$\left| \frac{\mathfrak{D}^{n+1} h(z)}{\mathfrak{D}^n h(z) - \mathfrak{D}^n h(-z)} - 1 \right| < \varsigma \left| \vartheta \frac{\mathfrak{D}^{n+1} h(z)}{\mathfrak{D}^n h(z) - \mathfrak{D}^n h(-z)} + 1 \right|, \tag{1.17}$$

where $n \in \mathbb{N}_0$, $0 \leq \vartheta \leq 1,\ 0 < \varsigma \leq 1,\ 0 \leq \frac{2(1-\varsigma)}{1+\vartheta\varsigma} < 1$, $\mathfrak{D}^n$ is defined by 1.7 and $z \in \Delta$.

Using the operator $\mathfrak{D}_q^n$ given by (1.6), we introduce the subclass $\mathfrak{S}^*_{s,n,q}\mathfrak{T}(\vartheta,\varsigma)$ as follows:

Definition 1.6. Let the function $h(z)$ be defined by (1.8). Then $h(z)$ is said to be in the class $\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta,\varsigma)$ if it satisfies the following condition:

$$\left|\frac{\mathfrak{D}_q^{n+1}h(z)}{\mathfrak{D}_q^n h(z)-\mathfrak{D}_q^n h(-z)}-1\right|<\varsigma\left|\vartheta\frac{\mathfrak{D}_q^{n+1}h(z)}{\mathfrak{D}_q^n h(z)-\mathfrak{D}_q^n h(-z)}+1\right|, \tag{1.18}$$

where $n\in\mathbb{N}_0$, $0\le\vartheta\le 1,\ 0<\varsigma\le 1,\ 0<q<1,\ 0\le\frac{2(1-\varsigma)}{1+\vartheta\varsigma}<1$, $\mathfrak{D}_q^n$ is defined by 1.6 and $z\in\Delta$.

We note that:

(i) $\lim\limits_{q\to 1^-}\mathfrak{S}^*\mathfrak{T}_{s,0,q}(\vartheta,\varsigma)=\mathfrak{S}^*_s(\vartheta,\varsigma)$ and $\lim\limits_{q\to 1^-}\mathfrak{S}^*\mathfrak{T}_{s,1,q}(\vartheta,\varsigma)=\mathfrak{C}_s(\vartheta,\varsigma)\quad(0\le\vartheta\le 1,\ 0<\varsigma\le 1)$;

(ii) $\lim\limits_{q\to 1^-}\mathfrak{S}^*\mathfrak{T}_{s,0,q}(\vartheta,\varsigma)=\mathfrak{S}^*_{s,q}(\vartheta,\varsigma)$ and $\lim\limits_{q\to 1^-}\mathfrak{S}^*\mathfrak{T}_{s,1,q}(\vartheta,\varsigma)=\mathfrak{C}_{s,q}(\vartheta,\varsigma)\quad(0\le\vartheta\le 1, 0<\varsigma\le 1, 0<q<1)$;

(iii) $\mathfrak{S}^*\mathfrak{T}_{s,n_1,q}(\vartheta,\varsigma)\subset\mathfrak{S}^*\mathfrak{T}_{s,n_2,q}(\vartheta,\varsigma)\quad(n_1>n_2\ge 0)$;

(iv) $\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta,\varsigma)\subset\mathfrak{S}^*\mathfrak{T}_{s,n-1,q}(\vartheta,\varsigma)\subset\cdots\subset\mathfrak{C}_{s,q}(\vartheta,\varsigma)\subset\mathfrak{S}^*_{s,q}(\vartheta,\varsigma)\quad(n\in\mathbb{N},\ 0<q<1, 0\le\vartheta\le 1,\ 0<\varsigma\le 1)$.

In this paper, we derive certain results associated with the quasi-Hadamard product of functions in the classes $\mathfrak{S}^*\mathfrak{T}_{s,q}(\vartheta,\varsigma)$, $\mathfrak{C}\mathfrak{T}_{s,q}(\vartheta,\varsigma)$ and $\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta,\varsigma)$, which extend the results obtained by Kumar [11, 12], Darwish [4] and Aouf [2].

2. Main results

Unless otherwise mentioned, we assume throughout this paper that: $n\in\mathbb{N}_0,\ 0\le\vartheta\le 1,\ 0<\varsigma\le 1,\ 0<q<1$ and $z\in\Delta$.

To prove our results, we need the following lemmas.

Lemma 2.1. *A function $h(z)$ be defined by* (1.8) *and $h(z)-h(-z)\neq 0$ for $z\neq 0$. Then $h(z)\in\mathfrak{S}^*\mathfrak{T}_{s,q}(\vartheta,\varsigma)$ if and only if*

$$\sum_{k=2}^{\infty}\left\{(1+\vartheta\varsigma)[k]_q+(\varsigma-1)\left[1-(-1)^k\right]\right\}a_k\le(\varsigma(2+\vartheta)-1)a_1. \tag{2.1}$$

Lemma 2.2. *A function $h(z)$ be defined by* (1.8) *and $h'(z)-h'(-z)\neq 0$ for $z\neq 0$. Then $h(z)\in\mathfrak{C}\mathfrak{T}_{s,q}(\vartheta,\varsigma)$ if and only if*

$$\sum_{k=2}^{\infty}[k]_q\left\{(1+\vartheta\varsigma)[k]_q+(\varsigma-1)\left[1-(-1)^k\right]\right\}a_k\le(\varsigma(2+\vartheta)-1)a_1. \tag{2.2}$$

Lemma 2.3. *A function $h(z)$ be defined by* (1.8) *and $\mathfrak{D}^n h(z)-\mathfrak{D}^n h(-z)\neq 0$ for $z\neq 0$. Then $h(z)\in\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta,\varsigma)$ if and only if*

$$\sum_{k=2}^{\infty}[k]_q^n\left\{(1+\vartheta\varsigma)[k]_q+(\varsigma-1)\left[1-(-1)^k\right]\right\}a_k\le(\varsigma(2+\vartheta)-1)a_1. \tag{2.3}$$

Proof. Let $|z|=1$. Then we have

$$\begin{aligned}&\left|\mathfrak{D}_q^{n+1}h(z)-(\mathfrak{D}_q^n h(z)-\mathfrak{D}_q^n h(-z))\right|-\varsigma\left|\vartheta\mathfrak{D}_q^{n+1}h(z)+(\mathfrak{D}_q^n h(z)-\mathfrak{D}_q^n h(-z))\right|\\&=\left|z+\sum_{k=2}^{\infty}[k]_q^n\left([k]_q-1+(-1)^k\right)a_k z^k\right|-\varsigma\left|(\vartheta+2)z-\sum_{k=2}^{\infty}[k]_q^n\left(\vartheta[k]_q+1-(-1)^k\right)a_k z^k\right|\\&\le\sum_{k=2}^{\infty}[k]_q^n\left\{(1+\vartheta\varsigma)[k]_q+(\varsigma-1)\left[1-(-1)^k\right]\right\}a_k-[\varsigma(\vartheta+2)-1]\le 0.\end{aligned}$$

Hence, by the maximum modules theorem, we have $f \in \mathfrak{S}^{\star}\mathfrak{T}_{s,n,q}(\vartheta, \varsigma)$.

For the converse, assume that

$$\left| \frac{\dfrac{\mathfrak{D}_q^{n+1}h(z)}{\left(\mathfrak{D}_q^n h(z) - \mathfrak{D}_q^n h(-z)\right)} - 1}{\vartheta \dfrac{\mathfrak{D}_q^{n+1}h(z)}{\left(\mathfrak{D}_q^n h(z) - \mathfrak{D}_q^n h(-z)\right)} + 1} \right| = \left| \frac{-z - \sum_{k=2}^{\infty} [k]_q^n \left([k]_q - 1 + (-1)^k\right) a_k z^k}{(\vartheta + 2)z - \sum_{k=2}^{\infty} [k]_q^n \left(\vartheta [k]_q + 1 - (-1)^k\right) a_k z^k} \right| < \varsigma.$$

Since $\left|\Re(z)\right| \leq |z|$ for all z, we have

$$\Re \left\{ \frac{z + \sum_{k=2}^{\infty} [k]_q^n \left([k]_q - 1 + (-1)^k\right) a_k z^k}{(\vartheta + 2)z - \sum_{k=2}^{\infty} [k]_q^n \left(\vartheta [k]_q + 1 - (-1)^k\right) a_k z^k} \right\} < \varsigma. \tag{2.4}$$

Take values of z on the real axis so that $\dfrac{\mathfrak{D}_q^{n+1}h(z)}{\left(\mathfrak{D}_q^n h(z) - \mathfrak{D}_q^n h(-z)\right)}$ is real and $\mathfrak{D}_q^n h(z) - \mathfrak{D}_q^n h(-z) \neq 0$ for $z \neq 0$. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we acquire

$$1 + \sum_{k=2}^{\infty} [k]_q^n \left([k]_q - 1 + (-1)^k\right) a_k \leq \varsigma(\vartheta + 2) - \varsigma \sum_{k=2}^{\infty} [k]_q^n \left(\vartheta [k]_q + 1 - (-1)^k\right) a_k.$$

This gives the required condition. □

Theorem 2.4. *The functions $h_i(z)$ be defined by* (1.9) *be in the class $\mathfrak{C}_{s,s}\mathfrak{T}(\vartheta, \varsigma)$ for every $i = 1, 2, \ldots, u$, and let the functions $g_j(z)$ be defined by* (1.11) *be in the class $\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta, \varsigma)$ for every $j = 1, 2, \ldots, v$. Then quasi-Hadamard product $h_1 * h_2 * \cdots * h_u * g_1 * g_2 * \cdots * g_v$ belongs to the class $\mathfrak{S}^*\mathfrak{T}_{s,2u+(n+1)v-1,q}(\vartheta, \varsigma)$.*

Proof. Let $\mathfrak{G}(z) = h_1 * h_2 * \cdots * h_u * g_1 * g_2 * \cdots * g_v$, then

$$\mathfrak{G}(z) = \left(\prod_{i=1}^{u} a_{1,i} \prod_{j=1}^{v} b_{1,j} \right) z - \sum_{k=2}^{\infty} \left(\prod_{i=1}^{u} a_{k,i} \prod_{j=1}^{v} b_{k,j} \right) z^k. \tag{2.5}$$

It is sufficient to show that

$$\sum_{k=2}^{\infty} [k]_q^{2u+(n+1)v-1} \left\{(1 + \vartheta\varsigma) [k]_q + (\varsigma - 1) \left[1 - (-1)^k\right]\right\} \left(\prod_{i=1}^{u} a_{k,i} \prod_{j=1}^{v} b_{k,j} \right) \leq (\varsigma (2 + \vartheta) - 1) \left(\prod_{i=1}^{u} a_{1,i} \prod_{j=1}^{v} b_{1,j} \right). \tag{2.6}$$

Since $h_i \in \mathfrak{C}\mathfrak{T}_{s,q}(\vartheta, \varsigma)$, by using Lemma 2.2, we have

$$\sum_{k=2}^{\infty} [k]_q \left\{(1 + \vartheta\varsigma) [k]_q + (\varsigma - 1) \left[1 - (-1)^k\right]\right\} a_{k,i} \leq (\varsigma (2 + \vartheta) - 1) a_{1,i}, \quad \forall\, i = 1, 2, \ldots, u. \tag{2.7}$$

Thus,

$$[k]_q \left\{(1 + \vartheta\varsigma) [k]_q + (\varsigma - 1) \left[1 - (-1)^k\right]\right\} a_{k,i} \leq (\varsigma (2 + \vartheta) - 1) a_{1,i}, \quad \forall\, i = 1, 2, \ldots, u.$$

The right-hand expression of the last inequality is not greater than $[k]_q^{-2} a_{1,i}$. Therefore,

$$a_{k,i} \leq [k]_q^{-2} a_{1,i}, \quad \forall\, i = 1, 2, \ldots, u. \tag{2.8}$$

Also, since $g_j \in \mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta, \varsigma)$, by using Lemma 2.3, we obtain

$$\sum_{k=2}^{\infty} [k]_q^n \left\{(1 + \vartheta\varsigma) [k]_q + (\varsigma - 1) \left[1 - (-1)^k\right]\right\} b_{k,j} \leq (\varsigma (2 + \vartheta) - 1) b_{1,j}, \quad \forall\, j = 1, 2, \ldots, v. \tag{2.9}$$

Hence, we obtain

$$b_{k,j} \leq [k]_q^{-(n+1)} b_{1,j}, \ \ \forall\ j = 1, 2, \ldots, v. \tag{2.10}$$

Using (2.8)-(2.10) for $i = 1, 2, \ldots, u$, $j = v$ and $j = 1, 2, \ldots, v-1$, respectively, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[[k]_q^{2u+(n+1)v-1}\left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\}\left(\prod_{i=1}^{u} a_{k,i}\prod_{j=1}^{v} b_{k,j}\right)\right] \\
\leq\ &\sum_{k=2}^{\infty}\left[[k]_q^{2u+(n+1)v-1}\,[k]_q^{-2u}\,[k]_q^{-(n+1)(v-1)}\left(\prod_{i=1}^{u} a_{1,i}\prod_{j=1}^{v-1} b_{1,j}\right)\right. \\
&\left.\times\ \left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} b_{k,v}\right] \\
\leq\ &\left(\prod_{i=1}^{u} a_{1,i}\prod_{j=1}^{v-1} b_{1,j}\right)\times\sum_{k=2}^{\infty}[k]_q^n\left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} b_{k,v} \\
\leq\ &(\varsigma(2+\vartheta)-1)\left(\prod_{i=1}^{u} a_{1,i}\prod_{j=1}^{v} b_{1,j}\right).
\end{aligned} \tag{2.11}$$

Thus, we have $G(z) \in \mathfrak{S}^*\mathfrak{T}_{s,2u+(n+1)v-1,q}(\vartheta,\varsigma)$. This completes the proof of Theorem 2.4. □

Putting $n = 1$ in Theorem 2.4, we obtain the following corollary:

Corollary 2.5. *Let the functions $h_i(z)$ be defined by* (1.9) *and the functions $g_j(z)$ be defined by* (1.11) *belong to the class $\mathfrak{C}\mathfrak{T}_{s,q}(\vartheta,\varsigma)$ for every $i = 1, 2, \ldots, u$ and $j = 1, 2, \ldots, v$. Then quasi-Hadamard product $h_1 * h_2 * \cdots * h_u * g_1 * g_2 * \cdots * g_v$ belongs to the class $\mathfrak{S}^*\mathfrak{T}_{s,2u+2v-1,q}(\vartheta,\varsigma)$.*

Theorem 2.6. *Let the functions $h_i(z)$ be defined by* (1.9) *be in the class $\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta,\varsigma)$ for every $i = 1, 2, \ldots, u$, and let the functions $g_j(z)$ be defined by* (1.11) *be in the class $\mathfrak{S}^*\mathfrak{T}_{s,q}(\vartheta,\varsigma)$ for every $j = 1, 2, \ldots, v$. Then quasi-Hadamard product $h_1 * h_2 * \cdots * h_u * g_1 * g_2 * \cdots * g_v$ belongs to the class $\mathfrak{S}^*\mathfrak{T}_{s,(n+1)u+v-1,q}(\vartheta,\varsigma)$.*

Proof. Let $\mathfrak{G}(z)$ is defined by (2.5). It is sufficient to show that

$$\sum_{k=2}^{\infty}[k]_q^{(n+1)u+v-1}\left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\}\left(\prod_{i=1}^{u} a_{k,i}\prod_{j=1}^{v} b_{k,j}\right) \leq (\varsigma(2+\vartheta)-1)\left(\prod_{i=1}^{u} a_{1,i}\prod_{j=1}^{v} b_{1,j}\right). \tag{2.12}$$

Since $h_i \in \mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta,\varsigma)$, by using Lemma 2.3, we have

$$\sum_{k=2}^{\infty}[k]_q^n\left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} a_{k,i} \leq (\varsigma(2+\vartheta)-1)\, a_{1,i}, \quad \forall\ i = 1, 2, \ldots, u. \tag{2.13}$$

Hence, we get

$$a_{k,i} \leq [k]_q^{-(n+1)} a_{1,i}, \quad \forall\ i = 1, 2, \ldots, u. \tag{2.14}$$

Furthermore, since $g_j \in \mathfrak{S}^*\mathfrak{T}_{s,q}(\vartheta,\varsigma)$, by using Lemma 2.1, we have

$$\sum_{k=2}^{\infty}\left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} b_{k,j} \leq (\varsigma(2+\vartheta)-1)\, b_{1,j}, \ \ \forall\ j = 1, 2, \ldots, v. \tag{2.15}$$

Hence, we obtain

$$b_{k,j} \leq k^{-1} b_{1,j}, \quad \forall\ j = 1, 2, \ldots, q. \tag{2.16}$$

Using (2.14)-(2.16) for $i = u,\ i = 1, 2, \ldots, u-1$ and $j = 1, 2, \ldots, v$, respectively, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} [k]_q^{(n+1)u+v-1} \left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} \prod_{i=1}^{u} a_{k,i} \prod_{j=1}^{v} b_{k,j} \\
\leq\ & \sum_{k=2}^{\infty} [k]_q^{(n+1)u+v-1} [k]_q^{-(n+1)(u-1)} [k]_q^{-v} \left(\prod_{i=1}^{u-1} a_{1,i} \prod_{j=1}^{v} b_{1,j}\right) \\
& \times \left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} a_{k,m} \\
\leq\ & \left(\prod_{i=1}^{u-1} a_{1,i} \prod_{j=1}^{v} b_{1,j}\right) \times \sum_{k=2}^{\infty} [k]_q^{n} \left\{(1+\vartheta\varsigma)[k]_q + (\varsigma-1)\left[1-(-1)^k\right]\right\} a_{k,u} \\
\leq\ & (\varsigma(2+\vartheta)-1)\left(\prod_{i=1}^{u} a_{1,i} \prod_{j=1}^{v} b_{1,j}\right). \qquad (2.17)
\end{aligned}$$

Thus, we have $\mathfrak{G}(z) \in \mathfrak{S}^*\mathfrak{T}_{s,(n+1)u+v-1}(\vartheta, \varsigma)$. This completes the proof of Theorem 2.6. □

Putting $n = 0$ in Theorem 2.6, we obtain the following corollary:

Corollary 2.7. *Let the functions $h_i(z)$ be defined by* (1.9) *and the functions $g_j(z)$ be defined by* (1.11) *belong to the class $\mathfrak{S}^*\mathfrak{T}_{s,q}(\vartheta, \varsigma)$ for every $i = 1, 2, \ldots, u$ and $j = 1, 2, \ldots, v$. Then quasi-Hadamard product $h_1 * h_2 * \cdots * h_u * g_1 * g_2 * \cdots * g_v$ belongs to the class $\mathfrak{S}^*\mathfrak{T}_{s,u+v-1,q}(\vartheta, \varsigma)$.*

Putting $n = 0$ in Theorem 2.4, or $n = 1$ in Theorem 2.6, we obtain the following corollary:

Corollary 2.8. *Let the functions $h_i(z)$ be defined by* (1.9) *be in the class $\mathfrak{C}\mathfrak{T}_{s,q}(\vartheta, \varsigma)$ for every $i = 1, 2, \ldots, u$, and let the functions $g_j(z)$ be defined by* (1.11) *be in the class $\mathfrak{S}^*\mathfrak{T}_{s,q}(\vartheta, \varsigma)$ for every $j = 1, 2, \ldots, v$. Then quasi-Hadamard product $h_1 * h_2 * \cdots * h_u * g_1 * g_2 * \cdots * g_v$ belongs to the class $\mathfrak{S}^*\mathfrak{T}_{s,2u+v-1}(\vartheta, \varsigma)$.*

3. Conclusions

In this paper, we used the conception of q-calculus operator $\mathfrak{D}_q^n$, and we demarcated and explored a new subclass $\mathfrak{S}^*\mathfrak{T}_{s,n,q}(\vartheta, \varsigma)$ of analytic functions in the open unit disk Δ. We also consequent several results of quasi-Hadamard product of recently defined subclass of analytic functions with respect to symmetric points. We have emphasized some significances of our main results as corollaries.

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