Montes Taurus J. Pure Appl. Math. 4 (3), 93–102, 2022

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

# Some Norm Inequalities for Fractional Integral Operators

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**Abstract**

In this paper, we introduce some new fractional integral operators and fractional area balance operators in the Banach spaces. The corresponding norm inequalities are established. They are significant improvement and generalizations of many known and new classes of fractional integral operators.

*Keywords:* Norm inequality, Fractional integral operator, Fractional area balance operator

*2010 MSC:* 47A30, 26A33

## 1. Introduction

It is well-known that fractional integral operator is one of the important operators in harmonic analysis with background of partial differential equations. In fact, the solution of the Laplace equation $\triangle g = f$ for good functions on $\mathbb{R}^n$ can be represented by using the fractional integral operators acting on $f$. In the recent past, different versions of fractional integral operators have been developed which are useful in the study of different classes of differential and integral equations. These fractional integral operators act as ready tools to study the classes of differential and integral equations. Fractional integral operators were utilized extensively to study the classical inequalities. A generalization of classical inequalities by means of fractional-order integral operators is considered as an interesting subject area. Many mathematician have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations. Hence, fractional integral inequalities are very important in the theory and applications of differential equations. Such inequalities are also of great importance in the mathematical modeling of the fractional boundary value problem (see [1]-[9], [11]-[13], [16]-[18]). First, we recall the following definitions and some related results.

**Definition 1.1** (*cf.* [1, 2, 6])**.** Let $f \in L[a, b]$, then Riemann-Liouville fractional integrals of $f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$T_1(f, x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \;\; x > a, \tag{1.1}$$

and

$$T_2(f, x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \;\; x < b, \tag{1.2}$$

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Received:10 January 2021, Accepted:9 September 2021, Published:7 November 2021
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respectively, where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{1.3}$$

is the Gamma function and when $\alpha = 0$, $T_1(f, x) = T_2(f, x) = f(x)$.

**Definition 1.2** (*cf.* [3])**.** Let $f \in L[a, b]$, then Riemann-Liouville $k$-fractional integrals of $f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$T_3(f, x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t) dt, \ x > a, \tag{1.4}$$

and

$$T_4(f, x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{(\alpha/k)-1} f(t) dt, \ x < b, \tag{1.5}$$

respectively, where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-(t^k/k)} dt, \ \alpha > 0, \tag{1.6}$$

is the $k$-Gamma function. Also, $\Gamma(x) = \lim_{k\to 1} \Gamma_k(x)$, $\Gamma_k(\alpha) = k^{(\alpha/k)-1}\Gamma(\alpha/k)$ and $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$.

It is well known that the Mellin transform of $f$ is defined by

$$M(f, \alpha) = \int_0^\infty t^{\alpha-1} f(t) dt, \ (\alpha > 0).$$

Therefore, the Mellin transform of the exponential function $e^{-(t^k/k)}$ is the $k$-Gamma function.

**Definition 1.3** (*cf.* [4, 5])**.** Let $t^r f(t) \in L^1[a, b]$, $a \geq 0$, then the generalized Riemann-Liouville fractional integral of $f$ of order $\alpha \geq 0$ and $r \geq 0$ are defined by

$$T_5(f, x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t) dt, \ x > a, \tag{1.7}$$

$$T_6(f, x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{\alpha-1} t^r f(t) dt, \ x < b, \tag{1.8}$$

and there are also $(k, r)$ fractional of Riemann type operators:

$$T_7(f, x) = \frac{(r+1)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{(\alpha/k)-1} t^r f(t) dt, \ x > a, \tag{1.9}$$

$$T_8(f, x) = \frac{(r+1)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{(\alpha/k)-1} t^r f(t) dt, \ x < b, \tag{1.10}$$

respectively, where $k, \alpha > 0$, $r \geq 0$, $x \in [a, b]$.

In particular, if $r = 0$, then Definition 1.3 reduces to Definition 1.1 and Definition 1.2.

**Definition 1.4** (*cf.* [11, 12])**.** Let $f$ be a conformable integrable function on $[a, b] \subset [0, \infty)$. The right-sided and left-sided generalized conformable fractional integrals $T_9$ and $T_{10}$ of $f$ of order $\alpha > 0$ are defined by

$$T_9(f, x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{x^{r+s} - t^{r+s}}{r+s}\right)^{\alpha-1} t^{r+s-1} f(t) dt, \ x > a, \tag{1.11}$$

and

$$T_{10}(f, x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\frac{t^{r+s} - x^{r+s}}{r+s}\right)^{\alpha-1} t^{r+s-1} f(t) dt, \ \ x < b, \tag{1.12}$$

respectively, where $r \in \mathbb{R}^1$, $s \in (0, 1]$, $r + s \neq 0$.

In particular, if $s = 1$, then $T_9, T_{10}$ reduce to $T_5, T_6$, respectively.

**Definition 1.5** (*cf.* [8, 9])**.** Let $f \in L[a,b]$, $g : [a,b] \to (0,\infty)$ be an increasing function, and $g' \in C[a,b], \alpha > 0$. Then $g-$ Riemann-Liouville fractional integrals of $f$ with respect to the function $g$ on $[a,b]$ are defined by

$$T_{11}(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x g'(t)[g(x) - g(t)]^{\alpha-1} f(t)dt, \;\; x > a, \tag{1.13}$$

and

$$T_{12}(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b g'(t)[g(t) - g(x)]^{\alpha-1} f(t)dt, \;\; x < b, \tag{1.14}$$

respectively.

In 2018, S. S. Dragomir [10] introduced the new notion of the area balance function:

**Definition 1.6** (*cf.* [10])**.** Let $f \in L[a,b]$, then the area balance function of $f$ is defined by

$$T_{13}(f,x) = \frac{1}{2}\left\{\int_x^b f(t)dt - \int_a^x f(t)dt\right\}, \;\; x \in [a,b]. \tag{1.15}$$

In the following, let $BV[a,b]$ and $AC[a,b]$ denote the class of bounded variation and absolutely continuous functions on $[a,b]$, respectively (see [10]). Dragomir [10] also proved some inequalities for the area balance function $T_{13}(f,x)$ is defined by (1.15):

Let $f \in AC[a,b]$. If $f \in BV[a,b]$, then

$$|T_{13}(f,x) - (\frac{a+b}{2} - x)f(x) - \frac{f'(a) + f'(b)}{4}[(x - \frac{a+b}{2})^2 + \frac{1}{4}(b-a)^2]| \le \frac{1}{4}[\frac{1}{4}(b-a)^2 + (x - \frac{a+b}{2})^2]V_a^b(f'), x \in [a,b];$$

If there exists the real numbers $m, M$ such that

$$m \le f'(x) \le M, \text{ for } a.e. x \in [a,b]$$

then also

$$|T_{13}(f,x) - (\frac{a+b}{2} - x)f(x) - \frac{m+M}{4}[(x - \frac{a+b}{2})^2 + \frac{1}{4}(b-a)^2]| \le \frac{1}{4}[\frac{1}{4}(b-a)^2 + (x - \frac{a+b}{2})^2](M-m),$$

for any $x \in [a,b]$. In 2020, Kuang [16] introduced the new notion of the generalized fractional integral operators and fractional area balance operators:

**Definition 1.7** (*cf.* [16])**.** Let $f \in L[a,b]$, $g : [a,b] \to (0,\infty)$ be an increasing function, and $g \in AC[a,b], k, c, \alpha > 0$, $a \ge 0$. Then the generalized fractional integral operator $T_{14}$ with respect to the function $g$ on $[a,b]$ is defined by

$$T_{14}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_a^b g'(t)|g(x) - g(t)|^{(\alpha/k)-1} f(t)dt, \tag{1.16}$$

where $\Gamma_k(\alpha)$ is defined by (1.6).

Let

$$T_{15}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_a^x g'(t)[g(x) - g(t)]^{(\alpha/k)-1} f(t)dt, \;\; x > a, \tag{1.17}$$

and

$$T_{16}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_x^b g'(t)[g(t) - g(x)]^{(\alpha/k)-1} f(t)dt, \;\; x < b. \tag{1.18}$$

Then

$$T_{14}(f,x) = T_{15}(f,x) + T_{16}(f,x). \tag{1.19}$$

Just indicate that for suitable and appropriate choice of the parameters and function, one can obtain various new and old results. For example:
If $c = k = 1$ in (1.19), then (1.19) reduces to

$$T_{14}(f, x) = T_{11}(f, x) + T_{12}(f, x). \tag{1.20}$$

If $c = (r + s)^{-\alpha},\ g(t) = t^{r+s},\ r \in \mathbb{R}^1,\ s \in (0, 1], r + s \neq 0,\ k = 1$ in (1.19), then (1.19) reduces to

$$T_{14}(f, x) = T_9(f, x) + T_{10}(f, x). \tag{1.21}$$

If $c = (r + 1)^{-(\alpha/k)},\ g(t) = t^{r+1},\ r \geq 0$, in (1.19), then (1.19) reduces to

$$T_{14}(f, x) = T_7(f, x) + T_8(f, x). \tag{1.22}$$

If $s = 1$ in (1.21), then (1.21) reduces to

$$T_{14}(f, x) = T_5(f, x) + T_6(f, x). \tag{1.23}$$

If $r = 0$ in (1.22), then (1.22) reduces to

$$T_{14}(f, x) = T_3(f, x) + T_4(f, x). \tag{1.24}$$

If $k = 1$ in (1.24), then (1.24) reduces to

$$T_{14}(f, x) = T_1(f, x) + T_2(f, x). \tag{1.25}$$

We can also rewrite $T_9$ and $T_{10}$ as

$$T_9(f, x) = \frac{(r + s)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+s} - t^{r+s})^{\alpha-1} t^{r+s-1} f(t)dt,\ \ x > a,$$

and

$$T_{10}(f, x) = \frac{(r + s)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{r+s} - x^{r+s})^{\alpha-1} t^{r+s-1} f(t)dt,\ \ x < b,$$

and then generalize them to

$$T_{17}(f, x) = \frac{(r + s)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_a^x (x^{r+s} - t^{r+s})^{(\alpha/k)-1} t^{r+s-1} f(t)dt,\ \ x > a, \tag{1.26}$$

and

$$T_{18}(f, x) = \frac{(r + s)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_x^b (t^{r+s} - x^{r+s})^{(\alpha/k)-1} t^{r+s-1} f(t)dt,\ \ x < b. \tag{1.27}$$

If $c = (r + s)^{-(\alpha/k)},\ g(t) = t^{r+s},\ r \in \mathbb{R}^1,\ s \in (0, 1],\ r + s \neq 0$ in (1.19), then (1.19) reduces to

$$T_{14}(f, x) = T_{17}(f, x) + T_{18}(f, x). \tag{1.28}$$

**Definition 1.8** (*cf.* [2])**.** Let $f \in L[a, b],\ a \geq 0$. The left-sided and right-sided Hadamard fractional integrals $T_{19}$ and $T_{20}$ of $f$ of order $\alpha > 0$ are defined by

$$T_{19}(f, x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\log x - \log t)^{\alpha-1} t^{-1} f(t)dt,\ \ x > a,$$

and

$$T_{20}(f, x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\log t - \log x)^{\alpha-1} t^{-1} f(t)dt,\ \ x < b,$$

respectively.

We can generalize them to

$$T_{21}(f,x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(\log x-\log t)^{(\alpha/k)-1}t^{-1}f(t)dt,\ \ x>a,$$

and

$$T_{22}(f,x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(\log t-\log x)^{(\alpha/k)-1}t^{-1}f(t)dt,\ \ x<b.$$

If $c=1,\ g(t)=\log t$ in (1.19), then (1.19) reduces to

$$T_{14}(f,x)=T_{21}(f,x)+T_{22}(f,x).$$

In particular, if $k=1$, then

$$T_{14}(f,x)=T_{19}(f,x)+T_{20}(f,x).$$

**Definition 1.9** (*cf.* [16])**.** Under the assumptions of Definition 1.7, the fractional area balance operators $T_{24}$ with respect to the function $g$ on $[a,b]$ is defined by

$$T_{24}(f,x)=\frac{c}{k\Gamma_k(\alpha)}\left\{\int_x^b g'(t)[g(t)-g(x)]^{(\alpha/k)-1}f(t)dt-\int_a^x g'(t)[g(x)-g(t)]^{(\alpha/k)-1}f(t)dt\right\}, \tag{1.29}$$

where $\Gamma_k(\alpha)$ is defined by (1.6).

Using (1.17) and (1.18), we have

$$T_{24}(f,x)=T_{16}(f,x)-T_{15}(f,x). \tag{1.30}$$

Just indicate that for suitable and appropriate choice of the parameters and function, one can obtain various new and old results. For example:
If $c=(r+s)^{-(\alpha/k)}, g(t)=t^{r+s},\ r\in\mathbb{R}^1,\ s\in(0,1],\ r+s\neq 0$ in (1.30), then (1.30) reduces to

$$T_{24}(f,x)=T_{18}(f,x)-T_{17}(f,x). \tag{1.31}$$

If $g(t)=t,\ \alpha=k=1,\ c=1/2$ in (1.30), then $T_{24}$ reduces to $T_{13}$. If $c=k=1$ in (1.30), then (1.30) reduces to

$$T_{24}(f,x)=T_{12}(f,x)-T_{11}(f,x). \tag{1.32}$$

If $c=(r+s)^{-\alpha},\ g(t)=t^{r+s},\ r\in\mathbb{R}^1,\ s\in(0,1], r+s\neq 0,\ k=1$ in (1.30), then (1.30) reduces to

$$T_{24}(f,x)=T_{10}(f,x)-T_{9}(f,x). \tag{1.33}$$

If $c=(r+1)^{-(\alpha/k)},\ g(t)=t^{r+1},\ r\geq 0$ in (1.30), then (1.30) reduces to

$$T_{24}(f,x)=T_{8}(f,x)-T_{7}(f,x). \tag{1.34}$$

If $k=1$ in (1.34), then (1.34) reduces to

$$T_{24}(f,x)=T_{6}(f,x)-T_{5}(f,x). \tag{1.35}$$

If $r=0$ in (1.34), then (1.34) reduces to

$$T_{24}(f,x)=T_{4}(f,x)-T_{3}(f,x). \tag{1.36}$$

If $k=1$ in (1.36), then (1.36) reduces to

$$T_{24}(f,x)=T_{2}(f,x)-T_{1}(f,x). \tag{1.37}$$

If $c=1, g(t)=\log t$ in (1.30), then (1.30) reduces to

$$T_{24}(f,x)=T_{22}(f,x)-T_{21}(f,x).$$

In particular, if $k = 1$, then

$$T_{24}(f, x) = T_{20}(f, x) - T_{19}(f, x).$$

Hence, Definition 1.7 and Definition 1.9 unified and generalized many known and new classes of fractional integral operators. It is noted that all the fractional integral operators above are established for functions of one variable. We naturally ask, how do we generalize these results to functions of several variable? In 2014, Sarikaya [17] gives the definitions Riemann-Liouville fractional integrals of two variable functions:

**Definition 1.10** (*cf.* [17, 18, 19])**.** Let $Q = [a, b]\times[c, d] = \bigcup_{k=1}^{4} Q_k$, where $Q_1 = [a, x]\times[c, y],\ Q_2 = [a, x]\times[y, d],\ Q_3 = [x, b] \times [c, y],\ Q_4 = [x, b] \times [y, d]$. The Riemann-Liouville fractional integrals $I_k (1 \leq k \leq 4)$ are defined by

$$I_1(f; x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{Q_1} (x - t)^{\alpha-1}(y - s)^{\beta-1} f(t, s)dsdt,$$

$$I_2(f; x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{Q_2} (x - t)^{\alpha-1}(s - y)^{\beta-1} f(t, s)dsdt,$$

$$I_3(f; x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{Q_3} (t - x)^{\alpha-1}(y - s)^{\beta-1} f(t, s)dsdt,$$

and

$$I_4(f; x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{Q_4} (t - x)^{\alpha-1}(s - y)^{\beta-1} f(t, s)dsdt.$$

If the $Q$ in the above Definition 1.10 is applied to the $n-$dimensional interval $Q = \bigcup_{k=1}^{n} Q_k$ in $\mathbb{R}^n_+$, where $Q_k = [a_k, b_k]$, then the problem becomes very complicated when $n$ is large. The aim of this paper is to introduce some new generalized fractional integral operators and fractional area balance operators on the Banach spaces, which includes $\mathbb{R}^n_+$ as the special case. In Section 2, we define generalized fractional integral operators and fractional area balance operators on the Banach space. In Section 3, some Lemmas are derived. The corresponding norm inequalities are established in Section 4. They are significant improvement and generalizations of many known and new classes of fractional integral operators.

## 2. Generalized fractional integral operators and fractional area balance operators

Throughout this paper, we write

$$E_n = \left\{ x = (x_1, x_2, \cdots, x_n) : x_k \geq 0, 1 \leq k \leq n,\ \|x\| = \left( \sum_{k=1}^{n} |x_k|^r \right)^{1/r},\ r > 0 \right\}.$$

When $1 \leq r < \infty$, $E_n$ is a Banach space. In particular, when $r = 2$, $E_n$ is a $n$-dimensional Euclidean space $\mathbb{R}^n_+$. When $r = 1$, $\|x\| = \sum_{k=1}^{n} |x_k|$ is a Cartesian norm.

Let

$$D = \{x = (x_1, x_2, \cdots, x_n) : x_k \geq 0,\ 1 \leq k \leq n, 0 \leq a < \|x\| < b\},\ 0 < \mu(D) < \infty,$$

where $\mu$ is the Lebesgue measure. Let

$$D_1 = \{y = (y_1, y_2, \cdots, y_n) : y_k \geq 0,\ 1 \leq k \leq n,\ 0 \leq a < \|y\| < \|x\|,\ x \in D\},$$

$$D_2 = \{y = (y_1, y_2, \cdots, y_n) : y_k \geq 0,\ 1 \leq k \leq n,\ \|x\| < \|y\| < b,\ x \in D\}.$$

The norm of operator $T : L^\infty(D) \to L^\infty(D)$ is defined by

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty}.$$

**Definition 2.1.** Let $f \in L(D)$, $g : [a, b] \to (0, \infty)$ be an increasing function, and $g \in AC[a, b]$, $k$, $c, \alpha > 0$, $a \geq 0$. Then the generalized fractional integrals operator $T_{25}$ with respect to the function $g$ on $D$ is defined by

$$T_{25}(f, x) = \frac{c}{k\Gamma_k(\alpha)} \int_D g^{'}(\|y\|)|g(\|x\|) - g(\|y\|)|^{(\alpha/k)-1} f(y)dy, \tag{2.1}$$

where $\Gamma_k(\alpha)$ is defined by (1.6).

Let

$$T_{26}(f, x) = \frac{c}{k\Gamma_k(\alpha)} \int_{D_1} g^{'}(\|y\|)[g(\|x\|) - g(\|y\|)]^{(\alpha/k)-1} f(y)dy, \tag{2.2}$$

and

$$T_{27}(f, x) = \frac{c}{k\Gamma_k(\alpha)} \int_{D_2} g^{'}(\|y\|)[g(\|y\|) - g(\|x\|)]^{(\alpha/k)-1} f(y)dy. \tag{2.3}$$

Thus,

$$T_{25}(f, x) = T_{26}(f, x) + T_{27}(f, x). \tag{2.4}$$

The fractional area balance operator with respect to the function $g$ on $D$ is defined by

$$T_{28}(f, x) = T_{27}(f, x) - T_{26}(f, x). \tag{2.5}$$

In particular, if $n = r = 1$ in Definition 2.1, then Definition 2.1 reduces to Definition 1.7 and Definition 1.9.

## 3. Some Lemmas

We require the following Lemmas to prove our main results.

**Lemma 3.1** (*cf.* [15]). *If $a_k, b_k, p_k > 0, 1 \leq k \leq n$, $f$ be a measurable function on $(0, \infty)$, then*

$$\int_{B(r_1,r_2)} f\left(\sum_{k=1}^{n} (\frac{x_k}{a_k})^{b_k}\right) x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n = \frac{\prod_{k=1}^{n} a_k^{p_k}}{\prod_{k=1}^{n} b_k} \cdot \frac{\prod_{k=1}^{n} \Gamma(\frac{p_k}{b_k})}{\Gamma(\sum_{k=1}^{n} \frac{p_k}{b_k})} \int_{r_1}^{r_2} f(t) t^{(\sum_{k=1}^{n} \frac{p_k}{b_k} - 1)} dt,$$

*where $B(r_1, r_2) = \{x \in E_n : 0 \leq r_1 < \|x\| < r_2\}$.*

We get the following Lemma 3.2 by taking $a_k = 1, b_k = r > 0, p_k = 1$, $1 \leq k \leq n, r_1 = a, r_2 = b$ in Lemma 3.1.

**Lemma 3.2.** *Let $f$ be a measurable function on $(0, \infty)$, then*

$$\int_D f(\|x\|)dx = \frac{\Gamma^n(1/r)}{r^n\Gamma(n/r)} \int_a^b f(t^{1/r}) t^{(n/r)-1} dt. \tag{3.1}$$

Let

$$G_1(g^{'}, x) = \frac{c}{k\Gamma_k(\alpha)} \int_D g^{'}(\|y\|)|g(\|x\|) - g(\|y\|)|^{(\alpha/k)-1} dy, \tag{3.2}$$

$$G_2(g^{'}, x) = \frac{c}{k\Gamma(\alpha)} \int_{D_1} g^{'}(\|y\|)[g(\|x\|) - g(\|y\|)]^{(\alpha/k)-1} dy, \tag{3.3}$$

and

$$G_3(g^{'}, x) = \frac{c}{k\Gamma_k(\alpha)} \int_{D_2} g^{'}(\|y\|)[g(\|y\|) - g(\|x\|)]^{(\alpha/k)-1} dy. \tag{3.4}$$

Thus,

$$G_1(g^{'}, x) = G_2(g^{'}, x) + G_3(g^{'}, x). \tag{3.5}$$

Let

$$G_4(g^{'}, x) = G_3(g^{'}, x) - G_2(g^{'}, x), \tag{3.6}$$

$$G_5(t) = \frac{c}{k\Gamma_k(\alpha)} \times \frac{\Gamma^n(1/r)}{r^n\Gamma(n/r)} \int_a^t [g(\|x\|) - g(u^{1/r})]^{(\alpha/k)-1} u^{(n/r)-1} du, \tag{3.7}$$

$$G_6(t) = \frac{c}{k\Gamma_k(\alpha)} \times \frac{\Gamma^n(1/r)}{r^n\Gamma(n/r)} \int_t^b [g(u^{1/r}) - g(\|x\|)]^{(\alpha/k)-1} u^{(n/r)-1} du. \tag{3.8}$$

**Lemma 3.3.** *Let $[a,b] \subset [0,\infty)$, $g^{'} : [a,b] \to [0,\infty)$ be an increasing function, and $g^{'} \in AC[a,b]$, $k, \ \alpha, \ c > 0$, then*

$$G_1(g^{'},x) = [G_2(1,x) + G_3(1,x)]g^{'}(\|x\|^{1/r}) + r(c_1(\|x\|) - c_2(\|x\|)), \tag{3.9}$$

*and*

$$G_4(g^{'},x) = [G_3(1,x) - G_2(1,x)]g^{'}(\|x\|^{1/r}) + r(c_1(\|x\|) + c_2(\|x\|), \tag{3.10}$$

*where*

$$c_1(\|x\|) = \int_{\|x\|}^{b} G_6(t)t^{1-(1/r)}g^{''}(t^{1/r})dt, \tag{3.11}$$

$$c_2(\|x\|) = \int_{a}^{\|x\|} G_5(t)t^{1-(1/r)}g^{''}(t^{1/r})dt. \tag{3.12}$$

*Proof.* From (3.3), (3.4) and Lemma 3.2, we have

$$G_2(g^{'},x) = \frac{c}{k\Gamma_k(\alpha)} \times \frac{\Gamma^n(1/r)}{r^n\Gamma(n/r)} \int_a^{\|x\|} g^{'}(t^{1/r})[g(\|x\|) - g(t^{1/r})]^{(\alpha/k)-1}t^{(n/r)-1}dt, \tag{3.13}$$

and

$$G_3(g^{'},x) = \frac{c}{k\Gamma_k(\alpha)} \times \frac{\Gamma^n(1/r)}{r^n\Gamma(n/r)} \int_{\|x\|}^{b} g^{'}(t^{1/r})[g(t^{1/r}) - g(\|x\|)]^{(\alpha/k)-1}t^{(n/r)-1}dt. \tag{3.14}$$

Thus,

$$G_2(1,x) = G_5(\|x\|), G_3(1,x) = G_6(\|x\|). \tag{3.15}$$

Then making use of integration by parts, we get

$$\begin{aligned}
\int_a^{\|x\|} G_5(t)(rt^{1-(1/r)})g^{''}(t^{1/r})dt &= \int_a^{\|x\|} G_5(t)dg^{'}(t^{1/r}) = G_5(t)g^{'}(t^{1/r})|_a^{\|x\|} - \int_a^{\|x\|} G_5^{'}(t)g^{'}(t^{1/r})dt = G_5(\|x\|)g^{'}(\|x\|^{1/r}) \\
&- \frac{c}{k\Gamma_k(\alpha)} \times \frac{\Gamma^n(1/r)}{r^n\Gamma(n/r)} \int_a^{\|x\|} g^{'}(t^{1/r})[g(\|x\|) - g(t^{1/r})]^{(\alpha/k)-1}t^{(n/r)-1}dt \\
&= G_2(1,x)g^{'}(\|x\|^{1/r}) - G_2(g^{'},x),
\end{aligned}$$

which leads to

$$G_2(g^{'},x) = G_2(1,x)g^{'}(\|x\|^{1/r}) - r\int_a^{\|x\|} G_5(t)g^{''}(t^{1/r})t^{1-(1/r)}dt. \tag{3.16}$$

Similarly, we have

$$G_3(g^{'},x) = G_3(1,x)g^{'}(\|x\|^{1/r}) + r\int_{\|x\|}^{b} G_6(t)g^{''}(t^{1/r})t^{1-(1/r)}dt. \tag{3.17}$$

Hence, (3.9)follows from (3.5), (3.16) and (3.17), as well as (3.10) follows from (3.6), (3.16) and (3.17). The proof is completed. □

## 4. Some norm inequalities for operator $T_{25}$ and $T_{28}$

**Theorem 4.1.** *Under the assumptions of Lemma 4, let $f \in L^\infty(D)$, then*

$$\begin{aligned}
&\frac{1}{\mu(D)}\left\{r(c_1(\|x\|) - c_2(\|x\|)) + [G_2(1,x) + G_3(1,x)]g^{'}(\|x\|^{1/r})\right\} \\
\leq \ & \|T_{25}\| \leq r(c_1(\|x\|) + c_2(\|x\|)) + \{G_2(1,x) + G_3(1,x)\}g^{'}(\|x\|^{1/r}),
\end{aligned} \tag{4.1}$$

*where $c_1(\|x\|)$ and $c_2(\|x\|)$ are defined by* (3.11) *and* (3.12)*, respectively.*

In particular, if $\|x\| = a$, then

$$c_1(a) = \int_a^b G_6(t) g''(t^{1/r}) t^{1-(1/r)} dt, c_2(a) = 0. \tag{4.2}$$

Thus,

$$\frac{1}{\mu(D)}\{rc_1(a) + G_3(1, a)g'(a^{1/r})\} \le \|T_{25}\| \le rc_1(a) + G_3(1, a)g'(a^{1/r}).$$

If $0 < \mu(D) \le 1$, then

$$\|T_{25}\| = rc_1(a) + G_3(1, a)g'(a^{1/r}). \tag{4.3}$$

*Proof.* From (2.1), (3.2), (3.5), (3.9) and Lemma 3.3, we have

$$\|T_{25} f\|_\infty \le \|f\|_\infty G_1(g', x) = \|f\|_\infty \left\{r(c_1(\|x\|) - c_2(\|x\|)) + [G_2(1, x) + G_3(1, x)]g'(\|x\|^{1/r})\right\}.$$

This implies that

$$\|T_{25}\| \le r(c_1(\|x\|) + c_2(\|x\|)) + \{G_2(1, x) + G_3(1, x)\} g'(\|x\|^{1/r}).$$

To prove the opposite inequality, we take $f_0(x) = \frac{1}{\mu(D)}$, thus we get $\|f_0\|_\infty = 1$, and

$$\|T_{25}\| \ge \|T_{25} f_0\|_\infty \ge \frac{1}{\mu(D)} \left\{r(c_1(\|x\|) - c_2(\|x\|)) + [G_2(1, x) + G_3(1, x)]g'(\|x\|^{1/r})\right\}.$$

The proof is completed. □

**Theorem 4.2.** *Under the assumptions of* Lemma 3.3*, let $f \in L^\infty(D)$, then*

$$\begin{aligned} &\frac{1}{\mu(D)} \left\{r(c_1(\|x\|) + c_2(\|x\|)) + [G_3(1, x) - G_2(1, x)]g'(\|x\|^{1/r})\right\} \\ \le\ & \|T_{28}\| \le r(c_1(\|x\|) + c_2(\|x\|)) + \{G_2(1, x) + G_3(1, x)\} g'(\|x\|^{1/r}), \end{aligned} \tag{4.4}$$

*where $c_1(\|x\|)$ and $c_2(\|x\|)$ are defined by* (3.11) *and* (3.12)*, respectively.*

In particular, if $\|x\| = a$, then

$$\frac{1}{\mu(D)}\{rc_1(a) + G_3(1, a)g'(a^{1/r})\} \le \|T_{28}\| \le rc_1(a) + G_3(1, a)g'(a^{1/r}).$$

If $0 < \mu(D) \le 1$, then

$$\|T_{28}\| = rc_1(a) + G_3(1, a)g'(a^{1/r}), \tag{4.5}$$

where $c_1(a)$ is defined by (4.2).

*Proof.* From (2.5), (3.6), (3.10) and Lemma 3.3, we have

$$\|T_{28} f\|_\infty \le \|f\|_\infty G_4(g', x) = \|f\|_\infty \{r(c_1(\|x\|) + c_2(\|x\|)) + [G_3(1, x) - G_2(1, x)]g'(\|x\|^{1/r})\}.$$

This implies that

$$\|T_{28}\| \le r(c_1(\|x\|) + c_2(\|x\|)) + \{G_3(1, x) - G_2(1, x)\}g'(\|x\|^{1/r}).$$

To prove the opposite inequality, we take $f_0(x) = \frac{1}{\mu(D)}$, thus we get $\|f_0\|_\infty = 1$, and

$$\|T_{28}\| \ge \|T_{28} f_0\|_\infty \ge \frac{1}{\mu(D)} \left\{r(c_1(\|x\|) + c_2(\|x\|)) + [G_3(1, x) - G_2(1, x)]g'(\|x\|^{1/r})\right\}$$

The proof is completed. □

*Remark* 4.3. $T_{25}$ and $T_{28}$ are very general fractional integral operators in Definition 2.1. Hence, by giving particular values to the functions and parameters in Theorem 4.1 and Theorem 4.2, we get the corresponding norm inequalities for different fractional integral operators. Such as, if $g(t) = t$, then (2.1) reduces to

$$T_{25}(f, x) = \frac{c}{k\Gamma_k(\alpha)} \int_D |\|x\| - \|y\||^{(\alpha/k)-1} f(y) dy, \tag{4.6}$$

where $\Gamma_k(\alpha)$ is defined by (1.6). Taking $g(t) = t$ and $0 < \mu(D) \leq 1$ in Theorem 4.1 and Theorem 4.2, we get

$$\|T_{25}\| = \|T_{28}\| = \frac{c}{k\Gamma_k(\alpha)} \times \frac{\Gamma^n(1/r)}{r^n \Gamma(n/r)} \int_a^b (t^{1/r} - a^{1/r})^{(n/r)-1} t^{(n/r)-1} dt. \tag{4.7}$$

In particular, if $c = n = r = k = 1$ in (4.6) and (4.7), then (4.6) reduces to

$$T_{25}(f, x) = T_1(f, x) + T_2(f, x) = \frac{1}{\Gamma(\alpha)} \int_a^b |x - y|^{\alpha-1} f(y) dy$$

and

$$\|T_{25}\| = \frac{1}{\Gamma(\alpha)} \int_a^b (t - a)^{\alpha-1} dt = \frac{1}{\alpha\Gamma(\alpha)} (b - a)^{\alpha}.$$


## Acknowledgments

This paper is dedicated to Professor Themistocles M. Rassias on the occasion of his 70th birthday.


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