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Truncated exponential based Frobenius-Genocchi and truncated exponential based Apostol type Frobenius-Genocchi polynomials

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Abstract

The aim of this article is to introduce Frobenius-Genocchi polynomials based on truncated-exponentials and to investigate several of their features, including their summation formulae and monomiality principle formalism. Also, we propose and investigate Apostol type Frobenius-Genocchi polynomials based on truncated exponentials, demonstrating their quasi-monomial aspects and providing various identities for these polynomials via umbral calculus.

Keywords: Truncated exponential polynomials, Frobenius-Genocchi polynomials, Apostol type polynomials, summation formulae, monomiality principle, umbral calculus

2010 MSC: 11B83, 33B10, 05A40, 33E20, 33E30

1. Introduction and Preliminaries

Numerous polynomials, numbers, and functions, as well as their generalizations and variations, have been developed and studied, owing primarily to their potential use and direct applications in a broad variety of scientific disciplines. The purpose of this article is to present Frobenius-Genocchi polynomials based on truncated-exponentials and to study some of their properties, such as their summation formulas and monomiality principle formalism. Additionally, we introduce and explore Apostol type Frobenius-Genocchi polynomials based on truncated-exponentials and demonstrate their quasi-monomial properties and use umbral calculus to provide several identities for these polynomials.

We begin by recalling the Frobenius-Genocchi polynomials $G_n^F(x|u)$ defined by the following generating function (see [44]):

$$\mathcal{F}(x,u;t) := \frac{(1-u)t}{e^t-u}\, e^{xt} = \sum_{n=0}^{\infty} G_n^F(x|u)\frac{t^n}{n!}. \tag{1.1}$$

In particular, $G_n^F(u) := G_n^F(0|u)$ are called the Frobenius-Genocchi numbers, which are generated by the following function :

$$\frac{(1-u)t}{e^t-u} = \sum_{n=0}^{\infty} G_n^F(u)\frac{t^n}{n!} \tag{1.2}$$

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(cf. [44, p. 175]).

Remark 1.1. We consider the restrictions in (1.1).

(i) **The case $u = 0$.** $\mathcal{F}(x, 0; t) = t\, e^{(x-1)t}$ is an entire function of both variables x and t. Therefore the restriction for (1.1) is $|t| < \infty$ and $x \in \mathbb{C}$. Here and elsewhere, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, rational numbers, integers, and positive integers, respectively. Also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.

(ii) **The case $u = 1$.** Then the generating function is

$$\mathcal{F}(x, 1; t) = 0 \cdot \frac{t}{e^t - 1} \cdot e^{xt}$$

Note that $e^t - 1 = 0$ implies $t = i2k\pi$ ($k \in \mathbb{Z}$). Since $\lim_{t \to 0} t/\left(e^t - 1\right) = 1$, we find that the generating function $\mathcal{F}(x, 1; t) = 0$ for $|t| < 2\pi$ and all $x \in \mathbb{C}$. When $t = i2k\pi$ ($k \in \mathbb{Z} \setminus \{0\}$), the generating function

$$\mathcal{F}(x, 1; t) = \frac{0}{0} \cdot t \cdot e^{xt}$$

are found to be undefined at those points. Accordingly,

$$\mathcal{F}(x, 1; t) = 0 = \sum_{n=0}^{\infty} G_n^F(x|1) \frac{t^n}{n!} \quad (|t| < 2\pi,\ x \in \mathbb{C}),$$

which implies that

$$G_n^F(x|1) = 0 \quad (n \in \mathbb{N}_0,\ x \in \mathbb{C})\,.$$

(iii) **The case $u \neq 0$ and $u \neq 1$.** Note that $e^t - u = 0$ implies

$$t = \log u = \ln |u| + i \arg u = \ln |u| + i(\mathrm{Arg\,u} + 2k\pi) \quad (k \in \mathbb{Z}).$$

Then the generating function $\mathcal{F}(x, u; t)$ is analytic in $|t| < |\mathrm{Log}\, u|$ and for all $x \in \mathbb{C}$, $\mathrm{Log}\, u$ being the principal logarithm of $\log u$. In this case, we have

$$\mathcal{F}(x, u; t) := \frac{(1 - u)t}{e^t - u}\, e^{xt} = \sum_{n=0}^{\infty} G_n^F(x|u) \frac{t^n}{n!} \quad (|t| < |\mathrm{Log}\, u|,\ x \in \mathbb{C})\,.$$

(iv) From (i) and (iii), assuming that

$$|\mathrm{Log}\, 0| = \lim_{u \in \mathbb{R} \to 0^+} |\mathrm{Log}\, u| = \infty,$$

we can give the generating function for the Frobenius-Genocchi polynomials $G_n^F(x|u)$ in (1.1) the following restrictions:

$$\frac{(1 - u)t}{e^t - u}\, e^{xt} = \sum_{n=0}^{\infty} G_n^F(x|u) \frac{t^n}{n!} \tag{1.3}$$

$$\left(u \in \mathbb{C} \setminus \{1\},\ |t| < |\mathrm{Log}\, u|,\ x \in \mathbb{C}\right).$$

Factoring the generating function in (1.1) as follows:

$$\mathcal{F}(x, u; t) = e^{xt} \cdot \frac{(1 - u)t}{e^t - u},$$

the first one and the second one (with (1.2)) of which being able to be expanded as Maclaurin series, and using a series manipulation technique (or Cauchy product) to combine the resulting double series into a single series, it is routine to obtain

$$G_n^F(x|u) = \sum_{k=0}^{n} \binom{n}{k} G_k^F(u)\, x^{n-k} \quad (n \in \mathbb{N}_0)\,. \tag{1.4}$$

It is noted that Eq. (1.4) can be used to define the Frobenius-Genocchi polynomials $G_n^F(x|u)$ in terms of the Frobenius-Genocchi numbers $G_k^F(u)$, recursively.

It is also commented that the Frobenius-Genocchi polynomials $G_n^F(x|u)$ when $u = -1$ reduces to the classical Genocchi polynomials $G_n(x)$ defined by the following generating function (see, e.g., [40, p. 90]):

$$\frac{2t}{e^t+1}\, e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \quad (|t| < \pi). \tag{1.5}$$

In particular, $G_n := G_n(0)$ are the classical Genocchi numbers.

The truncated exponential polynomials $\mathbf{e}_m(x)$ are defined by the series (see [2, 34, 35]):

$$\mathbf{e}_m(x) = \sum_{s=0}^{m} \frac{x^s}{s!} \quad (m \in \mathbb{N}_0)\,, \tag{1.6}$$

which are the first $(m+1)$ terms of the Maclaurin series for e^x. Dattoli et al. [13] conducted a systematic investigation of these polynomials. Additionally, they [13] informed that these polynomials appear in a large number of quantum mechanics and optics issues and are critical in estimating certain integrals related with products of special functions. It is intriguing to consider that $\mathbf{e}_m(x)$ $(m \in \mathbb{N})$ is irreducible in $\mathbb{Q}[x]$, which is a particular case of Schur's theorem (see [34, 35]; see also [11, 20]) : Irreducibility in $\mathbb{Q}[x]$ of any polynomial of the following form

$$1 + c_1\, x + c_2\, \frac{x^2}{2!} + \cdots + c_{m-1}\, \frac{x^{m-1}}{(m-1)!} \pm \frac{x^m}{m!} \quad (c_i \in \mathbb{Z},\ m \in \mathbb{N}).$$

$$(m \in \mathbb{N},\ c_i \in \mathbb{Z}\ (i = 1, \ldots, m-1))\,.$$

For additional properties and diverse applications of these polynomials, refer to, for instance, [1], [4]–[6], [9, 10, 15, 17, 18, 22, 23, 25, 26, 28, 30, 39]. Employing the well-known Gamma function Γ defined by (see, e.g., [2] and [40, Section 1.1]):

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, \mathrm{d}t \quad (\Re(z) > 0), \tag{1.7}$$

in particular,

$$\Gamma(m+1) = m! = \int_0^{\infty} e^{-t}\, t^m\, \mathrm{d}t \quad (m \in \mathbb{N}_0)\,, \tag{1.8}$$

we find (see, e.g., [13, Eq. (2)]):

$$\mathbf{e}_m(x) = \frac{1}{m!} \int_0^{\infty} e^{-t}\, (x+t)^m\, \mathrm{d}t \quad (m \in \mathbb{N}_0)\,. \tag{1.9}$$

In addition, the truncated exponential polynomials $\mathbf{e}_m(x)$ are defined by the generating function (see [13, p. 596, Eq. (4)]):

$$\frac{e^{x\,t}}{1-t} = \sum_{m=0}^{\infty} \mathbf{e}_m(x)\, t^m \quad (|t| < 1). \tag{1.10}$$

The generating function (1.10) may be facilely obtained by taking the Cauchy product of two Maclaurin series $1/(1-t)$ and $e^{x\,t}$. Differentiating both sides of the formula (1.10) with respect to the variable t and x, respectively, provides the ensuing differential-recurrent relations (see [13, Eq. (5)]):

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathbf{e}_m(x) = \mathbf{e}_{m-1}(x) \quad (m \in \mathbb{N}) \tag{1.11}$$

and

$$\mathbf{e}_{m+1}(x) = \left[1 + \frac{x}{m+1}\left(1 - \frac{\mathrm{d}}{\mathrm{d}x}\right)\right]\mathbf{e}_m(x) \quad (m \in \mathbb{N}_0)\,. \tag{1.12}$$

These two relations are combined to afford the second-order differential equation (see [13, Eq. (8)]):

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (m+x)\frac{\mathrm{d}}{\mathrm{d}x} + m\right]\mathbf{e}_m(x) = 0 \quad (m \in \mathbb{N}_0)\,. \tag{1.13}$$

The associated truncated exponential polynomials $\mathbf{e}_m^{(\alpha)}(x)$ are defined by means of the generating function (see [13, Eq. (12)]):

$$\frac{e^{xt}\,\Gamma(\alpha+1)}{(1-t)^{\alpha+1}} = \sum_{m=0}^{\infty} \mathbf{e}_m^{(\alpha)}(x)\,t^m \quad (|t| < 1), \tag{1.14}$$

which are also known as the McBride or modified Laguerre polynomials (see, e.g., [41, p. 425, Eq. (46)]). The integral representation and explicit expression of $\mathbf{e}_m^{(\alpha)}(x)$ are given as follows (see [13, Eqs. (10) and (11)]):

$$\mathbf{e}_m^{(\alpha)}(x) = \frac{1}{m!}\int_0^{\infty} e^{-t}\,t^{\alpha}\,(x+t)^m\,\mathrm{d}t \quad (m \in \mathbb{N}_0) \tag{1.15}$$

and

$$\mathbf{e}_m^{(\alpha)}(x) = \sum_{s=0}^{m} \frac{\Gamma(m-s+\alpha+1)\,x^s}{s!\,(m-s)!} \quad (m \in \mathbb{N}_0)\,. \tag{1.16}$$

The truncated exponential polynomials of order ℓ, ${}_{[\ell]}\mathbf{e}_m(x)$, are defined by the generating function (see [13, p. 599, Eq. (25)]):

$$\frac{e^{xt}}{1-t^{\ell}} = \sum_{m=0}^{\infty} {}_{[\ell]}\mathbf{e}_m(x)\,t^m \quad (|t| < 1,\ \ell \in \mathbb{N}). \tag{1.17}$$

The associated truncated exponential polynomials of order ℓ, ${}_{[\ell]}\mathbf{e}_m^{(\alpha)}(x)$, are given as (see [13, p. 599, Eq. (27)]):

$${}_{[\ell]}\mathbf{e}_m^{(\alpha)}(x) = \sum_{r=0}^{m} \frac{\Gamma(\alpha+r+1)\,x^{m-\ell r}}{r!\,(m-\ell r)!} \quad (m \in \mathbb{N}_0,\ \ell \in \mathbb{N})\,, \tag{1.18}$$

which are, like (1.14), found to be generated by the following function

$$\frac{e^{xt}\,\Gamma(\alpha+1)}{(1-t^{\ell})^{\alpha+1}} = \sum_{m=0}^{\infty} {}_{[\ell]}\mathbf{e}_m^{(\alpha)}(x)\,t^m \quad (|t| < 1,\ \ell \in \mathbb{N}). \tag{1.19}$$

The 2-variable truncated exponential polynomials ${}_{[\ell]}\mathbf{e}_m(x,y)$ of order ℓ are defined by the following generating function (see, e.g., [25, Eq. (1.24)]; see also [16, Eqs. (29) and (30)])

$$\frac{e^{xt}}{1-yt^{\ell}} = \sum_{m=0}^{\infty} {}_{[\ell]}\mathbf{e}_m(x,y)\,\frac{t^m}{m!} \tag{1.20}$$

$$\left(\ell \in \mathbb{N},\ x \in \mathbb{C};\ y \in \mathbb{C}\setminus\{0\},\ |t| < \sqrt[\ell]{1/|y|};\ y = 0,\ |t| < \infty\right).$$

The explicit expression of ${}_{[\ell]}\mathbf{e}_m(x,y)$ is given by

$${}_{[\ell]}\mathbf{e}_m(x,y) = m!\sum_{s=0}^{\left[\frac{m}{\ell}\right]} \frac{x^{m-\ell s}\,y^s}{(m-\ell s)!} \quad (m \in \mathbb{N}_0,\ \ell \in \mathbb{N},\ x,\ y \in \mathbb{C})\,. \tag{1.21}$$

The analogous particular cases of (1.20) and (1.21), without $m!$, when $\ell = 2$ are provided [13, Eqs. (32) and (31)], respectively.

Two variable special polynomials are significant from the viewpoint of applications. The two variable polynomials may allow one to derive a number of useful identities in a fairly straight forward way and enable one to introduce new families of special polynomials. For example, Bretti et al. [7] introduced general classes of the Appell polynomials of two variables by using properties of an iterated isomorphism, related to the Laguerre-type exponentials. The two variable forms of the Hermite, Laguerre and truncated exponential polynomials as well as their generalizations have been investigated by several researchers (see, e.g., [3], [12]–[14]).

Steffenson [42] proposed the concept of poweroids (see also [24]). Dattoli [12] reformulated the idea of poweroids to develop the monomiality principle as follows: There exist two operators $\hat{M}$ and $\hat{P}$ (called, respectively, multiplicative and derivative operators) acting on a sequence of polynomials $p_n(x)$ ($x \in \mathbb{C}$, $n \in \mathbb{N}_0$) of degree n which satisfy the following identities, for all $n \in \mathbb{N}$,

$$\hat{M}\{p_n(x)\} = p_{n+1}(x) \tag{1.22}$$

and

$$\hat{P}\{p_n(x)\} = n\, p_{n-1}(x). \tag{1.23}$$

Then the sequence of polynomials $p_n(x)$ is called quasi-monomial with respect to the operators $\hat{M}$ and $\hat{P}$ (see, e.g., [24]). These multiplicative and derivative operators must satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}, \tag{1.24}$$

$\hat{1}$ being the identity operator, and therefore exhibits a Weyl group structure.

If a sequence of polynomials $p_n(x)$ is quasi-monomial with respect to the operators $\hat{M}$ and $\hat{P}$, several relationships between the polynomials and the operators can be established such as

(i) The polynomials $p_n(x)$ satisfy the differential equation:

$$\hat{M}\hat{P}\{p_n(x)\} = n\, p_n(x) \quad (n \in \mathbb{N}). \tag{1.25}$$

(ii) If $p_0(x) = 1$, then $p_n(x)$ is constructed by the operator $\hat{M}$ as follows:

$$p_n(x) = \hat{M}^n\, \{1\}. \tag{1.26}$$

(iii) Via (1.26), the polynomials $p_n(x)$ are generated by

$$e^{t\hat{M}}\{1\} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} \quad (|t| < \infty). \tag{1.27}$$

Most properties of polynomial families, identified as quasi-monomials, can be derived by utilizing operational rules associated with the corresponding multiplicative and derivative operators (see, e.g., [12, 43]).

Remark 1.2. The 2-variable truncated exponential polynomials of order ℓ ${}_{[\ell]}\mathbf{e}_n(x, y)$ are quasi-monomials with respect to the following multiplicative and derivative operators:

$$\hat{M}_{e^{(\ell)}} = x + \ell y D_y y D_x^{\ell-1} \tag{1.28}$$

and

$$\hat{P}_{e^{(\ell)}} = D_x := \frac{\mathrm{d}}{\mathrm{d}x}. \tag{1.29}$$

It is noted that (1.28) is recorded in [25, Eq. (1.31)] and the Equation (1.32) in [25] may be revised as in (1.29).

2. Truncated-exponential based Frobenius-Genocchi polynomials

This section begins by introducing a sequence of new polynomials in Definition 2.1 and exploring some of their features.

Definition 2.1. Truncated-exponential based Frobenius-Genocchi polynomials ${}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)$ are defined by means of the following generating function:

$$\mathcal{H}(x,y,u;t) := \frac{(1-u)t}{(e^t-u)(1-yt^\ell)}e^{xt} = \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\frac{t^n}{n!} \tag{2.1}$$

$$\Big(\ell \in \mathbb{N},\ x \in \mathbb{C};\ u \in \mathbb{C}\setminus\{1\},\ y \in \mathbb{C}\setminus\{0\},\ |t| < \min\{|\mathrm{Log}\,u|,\ \sqrt[\ell]{1/|y|}\}; u \in \mathbb{C}\setminus\{1\},\ y=0,\ |t|<|\mathrm{Log}\,u|\Big).$$

Theorem 2.2. *The following formula holds true:*

$${}_{e^{(\ell)}}\mathcal{G}_n(x,y|u) = \sum_{m=0}^{n}\binom{n}{m} G^F_{n-m}(u)\,{}_{[\ell]}\boldsymbol{e}_m(x,y) \quad (n \in \mathbb{N}_0). \tag{2.2}$$

Proof. Factor the generating function as follows:

$$\mathcal{H}(x,y,u;t) := \frac{(1-u)t}{e^t-u}\cdot\frac{e^{xt}}{1-yt^\ell}. \tag{2.3}$$

Then, using the right-handed summations of (1.2) and (1.20) in the above respective factors, with the aid of double series manipulation (or Cauchy product), we obtain a single series. Finally, equating the coefficients of t^n for the rightmost summation of (2.1) and the above single series, we can get the identity (2.2). □

The following theorem shows that the sequence ${}_{(\ell)}\mathcal{G}_n(x,y|u)$ is quasi-monomial with respect to the given multiplicative and derivative operators.

Theorem 2.3. *The sequence of polynomials ${}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)$ is quasi-monomial with respect to the multiplicative operator $\hat{M}_{e^{(\ell)}\mathcal{G}}$ and the derivative operator $\hat{P}_{e^{(\ell)}}$ which are given as follows:*

$$\hat{M}_{e^{(\ell)}\mathcal{G}} = x + \ell y D_y y D_x^{\ell-1} + \frac{e^{D_x}(1-D_x)-u}{D_x(e^{D_x}-u)} \tag{2.4}$$

and

$$\hat{P}_{e^{(\ell)}\mathcal{G}} = D_x. \tag{2.5}$$

Proof. In view of Remark (1.2), since ${}_{[\ell]}\mathbf{e}_0(x,y) = 1$, we find from (1.26) and (1.27) that

$$\frac{e^{xt}}{1-yt^\ell} = e^{t\hat{M}_{e^{(\ell)}}}\{1\}. \tag{2.6}$$

Using (2.6) in (2.3), we get

$$\frac{(1-u)t}{e^t-u}\cdot e^{t\hat{M}_{e^{(\ell)}}} = \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\frac{t^n}{n!}. \tag{2.7}$$

Differentiating both sides of (2.7) with respect to t, we obtain

$$\left(\hat{M}_{e^{(\ell)}} + \frac{e^t(1-t)-u}{t(e^t-u)}\right)\frac{(1-u)t}{e^t-u}e^{t\hat{M}_{e^{(\ell)}}} = \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+1}(x,y|u)\frac{t^n}{n!}. \tag{2.8}$$

Now, replacing t in the first factor of (2.8) with D_x, with the aid of (1.28), we find from (2.7) that

$$\left(\hat{M}_{e^{(\ell)}} + \frac{e^{D_x}(1-D_x)-u}{D_x(e^{D_x}-u)}\right)\sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+1}(x,y|u)\frac{t^n}{n!}. \tag{2.9}$$

It is noted that the multiplicative operator in the left member of (2.9) can be justified by employing [25, Theorem 2.2]. Here, let

$$g(t) = \frac{(1-u)t}{e^t - u}.$$

Then we have

$$\frac{g'(t)}{g(t)} = \frac{e^t(1-t)-u}{t(e^t-u)}.$$

From (2.9), we obtain

$$\hat{M}_{e^{(\ell)}\mathcal{G}}\left\{{}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\right\} = {}_{e^{(\ell)}}\mathcal{G}_{n+1}(x,y|u) \quad (n \in \mathbb{N}_0). \tag{2.10}$$

Differentiating the last member of (2.1) with respect to x,

$$\sum_{n=1}^{\infty} D_x\left\{{}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\right\}\frac{t^n}{n!} = \sum_{n=1}^{\infty} {}_{e^{(r)}}\mathcal{G}_{n-1}(x,y|u)\frac{t^n}{(n-1)!}. \tag{2.11}$$

Equating the coefficients of t^n on both sides of (2.11), we get

$$\hat{P}_{e^{(\ell)}\mathcal{G}}\left\{{}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\right\} = D_x\left\{{}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\right\} = n\,\mathcal{G}_{n-1}(x,y|u) \quad (n \in \mathbb{N}). \tag{2.12}$$

□

Remark 2.4. Using (2.4) and (2.5) in (1.25), we derive the following differential equation for the truncated-exponential based Frobenius-Genocchi polynomials ${}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)$:

$$\left(xD_x + \ell y D_y y D_x^{\ell} + \frac{e^{D_x}(1-D_x)-u}{D_x(e^{D_x}-u)}D_x - n\right){}_{e^{(\ell)}}\mathcal{G}_n(x,y|u) = 0. \tag{2.13}$$

3. Addition and summation formulae

In this section we offer two summation formulae for the truncated-exponential based Frobenius-Genocchi polynomials ${}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)$, along with several special instances in the corollaries.

Theorem 3.1. *The following addition formula holds true:*

$${}_{e^{(\ell)}}\mathcal{G}_n(x+\nu,y|u) = \sum_{k=0}^{n}\binom{n}{k}\nu^k\, {}_{e^{(\ell)}}\mathcal{G}_{n-k}(x,y|u). \tag{3.1}$$

Proof. Replacing x by $x+\nu$ in the generating function (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x+\nu,y|u)\frac{t^n}{n!} &= \frac{(1-u)t}{(e^t-u)(1-yt^{\ell})}e^{(x+\nu)t}\\ &= \frac{(1-u)t}{(e^t-u)(1-yt^{\ell})}e^{xt}\cdot e^{\nu t}\\ &= \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x,y|u)\frac{t^n}{n!}\cdot\sum_{k=0}^{\infty}\frac{\nu^k}{k!}t^k\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\nu^k\, {}_{e^{(\ell)}}\mathcal{G}_{n-k}(x,y|u)}{(n-k)!k!}t^n,\end{aligned} \tag{3.2}$$

where the Cauchy product is utilized for the final equality. Then, equating the coefficients of t^n on both sides of the first and final series expansions in (3.2) yields (3.1). □

In the following corollary, the specific case $v = 1$ of (3.1) affords a summation formula for ${}_{e^{(\ell)}}\mathcal{G}_n(x, y|u)$.

Corollary 3.2. *The following formula holds true:*

$$ {}_{e^{(\ell)}}\mathcal{G}_n(x+1, y|u) = \sum_{k=0}^{n} \binom{n}{k} {}_{e^{(\ell)}}\mathcal{G}_{n-k}(x, y|u). \tag{3.3}$$

Theorem 3.3. *The following implicit summation formula for ${}_{e^{(\ell)}}\mathcal{G}_n(x, y|u)$ holds true:*

$$ {}_{e^{(\ell)}}\mathcal{G}_{n+k}(\eta, y|u) = \sum_{l=0}^{n} \sum_{m=0}^{k} \binom{k}{m} (\eta - x)^{l+m} {}_{e^{(\ell)}}\mathcal{G}_{n+k-l-m}(x, y|u), \tag{3.4}$$

where the variables x and η are independent.

Proof. Substituting $t+w$ for t in the generating function (2.1) and multiplying the resultant identity by $\exp(-x(t+w))$, we obtain

$$\frac{(1-u)(t+w)}{(e^{t+w}-u)(1-y(t+w)^r)} = \exp\left(-x(t+w)\right) \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x, y|u) \frac{(t+w)^n}{n!}. \tag{3.5}$$

Using the following known summation formula

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{l,m=0}^{\infty} f(l+m) \frac{x^l\, y^m}{l!\, m!} \tag{3.6}$$

in (3.5), we get

$$\frac{(1-u)(t+w)}{(e^{t+w}-u)(1-y(t+w)^r)} = \exp\left(-x(t+w)\right) \sum_{n,k=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(x, y|u) \frac{t^n\, w^k}{n!\, k!}. \tag{3.7}$$

It is noted that the left member of (3.7) is independent of the variable x. As a result, setting any other variable instead of the variable x in the right member of (3.7) gives the same result. Here, choosing a variable η instead of x and multiplying the resultant identity by $\exp(\eta(t+w))$, we derive

$$\sum_{n,k=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(\eta, y|u) \frac{t^n\, w^k}{n!\, k!} = \exp\left((\eta - x)(t+w)\right) \sum_{n,k=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(x, y|u) \frac{t^n\, w^k}{n!\, k!}. \tag{3.8}$$

Expanding the exponential term in the right member of (3.8) and using the formula (3.6), we find

$$\begin{aligned} \sum_{n,k=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(\eta, y|u) \frac{t^n\, w^k}{n!\, k!} &= \sum_{l,m=0}^{\infty} (\eta - x)^{l+m} \frac{t^l\, w^m}{l!\, m!} \sum_{n,k=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(x, y|u) \frac{t^n\, w^k}{n!\, k!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (\eta - x)^{l+m} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(x, y|u) \frac{t^{n+l}}{n!\, l!} \frac{w^{k+m}}{k!\, m!}. \end{aligned} \tag{3.9}$$

Applying Cauchy product in the double series of the first and second ones and then the third and fourth ones, we get

$$\sum_{n,k=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_{n+k}(\eta, y|u) \frac{t^n\, w^k}{n!\, k!} = \sum_{n,k=0}^{\infty} \sum_{l=0}^{n} \sum_{m=0}^{k} (\eta - x)^{l+m} {}_{e^{(\ell)}}\mathcal{G}_{n+k-l-m}(x, y|u) \frac{t^n}{(n-l)!\, l!} \frac{w^k}{(k-m)!\, m!},$$

both sides of which, upon equating the coefficients of $t^n\, w^k$, we obtain the desired identity (3.4). □

The summation formulas (3.10), (3.11) and (3.12) are particular instances of (3.4), respectively, when $n = 0$, $\eta \to \eta + x$ and $y = 0$, and $\eta = 0$.

Corollary 3.4. *The following implicit summation formula holds true:*

$$ {}_{e^{(\ell)}}\mathcal{G}_k(\eta, y|u) = \sum_{m=0}^{k} \binom{k}{m} (\eta - x)^m \, {}_{e^{(\ell)}}\mathcal{G}_{k-m}(x, y|u). \tag{3.10} $$

Corollary 3.5. *The implicit summation formula shown below is true:*

$$ {}_{e^{(\ell)}}\mathcal{G}_{n+k}(\eta + x, 0|u) = \sum_{l,m=0}^{n,k} \binom{n}{l}\binom{k}{m} (\eta)^{l+m} \, {}_{e^{(\ell)}}\mathcal{G}_{n+k-l-m}(x, |u). \tag{3.11} $$

Corollary 3.6. *The following implicit summation formula holds true:*

$$ {}_{e^{(\ell)}}\mathcal{G}_{n+k}(0, y|u) = \sum_{l,m=0}^{n,k} \binom{n}{l}\binom{k}{m} (-x)^{l+m} \, {}_{e^{(\ell)}}\mathcal{G}_{n+k-l-m}(x, y|u). \tag{3.12} $$

4. Further generalizations

By adding one more parameter, the following definition provides further extensions of the truncated-exponential based Frobenius-Genocchi polynomials ${}_{e^{(\ell)}}\mathcal{G}_n(x, y|u)$ in Definition 2.1.

Definition 4.1. The truncated-exponential based Apostol type Frobenius-Genocchi polynomials ${}_{e^{(\ell)}}\mathcal{G}_n(x, y|u, \lambda)$ are defined by means of the following generating function:

$$ \frac{(1-u)t}{(\lambda e^t - u)(1 - yt^\ell)} \, e^{xt} = \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x, y|u, \lambda) \frac{t^n}{n!}. \tag{4.1} $$

$$ \left(\ell \in \mathbb{N},\ x \in \mathbb{C},\ \lambda \in \mathbb{C} \setminus \{0\},\ u \in \mathbb{C} \setminus \{1\};\ \ y \in \mathbb{C} \setminus \{0\}, |t| < \min\left\{\left|\mathrm{Log}\,\frac{u}{\lambda}\right|, \sqrt[\ell]{1/|y|}\right\};\ \ y = 0,\ |t| < \left|\mathrm{Log}\,\frac{u}{\lambda}\right|\right). $$

Remark 4.2. Obviously

$$ {}_{e^{(\ell)}}\mathcal{G}_n(x, y|u, 1) = {}_{e^{(\ell)}}\mathcal{G}_n(x, y|u). $$

The polynomials $\mathcal{G}_n(x|u, \lambda) := {}_{e^{(\ell)}}\mathcal{G}_n(x, 0|u, \lambda)$ are Frobenius-Genocchi polynomials of Apostol type, the kinds of which have been studied by many researchers (see, e.g., [19, 29, 31, 32]).

The following theorem explores the monomiality principle formalism.

Theorem 4.3. *The sequence of the polynomials ${}_{e^{(r)}}\mathcal{G}_n(x, y|u, \lambda)$ is quasi-monomials with regard to, respectively, the multiplicative and derivative operators:*

$$ \hat{M}_{e^{(r)}\mathcal{F}} = x + ryD_y yD_x^{r-1} + \frac{\lambda e^{D_x}(1 - D_x) - u}{D_x(\lambda e^{D_x} - u)} \tag{4.2} $$

and

$$ \hat{M}_{e^{(r)}\mathcal{F}} = D_x. \tag{4.3} $$

Proof. The proof would proceed in the same manner as that of Theorem 2.2. We omit specifics. □

An application to the theory of Appell sequences

Setting $\lambda \in \mathbb{C} \setminus \{0\}$, $u \in \mathbb{C} \setminus \{1\}$, and $\lambda = u$ in (4.1), we obtain

$$\frac{(1-\lambda)t}{\lambda(e^t-1)(1-yt^\ell)}\, e^{xt} = \sum_{n=0}^{\infty} {}_{e^{(\ell)}}\mathcal{G}_n(x,y|\lambda,\lambda)\frac{t^n}{n!}. \tag{4.4}$$

$$\left(\ell \in \mathbb{N},\ x \in \mathbb{C},\ \lambda \in \mathbb{C} \setminus \{0,1\},\ \ y \in \mathbb{C} \setminus \{0\},\ |t| < \min\left\{\sqrt[\ell]{1/|y|}, 2\pi\right\}\right).$$

Let

$$g(t) := \frac{\lambda(e^t-1)(1-yt^\ell)}{(1-\lambda)t} \quad \left(\ell \in \mathbb{N},\ x,y \in \mathbb{C},\ \lambda \in \mathbb{C} \setminus \{0,1\}\right). \tag{4.5}$$

Since $\lim\limits_{t\to 0} \frac{e^t-1}{t} = 1$, $t = 0$ is a removable singularity of $g(t)$. By defining $g(0) := \frac{\lambda}{1-\lambda}$, $g(t)$ can be continued to be analytic at $t = 0$. Therefore $g(t)$ has Maclaurin series expansion. Thus, since $g(0) \neq 0$, $g(t)$ is an invertible series. Hence we conclude that the sequence $\{{}_{e^{(\ell)}}\mathcal{G}_n(x,y|\lambda,\lambda)\}_{n\in\mathbb{N}_0}$ is Appell sequence for $g(t)$. For more information and applications on Appell sequences, one may refer to (for example) to the monograph [33] and [24, 25].

Then the following theorems are consequents of the results in [33, pp. 26-28]. So proofs are omitted. For the following theorems, we assume the restrictions in (4.4) and the function $g(t)$ is the same as in (4.5) along with its constraints. Also let $\partial x := \frac{\partial}{\partial x}$. Further let $\mathscr{F}$ denote the algebra of formal power series in the variable t over the field $\mathbb{C}$.

Theorem 4.4. *For any polynomial $p(x)$, we have*

$$p(x) = \sum_{k=0}^{\infty} g(\partial x)\{p^{(k)}(x)\}\,\frac{{}_{e^{(\ell)}}\mathcal{G}_k(x,y|\lambda,\lambda)}{k!}. \tag{4.6}$$

Theorem 4.5. *The following identity is the conjugate representation for the sequence ${}_{e^{(\ell)}}\mathcal{G}_n(x,y|\lambda,\lambda)$:*

$${}_{e^{(\ell)}}\mathcal{G}_n(x,y|\lambda,\lambda) = \sum_{k=0}^{n}\binom{n}{k} g(\partial x)^{-1}\{x^{n-k}\}\, x^k. \tag{4.7}$$

Theorem 4.6. *The following identity holds true:*

$${}_{e^{(\ell)}}\mathcal{G}_n(x,y|\lambda,\lambda) = g(\partial x)^{-1}\{x^n\}. \tag{4.8}$$

Theorem 4.7. *For any $h(t) \in \mathscr{F}$,*

$$h(\partial x)\{{}_{e^{(\ell)}}\mathcal{G}_n(x,y|\lambda,\lambda)\} = \sum_{k=0}^{n}\binom{n}{k} h(\partial x)\{{}_{e^{(\ell)}}\mathcal{G}_k(x,y|\lambda,\lambda)\}\, x^{n-k}. \tag{4.9}$$

Theorem 4.8. *The following Appell identity holds true:*

$${}_{e^{(\ell)}}\mathcal{G}_n(x+v,y|\lambda,\lambda) = \sum_{k=0}^{n}\binom{n}{k} {}_{e^{(\ell)}}\mathcal{G}_k(v,y|\lambda,\lambda)\, x^{n-k}. \tag{4.10}$$

It is worth noting in passing that the two identities (3.1) and (4.10) have a similar appearance. It is emphasized that the identity (4.10) is actually valid for Apostol type and differs in number of parameters.

5. Concluding remarks

We introduced the truncated exponential based Frobenius-Genocchi polynomials in Definition 2.1 and the truncated exponential based Apostol type Frobenius-Genocchi polynomials in Definition 4.1. Among various potential properties and identities associated these new polynomials, we investigated several of them such as summation formulae and monomiality principle formalism. Due to the fact that the polynomials introduced in this paper are coupled with several known generalized polynomials, they may be reduced to a number of known polynomials. Take some examples:

$$_{e^{(r)}}\mathcal{G}_n(x,0|u,\lambda)=\mathcal{G}_n(x|u,\lambda), \tag{5.1}$$

which are the Apostol type Frobenius-Genocchi polynomials (see, e.g., [19, 31, 32]);

$$_{e^{(r)}}\mathcal{G}_n(x,0|-1,\lambda)=\mathcal{G}_n(x|\lambda) \tag{5.2}$$

are the Apostol-Genocchi polynomials (see [29]);

$$_{e^{(r)}}\mathcal{G}_n(x,0|-1,1)=G_n(x) \tag{5.3}$$

which are the Genocchi polynomials;

$$_{e^{(r)}}\mathcal{G}_n(x,0|u,1)=\mathcal{G}_n(x|u) \tag{5.4}$$

which are the Frobenius-Genocchi polynomials.

Also, the constraints on the cited well-known polynomials and two newly presented polynomials were clarified.

Additional possible properties and identities for these polynomials, such as their integral representations, will be investigated in the future.

For Frobenius-Euler polynomials and numbers, and associated ones, one may refer (for example) to [27], [36]–[38].

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