Montes Taurus J. Pure Appl. Math. 4 (1), 97–106, 2022

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Multi-index Fubini-type polynomials

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Abstract

We introduce a new family of two-variable Fubini-type polynomials by utilizing the multi-index Mittag-Leffler function. By means of this latter function, we also define a new type of Stirling numbers of the second kind. Furthermore, some analytical properties of the above-mentioned polynomials and numbers are discussed.

Keywords: Fubini polynomials, Truncated Fubini polynomials, Truncated Bernoulli polynomials, Truncated Euler polynomials, Stirling numbers of the second kind, Mittag-Leffler function, Wiman function

2010 MSC: 11B83, 11B68, 33E12, 05A15

1. Introduction and preliminaries

Special functions play a significant role in many fields of mathematics, physics, engineering and other related research areas; for details, see [2, 5], and [7–28] also the references cited therein. The family of special polynomials is one of the most useful of special functions and the so-called Fubini polynomials are an important subset. These polynomials appear in combinatorial mathematics and many of their main properties have been obtained.

The truncated exponential polynomials have been shown to play a role of crucial significance in the assessment of integrals involving products of special functions of mathematical physics (see [3] and the references cited therein). In recent years, numerous researchers have presented and examined different sorts of truncated polynomials and numbers, for instance, truncated Bernoulli, Euler and exponential-based Apostol-type polynomials (for details, see [3, 6, 8, 9, 16, 24, 26, 29]). In this exploratory article, the familiar symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will be used to denote the arrangements of all regular numbers, real numbers and complex numbers, separately, and we let

$$\mathbb{N}_0 := \{0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\}.$$

For non-negative integer p, define the truncated Bernoulli polynomials (see [8]) by

$$\frac{z^p/p!}{e^z - \sum\limits_{h=0}^{p-1} z^h/h!} e^{uz} = \sum_{\ell=0}^{\infty} B_{p,\ell}(u) \frac{z^\ell}{\ell!}. \tag{1.1}$$

†*Article ID:* MTJPAM-D-21-00044
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Received:19 June 2021, Accepted:19 October 2021, Published:5 December 2021
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In the case when $p = 1$, (1.1) provides the standard Bernoulli polynomials $B_\ell(u)$, which are defined by the following generating function (see [23, 25]):

$$\frac{z}{e^z - 1} e^{uz} = \sum_{\ell=0}^{\infty} B_\ell(u) \frac{z^\ell}{\ell!} \qquad (|z| < 2\pi). \tag{1.2}$$

For non-negative integer p, define the truncated Euler polynomials (see [16]) by

$$\frac{2z^p/p!}{e^z + 1 - \sum\limits_{h=0}^{p-1} z^h/h!} e^{uz} = \sum_{\ell=0}^{\infty} E_{p,\ell}(u) \frac{z^\ell}{\ell!}. \tag{1.3}$$

In the case when $p = 0$, (1.3) provides the standard Euler polynomials $E_\ell(u)$, which are defined by the generating function (see [23, 25])

$$\frac{2}{e^z + 1} e^{uz} = \sum_{\ell=0}^{\infty} E_\ell(u) \frac{z^\ell}{\ell!}. \tag{1.4}$$

Obviously for $u = 0$, (1.1) and (1.3) give the truncated Bernoulli and Euler numbers, i.e

$$B_{p,\ell}(0) = B_{p,\ell} \text{ and } E_{p,\ell}(0) = E_{p,\ell}.$$

Recently, Duran *et al.* [4] presented the truncated Fubini polynomials $F_{p,\ell}(u, v)$ as follows:

$$\frac{z^p/p!}{1 - v(e^z - 1 - \sum\limits_{h=0}^{p-1} z^h/h!)} e^{uz} = \sum_{\ell=0}^{\infty} F_{p,\ell}(u, v) \frac{z^\ell}{\ell!}, \tag{1.5}$$

which for $p = 0$ reduces to the standard Fubini polynomials of two-variables (see [10, 11, 12, 27])

$$\frac{1}{1 - v(e^z - 1)} e^{uz} = \sum_{\ell=0}^{\infty} F_\ell(u, v) \frac{z^\ell}{\ell!}. \tag{1.6}$$

When $u = 0$, (1.6) provides the usual Fubini polynomials, which are defined by (see [10, 11, 12, 27])

$$\frac{1}{1 - v(e^z - 1)} = \sum_{\ell=0}^{\infty} F_\ell(v) \frac{z^\ell}{\ell!}. \tag{1.7}$$

With the help of (1.7) and the choice $v = 1$, we have the familiar Fubini numbers F_ℓ as follows (see [6, 10]):

$$\frac{1}{(2 - e^z)} = \sum_{\ell=0}^{\infty} F_\ell \frac{z^\ell}{\ell!}. \tag{1.8}$$

In addition, Duran *et al.* [4] defined the truncated Stirling numbers of the second kind $S_{2,p}(\ell, k)$ as follows:

$$\frac{\left(e^z - 1 - \sum\limits_{h=0}^{p-1} z^h/h!\right)^k}{k!} = \sum_{\ell=0}^{\infty} S_{2,p}(\ell, k) \frac{z^\ell}{\ell!}. \tag{1.9}$$

When $p = 0$, (1.9) yields the classical Stirling numbers of the second kind $S_2(\ell, k)$ (see [12–17]), which are given by

$$\frac{(e^z - 1)^k}{k!} = \sum_{\ell=0}^{\infty} S_2(\ell, k) \frac{z^\ell}{\ell!}. \tag{1.10}$$

The Mittag-Leffler function $E_\alpha(z)$, introduced by the Swedish mathematician Mittag-Leffler [18], is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \tag{1.11}$$

where $z \in \mathbb{C}$ and $\alpha \geq 0$, and is a generalization of the exponential function. Its importance has been realized during the last two decades due to its involvement in problems of physics, chemistry, biology, engineering and the applied sciences. Subsequently, the following two-parameter extension of $E_\alpha(z)$ was introduced:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+\alpha n)}, \tag{1.12}$$

which is also referred to as a Mittag-Leffler function and which reduces to (1.11) when $\beta = 1$. This function was introduced by Agarwal in 1953 [1] though some facts were mentioned earlier by Wiman in [28]. Some special cases of $E_{\alpha,\beta}(z)$ are given below [14]:

$$E_{1,2}(z) = \frac{e^z - 1}{z},\ E_{1,1}(z) = e^z,\ E_{2,1}(z^2) = \cosh z \text{ and } E_{2,1}(-z^2) = \cos z.$$

The properties of the functions $E_{\alpha,\beta}(z)$ and $E_\alpha(z)$ are very similar. These functions play a very important role in the solution of differential equations of fractional order.

Recently, Ghayasuddin *et al.* [5] presented the truncated Fubini polynomials $F_{p,\ell}^{(\alpha,\beta)}(u,v)$ as follows:

$$\frac{z^p/p!}{1 - v(z^p\, E_{\alpha,\beta}(z) - 1)} e^{uz} = \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha,\beta)}(u,v) \frac{z^\ell}{\ell!}, \tag{1.13}$$

$$(\Re(\alpha) > 0, \Re(\beta) > 0, p, \ell \in \mathbb{N}_0).$$

In the case $\alpha = 1$ and $\beta = p+1$ in (1.13) and by using the fact

$$E_{1,p+1}(z) = z^{-p}(e^z - \sum_{h=0}^{p-1} z^h/h!),$$

we get the two-variable truncated Fubini polynomials $F_{p,\ell}(u,v)$ defined by Duran *et al.* [4]. In addition, the case $p = 0$ gives the standard Fubini polynomials of two variables $F_\ell(u,v)$ defined by (1.6). Moreover, if we set $p = 0$ and $\alpha = \beta = 1$ in (1.13) then we can also obtain the classical Fubini polynomials $F_\ell(u,v)$ given in (1.6).

Furthermore, Ghayasuddin *et al.* [5] defined the truncated Stirling numbers of the second kind $S_{2,p}^{(\alpha,\beta)}(\ell,k)$ as follows:

$$\frac{\left(z^p\, E_{\alpha,\beta}(z) - 1\right)^k}{k!} = \sum_{\ell=0}^{\infty} S_{2,p}^{(\alpha,\beta)}(\ell,k) \frac{z^\ell}{\ell!}. \tag{1.14}$$

Upon setting $\alpha = 1$ and $\beta = p+1$, (1.14) yields the truncated Stirling numbers of the second kind defined by Duran *et al.* [4], which when $p = 0$ reduce to the classical Stirling numbers of the second kind given in (1.10).

Kiryakova (see [13, 14, 20] and additionally [19]) introduced the following new class of multi-index Mittag-Leffler functions (involving $2s$ parameters)

$$E_{(\alpha_i),(\beta_i)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta_1+\alpha_1 n)\cdots\Gamma(\beta_s+\alpha_s n)}, \tag{1.15}$$

where $s > 1$ is an integer and it is understood that $1 \leq i \leq s$. The parameters $\alpha_1, \ldots, \alpha_s > 0$ and $\beta_1, \ldots, \beta_s$ are arbitrary real numbers. The primary object of this research paper is to propose another class of two-variable Fubini-type polynomials and another kind of Stirling numbers of the second kind by utilizing the multi-index Mittag-Leffler functions. For other types of multi-index and multi-variable generalizations of the classical Mittag-Leffler function we refer to the recent surveys [2, 21, 15], the monograph [19] and the last edition of the monograph [7].

2. Extended polynomials and numbers

In this section, we present the two-variable Fubini polynomials and a new type of the Stirling numbers of the second kind based on the multi-index Mittag-Leffler functions.

Definition 2.1. Let $p, \ell \in \mathbb{N}_0$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$ such that $v(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1) \neq 1$. Then the extended Fubini-type polynomials are defined by the following exponential generating function:

$$\frac{z^p/p!}{1 - v(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1)} e^{uz} = \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_1,\beta_i)}(u, v)\, \frac{z^\ell}{\ell!}, \tag{2.1}$$

where in order not to overburden the notation we write

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u, v) = F_{p,\ell}^{(\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_s)}(u, v)$$

and similarly for other quantities.

In the case when $u = 0$ in (2.1), we have a new type of Fubini polynomial, which we shall call the truncated Fubini polynomials given by

$$\frac{z^p/p!}{1 - v(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1)} = \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(v)\, \frac{z^\ell}{\ell!}. \tag{2.2}$$

Further, if we set $v = 1$ in (2.2) then we have the new extended Fubini-type numbers:

$$\frac{z^p/p!}{2 - z^p\, E_{(\alpha_i),(\beta_i)}(z)} = \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}\, \frac{z^\ell}{\ell!}. \tag{2.3}$$

Remark 2.2. We note that the case $s = 2$, by setting $\alpha_1 = \alpha$, $\alpha_2 = 0$, and $\beta_1 = \beta$, $\beta_2 = 1$ in (2.1) then this definition clearly reduces to the two-variable truncated Fubini polynomials $F_{p,\ell}^{(\alpha,\beta)}(u, v)$ defined by Ghayasuddin *et al.* [5]. Further, for $\alpha = 1$ and $\beta = p + 1$ this gives the two-variable truncated Fubini polynomials $F_{p,\ell}(u, v)$ defined by Duran *et al.* [4]. Clearly, for $p = 0$ we easily recover the classical Fubini polynomials of two variables given in (1.6) (see [10, 11, 12, 27]).

Definition 2.3. For $p, \ell \in \mathbb{N}_0$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$, the extended Stirling numbers of the second kind are defined by

$$\sum_{\ell=0}^{\infty} S_{2,p}^{(\alpha_i,\beta_i)}(\ell, k)\, \frac{z^\ell}{\ell!} = \frac{(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1)^k}{k!}. \tag{2.4}$$

Now, when $s = 2$ and $\alpha_1 = \alpha$, $\alpha_2 = 0$, and $\beta_1 = \beta$, $\beta_2 = 1$ in (2.4) we obtain the truncated Stirling numbers of the second kind defined by Ghayasuddin *et al.* [5], which further for $\alpha = 1$ and $\beta = p + 1$ gives the truncated Stirling numbers of the second kind defined by Duran *et al.* [4]. Finally, when $p = 0$ we easily recover the classical Stirling numbers of the second kind $S_2(\ell, k)$ given in (1.10).

3. Properties of the extended polynomials and numbers

This section presents some useful properties (such as summation formulas, a differential formula and connection formulas) of our extended polynomials and numbers.

Theorem 3.1. *For $p, \ell \in \mathbb{N}_0$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$ and $w \in \mathbb{R}$, each of the following summation formulas holds true:*

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u + w, v) = \sum_{m=0}^{\ell} \binom{\ell}{m} w^m\, F_{p,\ell-m}^{(\alpha_i,\beta_i)}(u, v) \tag{3.1}$$

and

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u + w, v) = \sum_{m=0}^{\ell} \binom{\ell}{m} (u + w)^m\, F_{p,\ell-m}^{(\alpha_i,\beta_i)}(v). \tag{3.2}$$

Proof. In view of (2.1), we have

$$\begin{aligned} S = \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(u+w,v)\,\frac{z^\ell}{\ell!} &= \frac{z^p/p!}{1-v(z^p\,E_{(\alpha_i),(\beta_i)}(z)-1)}e^{(u+w)z} \\ &= \frac{z^p/p!}{1-v(z^p\,E_{(\alpha_i),(\beta_i)}(z)-1)}e^{uz}e^{wz} \\ &= \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v)\frac{z^\ell}{\ell!}\sum_{m=0}^{\infty} w^m\frac{z^m}{m!}. \end{aligned}$$

Now, by employing the lemma (see [23, p. 56])

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}A(m,\ell) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}A(m,\ell-m),$$

we have

$$S = \sum_{\ell=0}^{\infty}\left(\sum_{m=0}^{\ell}\binom{\ell}{m}w^m\,F_{p,\ell-m}^{(\alpha_i,\beta_i)}(u,v)\right)\frac{z^\ell}{\ell!}.$$

On equating like powers of z in the above expression we obtain our first result (3.1). Similarly, the second result (3.2) can be established with the assistance of (2.2). □

Corollary 3.2. *With the help of* (3.2)*, the choice* $w = 0$ *produces*

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v) = \sum_{m=0}^{\ell}\binom{\ell}{m}u^m\,F_{p,\ell-m}^{(\alpha_i,\beta_i)}(v). \tag{3.3}$$

Remark 3.3. Consideration of the identities (3.1) and (3.2) leads to the following interesting result:

$$\sum_{m=0}^{\ell}\binom{\ell}{m}w^m\,F_{p,\ell-m}^{(\alpha_i,\beta_i)}(u,v) = \sum_{m=0}^{\ell}\binom{\ell}{m}(u+w)^m\,F_{p,\ell-m}^{(\alpha_i,\beta_i)}(v). \tag{3.4}$$

Theorem 3.4. *For* $p,\ell \in \mathbb{N}_0$*,* $\alpha_i > 0$*,* $\beta_i \in \mathbb{R}$*, we have the following relation:*

$$S_{2,p}^{(\alpha_i,\beta_i)}(\ell,k+\rho) = \frac{k!\,\rho!}{(k+\rho)!}\sum_{r=0}^{\ell}\binom{\ell}{r}S_{2,p}^{(\alpha_i,\beta_i)}(\ell-r,k)\,S_{2,p}^{(\alpha_i,\beta_i)}(r,\rho). \tag{3.5}$$

Proof. From (2.4), we have

$$\begin{aligned} \sum_{\ell=0}^{\infty} S_{2,p}^{(\alpha_i,\beta_i)}(\ell,k+\rho)\,\frac{z^\ell}{\ell!} &= \frac{\{z^p\,E_{(\alpha_i),(\beta_i)}(z)-1\}^{k+\rho}}{(k+\rho)!}. \\ &= \frac{k!\rho!}{(k+\rho)!}\,\frac{\{z^pE_{\alpha_i,\beta_i}(z)-1\}^k}{k!}\,\frac{\{z^pE_{(\alpha_i),(\beta_i)}(z)-1\}^\rho}{\rho!} \\ &= \frac{k!\rho!}{(k+\rho)!}\sum_{\ell=0}^{\infty}S_{2,p}^{(\alpha_i,\beta_i)}(\ell,k)\,\frac{z^\ell}{\ell!}\sum_{r=0}^{\infty}S_{2,p}^{(\alpha_i,\beta_i)}(r,\rho)\,\frac{z^r}{r!} \\ &= \frac{k!\rho!}{(k+\rho)!}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\ell}\binom{\ell}{r}S_{2,p}^{(\alpha_i,\beta_i)}(\ell-r,k)\,S_{2,p}^{(\alpha_i,\beta_i)}(r,\rho)\,\frac{z^\ell}{\ell!}. \end{aligned}$$

On equating like powers of z in the above expression, we easily obtain our stated result (3.5). □

Theorem 3.5. *For $p, \ell \in \mathbb{N}_0$, with $\ell \geq p$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$, the following result holds true:*

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(v) = \sum_{k=0}^{\infty} \binom{\ell}{p} v^k\, k!\, S_{2,p}^{(\alpha_i,\beta_i)}(\ell - p, k). \tag{3.6}$$

Proof. We have

$$\begin{aligned}
\sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(v)\, \frac{z^\ell}{\ell!} &= \frac{z^p/p!}{1 - v(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1)} \\
&= \frac{z^p}{p!} \sum_{k=0}^{\infty} v^k\, (z^p E_{(\alpha_i),(\beta_i)}(z) - 1)^k \\
&= \frac{z^p}{p!} \sum_{k=0}^{\infty} v^k\, k! \sum_{\ell=0}^{\infty} S_{2,p}^{(\alpha_i,\beta_i)}(\ell, k)\, \frac{z^\ell}{\ell!} \\
&= \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} v^k\, k!\, S_{2,p}^{(\alpha_i,\beta_i)}(\ell, k)\, \frac{z^{\ell+p}}{\ell! p!},
\end{aligned}$$

which upon comparison of the coefficients of z^ℓ yields our stated result (3.6). □

Corollary 3.6. *If we set $p = 1$ in (3.6) then we have*

$$F_{1,\ell}^{(\alpha_i,\beta_i)}(v) = \sum_{k=0}^{\infty} \ell\, v^k\, k!\, S_{2,1}^{(\alpha_i,\beta_i)}(\ell - 1, k). \tag{3.7}$$

Theorem 3.7. *For $p, \ell \in \mathbb{N}_0$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$, the following differentiation formula holds true:*

$$\frac{\partial}{\partial u}\, F_{p,\ell}^{(\alpha_i,\beta_i)}(u, v) = \ell\, F_{p,\ell-1}^{(\alpha_i,\beta_i)}(u, v). \tag{3.8}$$

Proof. Let us consider

$$\begin{aligned}
\frac{\partial}{\partial u} \left[\sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(u, v)\, \frac{z^\ell}{\ell!} \right] &= \frac{\partial}{\partial u} \left[\frac{z^p/p!}{1 - v(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1)} e^{uz} \right] \\
&= \frac{z^{p+1}/p!}{1 - v(z^p\, E_{(\alpha_i),(\beta_i)}(z) - 1)} e^{uz} \\
&= \sum_{\ell=1}^{\infty} F_{p,\ell-1}^{(\alpha_i,\beta_i)}(u, v)\, \frac{z^\ell}{(\ell - 1)!},
\end{aligned}$$

which on comparison of the coefficients of z^ℓ on both sides of the above expression yields our stated result. □

For $m \in \mathbb{N}$, the Pochhammer symbol $(a)_m$ (or rising factorial) is given by

$$(a)_m = a(a + 1)(a + 2) \cdots (a + m - 1),$$

and the binomial expansion is given by

$$(z + \upsilon)^{-m} = \sum_{r=0}^{\infty} (-1)^r \binom{m + r - 1}{r} z^r\, \upsilon^{-(m+r)} \qquad (|z| < \upsilon). \tag{3.9}$$

Theorem 3.8. *For $p, \ell \in \mathbb{N}_0$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$, we have*

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u, v) = \sum_{k=0}^{\infty} \sum_{r=0}^{n} (u)_k \binom{\ell}{r} F_{p,\ell-r}^{(\alpha_i,\beta_i)}(-k, v)\, S_2(r, k). \tag{3.10}$$

Proof. By making use of (2.1) and (3.9), we have

$$\begin{aligned}
\sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v)\frac{z^\ell}{\ell!} &= \frac{z^p/p!}{1-v(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)}(e^{-z})^{-u}\\
&= \frac{z^p/p!}{1-v(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)}\sum_{k=0}^{\infty}\binom{u+k-1}{k}(1-e^{-z})^k\\
&= \frac{z^p/p!}{1-v(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)}\sum_{k=0}^{\infty}(u)_k\,\frac{(e^z-1)^k}{k!}\,e^{-kz}\\
&= \sum_{k=0}^{\infty}(u)_k\sum_{\ell=0}^{\infty}F_{p,\ell}^{(\alpha_i,\beta_i)}(-k,v)\frac{z^\ell}{\ell!}\sum_{r=0}^{\infty}S_2(r,k)\,\frac{z^r}{r!}\\
&= \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}(u)_k\,F_{p,\ell-r}^{(\alpha_i,\beta_i)}(-k,v)\,S_2(r,k)\,\frac{z^\ell}{(\ell-r)!\,r!},
\end{aligned}$$

which yields our stated result (3.10). □

Theorem 3.9. *For $p,\ell\in\mathbb{N}_0$, $\alpha_i>0$, $\beta_i\in\mathbb{R}$ and $v_1\neq v_2$, we have*

$$\begin{aligned}
&\frac{p!\ell!}{(\ell+p)!}\sum_{k=0}^{\ell+p}\binom{\ell+p}{k}F_{p,\ell+p-k}^{(\alpha_i,\beta_i)}(u_1,v_1)\,F_{p,k}^{(\alpha_i,\beta_i)}(u_2,v_2) \qquad (3.11)\\
&\qquad=\left(\frac{v_2}{v_2-v_1}\right)F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1+u_2,v_2)-\left(\frac{v_1}{v_2-v_1}\right)F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1+u_2,v_1).
\end{aligned}$$

Proof. Let us consider

$$\begin{aligned}
&\frac{z^p/p!}{1-v_1(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)}e^{u_1z}\;\frac{z^p/p!}{1-v_2(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)}e^{u_2z}\\
&\quad=\left(\frac{v_2}{v_2-v_1}\right)\left[\frac{z^{2p}/(p!)^2\,e^{(u_1+u_2)z}}{1-v_2(z^pE_{(\alpha_i),(\beta_i)}(z)-1)}\right]-\left(\frac{v_1}{v_2-v_1}\right)\left[\frac{z^{2p}/(p!)^2\,e^{(u_1+u_2)z}}{1-v_1(z^pE_{(\alpha_i),(\beta_i)}(z)-1)}\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1,v_1)\frac{z^\ell}{\ell!}\sum_{k=0}^{\infty}F_{p,k}^{(\alpha_i,\beta_i)}(u_2,v_2)\frac{z^k}{k!}\\
&\quad=\left(\frac{v_2}{v_2-v_1}\right)\sum_{\ell=0}^{\infty}F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1+u_2,v_2)\frac{z^{\ell+p}}{\ell!p!}-\left(\frac{v_1}{v_2-v_1}\right)\sum_{\ell=0}^{\infty}F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1+u_2,v_1)\frac{z^{\ell+p}}{\ell!p!}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell}F_{p,\ell-k}^{(\alpha_i,\beta_i)}(u_1,v_1)\,F_{p,k}^{(\alpha_i,\beta_i)}(u_2,v_2)\,\frac{z^\ell}{(\ell-k)!k!}\\
&\quad=\left(\frac{v_2}{v_2-v_1}\right)\sum_{\ell=0}^{\infty}F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1+u_2,v_2)\frac{z^{\ell+p}}{\ell!p!}-\left(\frac{v_1}{v_2-v_1}\right)\sum_{\ell=0}^{\infty}F_{p,\ell}^{(\alpha_i,\beta_i)}(u_1+u_2,v_1)\frac{z^{\ell+p}}{\ell!p!}.
\end{aligned}$$

On equating the coefficients of $z^{\ell+p}$ on both sides in the above expansions and after a little simplification, we get our stated result (3.11). □

Theorem 3.10. *Let $p,\ell,k,h\in\mathbb{N}_0$ such that $\ell\geq h+k$, $\alpha_i>0$, $\beta_i\in\mathbb{R}$. Then the following relation holds true:*

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v)=\frac{\ell!p!}{2(\ell+p)!}\sum_{k=0}^{\ell+p}\binom{\ell+p}{k}\frac{F_{p,\ell+p-k}^{(\alpha_i,\beta_i)}(v)\,E_{p,k}(u)}{[\ell+p]_p}+\frac{\ell!p!}{2}\sum_{h=0}^{\ell}\sum_{k=0}^{\ell-h}\frac{F_{p,\ell-h-k}^{(\alpha_i,\beta_i)}(v)\,E_{p,k}(u)}{k!(\ell-h-k)!(h+p)!}, \qquad (3.12)$$

where $E_{p,k}(u)$ are the truncated Euler polynomials given in (1.3) *and $[\ell+p]_p$ is the falling factorial defined by*

$$[\ell]_p = \ell(\ell-1)\dots(\ell-p+1).$$

Proof. From (2.1), we have

$$\begin{aligned}
\sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v)\frac{z^\ell}{\ell!} &= \frac{z^p/p!\, e^{uz}}{1-v(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)} \, \frac{2z^p/p!}{\left(e^z+1-\sum\limits_{h=0}^{p-1} z^h/h!\right)} \, \frac{\left(e^z+1-\sum\limits_{h=0}^{p-1} z^h/h!\right)}{2z^p/p!} \\
&= \frac{p!}{2z^p} \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(v)\frac{z^\ell}{\ell!} \sum_{k=0}^{\infty} E_{p,k}(u)\,\frac{z^k}{k!} \left(\sum_{h=p}^{\infty} z^h/h! + 1\right) \\
&= \frac{p!}{2} \sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell} F_{p,\ell-k}^{(\alpha_i,\beta_i)}(v)\, E_{p,k}(u)\,\frac{z^{\ell-p}}{k!(\ell-k)!} \left(\sum_{h=0}^{\infty} z^{(h+p)}/(h+p)! + 1\right) \\
&= \frac{p!}{2} \sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell} F_{p,\ell-k}^{(\alpha_i,\beta_i)}(v) E_{p,k}(u)\frac{z^{\ell-p}}{k!(\ell-k)!} + \frac{p!}{2} \sum_{\ell=0}^{\infty}\sum_{h=0}^{\ell}\sum_{k=0}^{\ell-h} F_{p,\ell-h-k}^{(\alpha_i,\beta_i)}(v) E_{p,k}(u)\frac{z^\ell}{q!(\ell-h-k)!(h+p)!}.
\end{aligned}$$

On equating like powers of z on both sides in the above expression and after some simplification, we arrive at our desired result (3.12). □

Theorem 3.11. *Let $p,\ell,k,h \in \mathbb{N}_0$ such that $\ell \geq h+k$, $\alpha_i > 0$, $\beta_i \in \mathbb{R}$. Then we have*

$$F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v) = \ell!p! \sum_{h=0}^{\ell}\sum_{k=0}^{\ell-h} \frac{F_{p,\ell-h-k}^{(\alpha_i,\beta_i)}(v)\, B_{p,k}(u)}{k!(\ell-h-k)!(h+p)!}, \tag{3.13}$$

where $B_{p,k}(u)$ are the truncated Bernoulli polynomials given in (1.1).

Proof. Consider

$$\begin{aligned}
\sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v)\frac{z^\ell}{\ell!} &= \frac{z^p/p!\, e^{uz}}{1-v(z^p\, E_{(\alpha_i),(\beta_i)}(z)-1)} \, \frac{z^p/p!}{\left(e^z-\sum\limits_{h=0}^{p-1} z^h/h!\right)} \, \frac{\left(e^z-\sum\limits_{h=0}^{p-1} z^h/h!\right)}{z^p/p!} \\
&= \frac{p!}{z^p} \sum_{\ell=0}^{\infty} F_{p,\ell}^{(\alpha_i,\beta_i)}(v)\frac{z^\ell}{\ell!} \sum_{k=0}^{\infty} B_{p,k}(u)\,\frac{z^k}{k!} \left(\sum_{h=p}^{\infty} z^h/h!\right) \\
&= p! \sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell} F_{p,\ell-k}^{(\alpha_i,\beta_i)}(v)\, B_{p,k}(u)\,\frac{z^\ell}{k!(\ell-k)!} \sum_{h=0}^{\infty} z^h/(h+p)! \\
&= p! \sum_{\ell=0}^{\infty}\sum_{h=0}^{\ell}\sum_{k=0}^{\ell-h} F_{p,\ell-h-k}^{(\alpha_i,\beta_i)}(v)\, B_{p,k}(u)\,\frac{z^\ell}{k!(\ell-h-k)!(h+p)!}.
\end{aligned}$$

Then upon comparison of the coefficients of z^ℓ this yields our stated result (3.13). □

4. Concluding remarks

In the present article, we have presented a new type of two-variable Fubini polynomials $F_{p,\ell}^{(\alpha_i,\beta_i)}(u,v)$ and Stirling numbers $S_{2,p}^{(\alpha_i,\beta_i)}(\ell,k)$ of the second kind by means of the multi-index Mittag-Leffler function $E_{(\alpha_i),(\beta_i)}(z)$. Furthermore, we have also studied some analytical properties, namely summation formulas, a differentiation formula and relations with some well-known polynomials and numbers.

To conclude, we briefly mention the variations in the generating functions of our introduced two-variable Fubini polynomials and Stirling numbers of the second kind. The multi-index Mittag-Leffler function $E_{(\alpha_i),(\beta_i)}(z)$ has the following connection with the Wright hypergeometric function ${}_p\Psi_q(z)$ and the Fox H-function $H^{m,n}_{p,q}(z)$ (see [14, 22] for the definitions of these functions):

$$E_{(\alpha_i),(\beta_i)}(z) = {}_1\Psi_m \left[\begin{array}{c} (1,1) \\ \\ (\beta_i,\ \alpha_i)_1^m \end{array} \mid z \right] \tag{4.1}$$

and

$$E_{(\alpha_i),(\beta_i)}(z) = H^{1,1}_{1,m+1} \left[-z \mid \begin{array}{c} (0,1) \\ \\ (0,1),\ (1-\beta_i,\ \alpha_i)_1^m \end{array} \right]. \tag{4.2}$$

Consequently, by making use of the above relations, it is possible to represent the generating functions of our introduced polynomials and numbers in terms of these generalized hypergeometric functions.

Acknowledgement

The authors wish to acknowledge helpful comments and suggestions for improvement by the referees.

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