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On the bounds for the spectral norms of geometric circulant matrices with generalized Jacobsthal and Jacobsthal Lucas numbers

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Abstract

The study is about the different norms of geometric circulant matrices with the sequences called (s, t)-Jacobsthal, (s, t)-Jacobsthal Lucas and hyperharmonic Jacobsthal numbers. In the paper we obtain the upper and lower bounds for the spectral norms of geometric circulant matrix with the (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas numbers and also with hyperharmonic Jacobsthal numbers.

Keywords: Jacobsthal numbers, Jacobsthal Lucas numbers, hyperharmonic numbers, geometric circulant matrix, norm

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1. Introduction

Circulant matrices are important for various reasons since they are widely used in probability, coding theory and numerical analysis. There have been several studies on properties of different number sequences. For example in [10] the author investigated the Fibonacci and Lucas sequences in detail. From these sequences, Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$; with the initial values of $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$; with the initial values of $c_0 = 2$, $c_1 = 1$ for $n \geq 2$; respectively in [6]. There are many articles in the literature that study on the norms of circulant, r-circulant matrices with different sequences and their generalized sequences. For example, in [14], Solak studied the spectral norms of circulant matrices with Fibonacci and Lucas numbers. In [13], Shen and Cen found bounds for the spectral norms of r-circulant matrices with Fibonacci and Lucas numbers. In [9], Kocer et al. obtained the norms of circulant and semicurculant matrices whose entries are Horadam numbers. Uslu and Uygun in [18] have given the relation among k-Fibonacci, k-Lucas and generalized k-Fibonacci numbers and the spectral norms of the matrices involving these numbers. In [21], Yazlik and Taskara calculated upper and lower bounds for r-circulant matrices with generalized k-Horadam numbers. In [3], Bahsi and Solak computed the norms of r-circulant matrices with the hyper-Fibonacci and Lucas numbers. In [5], the authors established the upper bound estimation on the spectral norm of r-circulant matrices with the Fibonacci and Lucas numbers. In [15, 17], Tuglu and Kizilates studied the spectral norms of circulant and r-circulant matrices with the harmonic, harmonic Fibonacci and hyperharmonic Fibonacci numbers. In [16], Tuglu and Kizilates computed the upper and lower bounds for the spectral norms r-circulant matrices with hyperharmonic Fibonacci numbers. In [1, 2], Bahsi gave the spectral

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norms of circulant and r-circulant matrices with the hyperharmonic numbers and generalized Fibonacci numbers. In [20], Uygun computed some bounds for the norms of circulant matrices with the k-Jacobsthal and k-Jacobsthal Lucas numbers. In [8], Kizilates, Tuglu calculated bounds for the spectral norms of geometric circulant matrices with generalized Fibonacci, Lucas and hyperharmonic Fibonacci numbers.

This paper essentially seeks to show the upper and lower bounds for the spectral norms of a special circulant matrix called geometric circulant matrix with the generalized Jacobsthal and Jacobsthal Lucas numbers including two variables and also with hyperharmonic Jacobsthal numbers.

2. Preliminaries

The sequences have been generalized in two ways mainly. Firstly, the initial conditions are preserved while in the second one, the recurrence relation is preserved instead.

Definition 2.1. For natural number $n \geq 2$, (s,t)-Jacobsthal $\{\hat{j}_n(s,t)\}_{n\in N}$ and the (s,t)-Jacobsthal Lucas $\{\hat{c}_n(s,t)\}_{n\in N}$ sequences are characterized by respectively,

$$\hat{j}_n(s,t) = s\hat{j}_{n-1}(s,t) + 2t\hat{j}_{n-2}(s,t), \quad \hat{j}_0(s,t) = 0, \hat{j}_1(s,t) = 1, \tag{2.1}$$

and

$$\hat{c}_n(s,t) = s\hat{c}_{n-1}(s,t) + 2t\hat{c}_{n-2}(s,t), \quad \hat{c}_0(s,t) = 2, \hat{c}_1(s,t) = s, \tag{2.2}$$

for any real numbers s,t; where $s > 0, t \neq 0$ and $s^2 + 8t > 0$.

From now on, for the sake of convenience we will utilize $\hat{j}_n$ instead of $\hat{j}_n(s,t)$ and $\hat{c}_n$ instead of $\hat{c}_n(s,t)$.

From (2.1) and (2.2) we get the first five elemets of (s,t)-Jacobsthal sequence are $\hat{j}_2 = s$, $\hat{j}_3 = s^2+2t$, $\hat{j}_4 = s^3+4st$, $\hat{j}_5 = s^4 + 6s^2t + 4t^2$, $\hat{j}_6 = s^5 + 8s^3t + 12st^2$. And similarly we get the first five (s,t)-Jacobsthal Lucas numbers are $\hat{c}_2 = s^2 + 4t$, $\hat{c}_3 = s^3 + 6st$, $\hat{c}_4 = s^4 + 8s^2t + 8t^2$, $\hat{c}_5 = s^5 + 10s^3t + 20st^2$, $\hat{c}_6 = s^6 + 12s^4t + 36s^2t^2 + 16t^3$.

Particular cases of the Definition 2.1 are:

- If $s = 1$, $t = 1/2$ and $\hat{j}_0(1,1/2) = 0$, $\hat{j}_1(1,1/2) = 1$,we get the Fibonacci sequence.
- If $s = 1$, $t = 1/2$ and $\hat{c}_0(1,1/2) = 2$, $\hat{c}_1(1,1/2) = 1$, we get the Lucas sequence.
- If $s = t = 1$ and $\hat{j}_0(1,1) = 0$, $\hat{j}_1(1,1) = 1$, we get the Jacobsthal sequence.
- If $s = t = 1$ and $\hat{c}_0(1,1) = 2$, $\hat{c}_1(1,1) = 1$, we get the Jacobsthal-Lucas sequence.

The characteristic equation for the recurrences (2.1) and (2.2) is

$$x^2 - sx - 2t = 0,$$

with roots

$$\alpha = \frac{s + \sqrt{s^2+8t}}{2}, \quad \beta = \frac{s - \sqrt{s^2+8t}}{2}.$$

From this result, we get

$$\alpha + \beta = s, \quad \alpha\beta = -2t, \quad \alpha - \beta = \sqrt{s^2+8t}.$$

Their Binet's formulas are given in [19] by

$$\hat{j}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.3}$$

and

$$\hat{c}_n = \alpha^n + \beta^n. \tag{2.4}$$

Lemma 2.2. *The summation of the squares of the first n terms of (s,t)-Jacobsthal sequences is demonstrated by:*

$$\sum_{i=0}^{n-1} \hat{j}_i^{\,2} = \frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{k,2n-2}-\hat{c}_{k,2n}-\hat{c}_{k,2}+2}{(2t-1)^2-s^2}+2\frac{(-2t)^n-1}{2t+1}\right). \tag{2.5}$$

Proof. By Binet formula for (s,t)-Jacobsthal sequence and the sum formula for the series we obtain,

$$\begin{aligned}\sum_{i=0}^{n-1} \hat{j}_i^{\,2} &= \sum_{i=0}^{n-1}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)^2 = \frac{1}{s^2+8t}\sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}-2(-2t)^i\right)\\ &= \frac{1}{s^2+8t}\left(\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}+2\frac{(-2t)^n-1}{2t+1}\right)\\ &= \frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}+2\frac{(-2t)^n-1}{2t+1}\right).\end{aligned}$$

□

Lemma 2.3. *The summation of the squares of the first n terms of (s,t)-Jacobsthal Lucas numbers is obtained as the following:*

$$\sum_{i=0}^{n-1} \hat{c}_i^{\,2} = \frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}+2\frac{(-2t)^n-1}{2t+1}. \tag{2.6}$$

Proof. By Binet formula for (s,t)-Jacobsthal Lucas sequence and the sum formula for the series we obtain,

$$\begin{aligned}\sum_{i=0}^{n-1} \hat{c}_i^{\,2} &= \sum_{i=0}^{n-1}(\alpha^i+\beta^i)^2 = \sum_{i=0}^{n-1}(\alpha^{2i}+\beta^{2i}-2(-2t)^i)\\ &= (\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}-2\frac{(-2t)^n-1}{2t+1} = \frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}.\end{aligned}$$

□

Definition 2.4. In [4], for positive integer r, Dil and Mezö defined hyper-Fibonacci numbers

$$F_n^{(r)} = \sum_{i=1}^{n} F_i^{(r-1)}$$

with $F_n^{(0)} = F_n, F_0^{(r)} = 0$, and $F_1^{(r)} = 1$. Similarly we define hyper-Jacobsthal numbers $J_n^{(r)}$, for positive integer r

$$J_n^{(r)} = \sum_{i=1}^{n} J_i^{(r-1)}$$

with $J_n^{(0)} = J_n, J_0^{(r)} = 0$, and $J_1^{(r)} = 1$.

Definition 2.5. In [17], for integer $n \geq 1$, Tuglu et al. defined harmonic Fibonacci numbers as $F_n = \sum\limits_{i=1}^{n}\frac{1}{F_i}$. Similarly, we define harmonic Jacobsthal numbers, displayed by $J_n = \sum\limits_{i=1}^{n}\frac{1}{J_i}$.

Definition 2.6. In [17], for integer $n, r \geq 1$, Tuglu et al. showed hyperharmonic Fibonacci numbers as

$$F_n^{(r)} = \sum_{i=1}^{n} F_i^{(r-1)}$$

where $F_n^{(0)} = \frac{1}{F_n}$, $F_0^{(r)} = 0$. Then the authors found the following inequalities

$$\frac{1}{\sqrt{n}} F_{n-1}^{(r+1)} \leq \sqrt{\sum_{i=0}^{n-1} (F_i^{(r)})^2} \leq F_{n-1}^{(r+1)}.$$

Definition 2.7. We expose the hyperharmonic Jacobsthal numbers associated by the Definition 2.6, for $n, r \geq 1$,

$$\mathfrak{I}_n^{(r)} = \sum_{i=0}^{n} \mathfrak{I}_i^{(r-1)},$$

where $\mathfrak{I}_n^{(0)} = \frac{1}{J_n}$, $\mathfrak{I}_0^{(r)} = 0$.

Definition 2.8. An $n \times n$ geometric circulant matrix C_{r^*} is defined as the following [20]

$$C_{r^*} = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ r^2 c_{n-2} & rc_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} c_1 & r^{n-2} c_2 & r^{n-3} c_3 & \dots & c_0 \end{bmatrix}.$$

For brevity, we denote the geometric circulant matrix with $C_{r^*} = circ(c_0, c_1, \dots, c_{n-1})$. If we choose $r = 1$, we get the circulant matrix.

Theorem 2.9. *Let $C = circ(\mathfrak{I}_0^{(r)}, \mathfrak{I}_1^{(r)}, \mathfrak{I}_2^{(r)}, \dots, \mathfrak{I}_{n-1}^{(r)})$ be an $n \times n$ circulant matrix with hyperharmonic Jacobsthal numbers. The spectral norm of C is demonsrated by*

$$\|C\|_2 = \mathfrak{I}_{n-1}^{(r+1)}. \tag{2.7}$$

Proof. Circulant matrices are normal so that the spectral norm of circulant C is equal to its spectral radius. Moreover, C is irreducible and its entries are nonnegative, we can say that the spectral radius of C is the same as its Perron root. Let ϑ be a vector with elements 1. Then

$$C\vartheta = \left(\sum_{i=0}^{n-1} \mathfrak{I}_i^{(r)} \right) \vartheta.$$

It is apparent that $\mathfrak{I}_{n-1}^{(r+1)}$ is an eigenvalue of C. ϑ is a positive eigenvector so, it must be the Perron root of the matrix C. The spectral norm of C is obtained easily by Definition 2.7. □

For any $A = [a_{ij}] \in M_{m,n}(C)$, the Frobenious (or Euclidean) norm of matrix A is displayed by the following equality:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \tag{2.8}$$

and the spectral norm of matrix A is also shown as

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)},$$

where A^H is the conjugate transpose of matrix A and $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$.

Corollary 2.10. *For the Euclidean norm of the circulant matrix C, as mentioned above, we get the following inequality*

$$\mathfrak{I}_{n-1}^{(r+1)} \leq \|C\|_E \leq \sqrt{n} \mathfrak{I}_{n-1}^{(r+1)}.$$

Corollary 2.11. *For the hyperharmonic Jacobsthal numbers, we obtain the following inequality*

$$\frac{1}{\sqrt{n}}\mathfrak{J}_{n-1}^{(r+1)} \leq \sqrt{\sum_{i=0}^{n-1}(\mathfrak{J}_i^{(r)})^2} \leq \mathfrak{J}_{n-1}^{(r+1)}. \tag{2.9}$$

Lemma 2.12. *Suppose that $A, B \in M_{m,n}(C)$, and the Hadamard product of A, B is the entrywise product defined by* [11, 12]

$$A \circ B = (a_{ij}b_{ij})$$

with

$$\|A \circ B\|_2 \leq \|A\|_2\|B\|_2$$

and $r_1(A)$, the maximum row length norm, $c_1(B)$, the maximum column length norm are given as $r_1(A) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2}$ and $c_1(B) = \max_{1\leq j\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2}$ with the following property

$$\|A \circ B\|_2 \leq r_1(A)c_1(B). \tag{2.10}$$

Lemma 2.13. *Let $A \in M_{m,n}(C)$ and $B \in M_{m,n}(C)$, then the Kronecker product of A, B is demonstrated by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

and has the following property [22]

$$\|A \otimes B\|_2 = \|A\|_2\|B\|_2.$$

Lemma 2.14. *Suppose that $A \in M_{m,n}(C)$, then the following inequalities are satisfied* [11]

$$\begin{aligned} \frac{1}{\sqrt{n}}\|A\|_F &\leq \|A\|_2 \leq \|A\|_F \\ \|A\|_2 &\leq \|A\|_F \leq \sqrt{n}\|A\|_2. \end{aligned} \tag{2.11}$$

3. Lower and Upper Bounds of Geometric Circulant Matrices Involving (s, t)-Jacobsthal, (s, t)-Jacobsthal-Lucas, Hyperharmonic Jacobsthal Numbers

Theorem 3.1. *Let $r \in \mathbb{C}$ and $J_{r^*} = circ_{r^*}(\hat{j}_0, \hat{j}_1, \dots, \hat{j}_{n-1})$ be an $n\times n$ geometric circulant matrix with (s,t)-Jacobsthal numbers, then the upper and lower bounds for spectral norm of J_{r^*} are obtained as follows.*

(i) If we choose $|r| \geq 1$, then

$$\sqrt{\frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)} \leq \|J_{r^*}\|_2.$$

$$\|J_{r^*1}\|_2 \leq \sqrt{\frac{(|r|^2 - |r|^{2n})}{(1-|r|^2)(s^2+8t)}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)}.$$

(ii) If we choose $|r| \leq 1$, then

$$\frac{|r|}{\sqrt{s^2+8t}}\sqrt{\left[\frac{2|r|^{2n+2} - \hat{c}_2|r|^{2n} - \hat{c}_{2n}|r|^2 + 4t^2\hat{c}_{2n-2}}{|r|^4 - \hat{c}_2|r|^2 + 4t^2} + 2\frac{(-2t)^n - |r|^{2n}}{2t+|r|^2}\right]} \leq \|J_{r^*}\|_2$$

$$\|J_{r^*1}\|_2 \leq \sqrt{\frac{n-1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)}.$$

Proof. The $n \times n$ geometric circulant matrix J_{r^*} is of the form

$$J_{r^*} = \begin{bmatrix} \hat{j}_0 & \hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_{n-1} \\ r\hat{j}_{n-1} & \hat{j}_0 & \hat{j}_1 & \dots & \hat{j}_{n-2} \\ r^2\hat{j}_{n-2} & r\hat{j}_{n-1} & \hat{j}_0 & \dots & \hat{j}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}\hat{j}_1 & r^{n-2}\hat{j}_2 & r^{n-3}\hat{j}_3 & \dots & \hat{j}_0 \end{bmatrix}.$$

(i) For $|r| \geq 1$, by using (2.5), (2.8) we have

$$\begin{aligned} \|J_{r^*}\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\hat{j}_k^2 + \sum_{k=1}^{n-1} k|r^{n-k}|^2 \hat{j}_k^2 \\ &\geq \sum_{k=0}^{n-1}(n-k)\hat{j}_k^2 + \sum_{k=1}^{n-1} k\hat{j}_k^2 = n\sum_{k=0}^{n-1}\hat{j}_k^2 \\ &= \frac{n}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right). \end{aligned}$$

From the equality (2.11),

$$\sqrt{\frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)} \leq \frac{1}{\sqrt{n}}\|J_{r^*}\|_F \leq \|J_{r^*}\|_2.$$

Otherwise, let $J_{r^*} = B \circ C$ where B and C are defined as

$$B = \begin{bmatrix} \hat{j}_0 & 1 & 1 & \dots & 1 \\ r & \hat{j}_0 & 1 & \dots & 1 \\ r^2 & r & \hat{j}_0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & \hat{j}_0 \end{bmatrix}, \quad C = \begin{bmatrix} \hat{j}_0 & \hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_{n-1} \\ \hat{j}_{n-1} & \hat{j}_0 & \hat{j}_1 & \dots & \hat{j}_{n-2} \\ \hat{j}_{n-2} & \hat{j}_{n-1} & \hat{j}_0 & \dots & \hat{j}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{j}_1 & \hat{j}_2 & \hat{j}_3 & \dots & \hat{j}_0 \end{bmatrix}.$$

By the maximum row and column length norm of these matrices,

$$\begin{aligned} r_1(B) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|b_{nj}|^2} = \sqrt{\hat{j}_0^{\,2} + |r|^2 + \ldots + |r^{n-1}|^2} = \sqrt{\frac{|r|^2 - |r|^{2n}}{1-|r|^2}} \\ c_1(C) &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|c_{in}|^2} = \sqrt{\frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)}. \end{aligned}$$

By using (2.10) we obtain

$$\|J_{r^*}\|_2 \leq r_1(B)c_1(C) = \sqrt{\frac{(|r|^{2n} - |r|^2)}{(|r|^2-1)(s^2+8t)}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)}.$$

The first part of the proof is completed.

(ii) For $|r| \leq 1$, by (2.5), (2.8) it is obtained that,

$$\begin{aligned}
\|J_{r^*}\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\hat{j}_k^2 + \sum_{k=1}^{n-1} k|r^{n-k}|^2 \hat{j}_k^2 \\
&\geq \sum_{k=0}^{n-1}(n-k)|r^{n-k}|^2 \hat{j}_k^2 + \sum_{k=1}^{n-1} k|r^{n-k}|^2 \hat{j}_k^2 = n|r|^{2n} \sum_{k=0}^{n-1}\left(\frac{\hat{j}_k}{|r|^k}\right)^2 \\
&= \frac{n|r|^{2n}}{s^2+8t}\left[\frac{\left(\frac{\alpha^2}{|r|^2}\right)^n - 1}{\left(\frac{\alpha^2}{|r|^2}\right)-1} + \frac{\left(\frac{\beta^2}{|r|^2}\right)^n - 1}{\left(\frac{\beta^2}{|r|^2}\right)-1} + 2\frac{\left(\frac{-2t}{|r|^2}\right)^n - 1}{\left(\frac{2t}{|r|^2}\right)+1}\right] \\
&= \frac{n|r|^2}{s^2+8t}\left[\frac{2|r|^{2n+2} - \hat{c}_2|r|^{2n} - \hat{c}_{2n}|r|^2 + 4t^2\hat{c}_{2n-2}}{|r|^4 - \hat{c}_2|r|^2 + 4t^2} + 2\frac{(-2t)^n - |r|^{2n}}{2t+|r|^2}\right].
\end{aligned}$$

From (2.11), we get

$$\frac{|r|}{s^2+8t}\sqrt{\left[\frac{2|r|^{2n+2} - \hat{c}_2|r|^{2n} - \hat{c}_{2n}|r|^2 + 4t^2\hat{c}_{2n-2}}{|r|^4 - \hat{c}_2|r|^2 + 4t^2} + 2\frac{(-2t)^n - |r|^{2n}}{2t+|r|^2}\right]} \leq \frac{\|J_{r^*}\|_F}{\sqrt{n}} \leq \|J_{r^*}\|_2.$$

On the other hand, let $J_{r^*} = B \circ C$ where B and C are given in the following forms

$$B = \begin{bmatrix} \hat{j}_0 & 1 & 1 & \dots & 1 \\ r & \hat{j}_0 & 1 & \dots & 1 \\ r^2 & r & \hat{j}_0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & \hat{j}_0 \end{bmatrix}, \quad C = \begin{bmatrix} \hat{j}_0 & \hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_{n-1} \\ \hat{j}_{n-1} & \hat{j}_0 & \hat{j}_1 & \dots & \hat{j}_{n-2} \\ \hat{j}_{n-2} & \hat{j}_{n-1} & \hat{j}_0 & \dots & \hat{j}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{j}_1 & \hat{j}_2 & \hat{j}_3 & \dots & \hat{j}_0 \end{bmatrix}.$$

By the maximum row and column length norm of these matrices,

$$\begin{aligned}
r_1(B) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\hat{j}_0^2 + (n-1)} = \sqrt{n-1} \\
c_1(C) &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}\hat{j}_k^2} = \sqrt{\frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)}.
\end{aligned}$$

By using (2.10), we obtain the second part of the proof

$$\|J_{r^*}\|_2 \leq r_1(B)c_1(C) = \sqrt{\frac{n-1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right)}.$$

□

Corollary 3.2. *Let $A = B = J_{r^*} = circ_{r^*}(\hat{j}_0, \hat{j}_1, \dots, \hat{j}_{n-1})$ be an $n \times n$ geometric circulant matrix with (s,t)-Jacobsthal numbers, then the upper and lower bounds for spectral norm of Kronecker product of A and B are shown with*

(i) If $|r| \geq 1$, then

$$\frac{1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right) \leq \|A \otimes B\|_2$$

$$\|A \otimes B\|_2 \leq \frac{(|r|^{2n} - |r|^2)}{(|r|^2 - 1)(s^2+8t)}\left(\frac{4t^2\hat{c}_{2n-2} - \hat{c}_{2n} - \hat{c}_2 + 2}{(2t-1)^2 - s^2} + 2\frac{(-2t)^n - 1}{2t+1}\right).$$

(ii) If $|r| \leq 1$, then

$$\frac{|r|^2}{s^2+8t}\left[\frac{2|r|^{2n+2}-\hat{c}_2|r|^{2n}-\hat{c}_{2n}|r|^2+4t^2\hat{c}_{2n-2}}{|r|^4-\hat{c}_2|r|^2+4t^2}+2\frac{(-2t)^n-|r|^{2n}}{2t+|r|^2}\right] \leq \|A\otimes B\|_2$$

$$\|A\otimes B\|_2 \leq \frac{n-1}{s^2+8t}\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}+2\frac{(-2t)^n-1}{2t+1}\right).$$

Proof. The proof is easily seen by using Lemma 2.12. □

Theorem 3.3. *Suppose that $r \in \mathbb{C}$ and $C_{r^*} = circ_{r^*}(\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n-1})$ be an $n \times n$ geometric circulant matrix with (s,t)-Jacobsthal Lucas numbers, then the bounds for spectral norm of C_{r^*} are obtained as*

(i) For $|r| \geq 1$,

$$\sqrt{\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}} \leq \|C_{r^*}\|_2$$

$$\|C_{r^*}\|_2 \leq \sqrt{\left(\frac{1-|r|^{2n}}{1-|r|^2}\right)\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}\right)}.$$

(ii) For $|r| \leq 1$,

$$|r|\sqrt{\left[\frac{2|r|^{2n+2}-\hat{c}_2|r|^{2n}-\hat{c}_{2n}|r|^2+4t^2\hat{c}_{2n-2}}{|r|^4-\hat{c}_2|r|^2+4t^2}-2\frac{(-2t)^n-|r|^{2n}}{2t+|r|^2}\right]} \leq \|C_{r^*}\|_2$$

$$\|C_{r^*}\|_2 \leq \sqrt{n\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}\right)}.$$

Proof. The matrix C_{r^*} is of the form

$$C_{r^*} = \begin{bmatrix} c_0 & c_1 & c_2 & \ldots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \ldots & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \ldots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \ldots & c_0 \end{bmatrix}.$$

(i) If $|r| \geq 1$, then by (2.6), (2.8) we have

$$\begin{aligned} \|C_{r^*}\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\hat{c}_k^2 + \sum_{k=1}^{n-1}k|r^{n-k}|^2\hat{c}_k^2 \\ &\geq n\sum_{k=0}^{n-1}\hat{c}_k^2 = n\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}\right). \end{aligned}$$

From (2.11)

$$\|C_{r^*}\|_2 \geq \frac{1}{\sqrt{n}}\|C_{r^*}\|_F \geq \sqrt{\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}}.$$

Otherwise, let $C_{r^*} = B \circ C$ where where B, C are given with the following forms Otherwise, let $J_{r^*} = B \circ C$ where B and C are defined as

$$B = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ r & 1 & 1 & \ldots & 1 \\ r^2 & r & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \ldots & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \hat{c}_0 & \hat{c}_1 & \hat{c}_2 & \ldots & \hat{c}_{n-1} \\ \hat{c}_{n-1} & \hat{c}_0 & \hat{c}_1 & \ldots & \hat{c}_{n-2} \\ \hat{c}_{n-2} & \hat{c}_{n-1} & \hat{c}_0 & \ldots & \hat{c}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \ldots & \hat{c}_0 \end{bmatrix}.$$

By the maximum row and column length norm of B and C matrices,

$$\begin{aligned} r_1(B) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{1+|r|^2+\ldots+|r^{n-1}|^2} = \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}} \\ c_1(C) &= \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}\hat{c}_k^2} = \sqrt{\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2} - 2\frac{(-2t)^n-1}{2t+1}}. \end{aligned}$$

By (2.10), we get

$$\|C_{r^*}\|_F^2 = \|B\circ C\|_2 \le \sqrt{\left(\frac{1-|r|^{2n}}{1-|r|^2}\right)\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2} - 2\frac{(-2t)^n-1}{2t+1}}.$$

(ii) For $|r|\le 1$,

$$\begin{aligned} \|C_{r^*}\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\hat{c}_k^2 + \sum_{k=1}^{n-1}k|r^{n-k}|^2\hat{c}_k^2 \\ &\ge \sum_{k=0}^{n-1}|r^{n-k}|^2(n-k)\hat{c}_k^2 + \sum_{k=1}^{n-1}k|r^{n-k}|^2\hat{c}_k^2 \\ &= n|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{\hat{c}_k}{|r|^k}\right)^2 \\ &= n|r|^{2n}\left(\sum_{k=0}^{n-1}\frac{\alpha^{2k}}{|r|^{2k}} + \sum_{k=0}^{n-1}\frac{\beta^{2k}}{|r|^{2k}} + 2\sum_{k=0}^{n-1}\frac{(-2t)^k}{|r|^{2k}}\right) \\ &= n|r|^{2n}\left[\frac{\left(\frac{\alpha^2}{|r|^2}\right)^n-1}{\left(\frac{\alpha^2}{|r|^2}\right)-1} + \frac{\left(\frac{\beta^2}{|r|^2}\right)^n-1}{\left(\frac{\beta^2}{|r|^2}\right)-1} - 2\frac{\left(\frac{-2t}{|r|^2}\right)^n-1}{\left(\frac{2t}{|r|^2}\right)+1}\right] \\ &= n|r|^2\left[\frac{2|r|^{2n+2}-\hat{c}_2|r|^{2n}-\hat{c}_{2n}|r|^2+4t^2\hat{c}_{2n-2}}{|r|^4-\hat{c}_2|r|^2+4t^2} - 2\frac{(-2t)^n-|r|^{2n}}{2t+|r|^2}\right]. \end{aligned}$$

From (2.11)

$$\begin{aligned} \|C_{r^*}\|_2 &\ge \frac{1}{\sqrt{n}}\|C_{r^*}\|_F \\ &\ge |r|\sqrt{\left[\frac{2|r|^{2n+2}-\hat{c}_2|r|^{2n}-\hat{c}_{2n}|r|^2+4t^2\hat{c}_{2n-2}}{|r|^4-\hat{c}_2|r|^2+4t^2} - 2\frac{(-2t)^n-|r|^{2n}}{2t+|r|^2}\right]}. \end{aligned}$$

Moreover, let $C_{r^*} = B\circ C$ where B, C are given in the following forms

$$B = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ r & 1 & 1 & \ldots & 1 \\ r^2 & r & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \ldots & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \hat{c}_0 & \hat{c}_1 & \hat{c}_2 & \ldots & \hat{c}_{n-1} \\ \hat{c}_{n-1} & \hat{c}_0 & \hat{c}_1 & \ldots & \hat{c}_{n-2} \\ \hat{c}_{n-2} & \hat{c}_{n-1} & \hat{c}_0 & \ldots & \hat{c}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \ldots & \hat{c}_0 \end{bmatrix}.$$

By the maximum row and column length norm of B and C,

$$r_1(B) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{n}$$

$$c_1(C) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}\hat{c}_k^2} = \sqrt{\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}}.$$

By (2.10), we get

$$\|C_{r^*}\|_2 = \|B\circ C\|_2 \le r_1(B)c_1(C) \le \sqrt{n\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}\right)}.$$

The proof is completed by this result.

□

Corollary 3.4. *Let $A = B = C_{r^*} = circ_{r^*}(\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n-1})$ be an $n\times n$ geometric circulant matrix with (s,t)-Jacobsthal Lucas numbers, then the upper and lower bounds for spectral norm of Kronecker product of A and B are held*

(i) For $|r|\ge 1$,

$$\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1} \le \|A\otimes B\|_2$$

$$\|A\otimes B\|_2 \le \frac{(1-|r|^{2n})}{(1-|r|^2)(s^2+8t)}\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}+2\frac{(-2t)^n-1}{2t+1}\right).$$

(ii) For $|r|\le 1$,

$$|r|^2\left[\frac{2|r|^{2n+2}-\hat{c}_2|r|^{2n}-\hat{c}_{2n}|r|^2+4t^2\hat{c}_{2n-2}}{|r|^4-\hat{c}_2|r|^2+4t^2}-2\frac{(-2t)^n-|r|^{2n}}{2t+|r|^2}\right] \le \|A\otimes B\|_2$$

$$\|A\otimes B\|_2 \le n\left(\frac{4t^2\hat{c}_{2n-2}-\hat{c}_{2n}-\hat{c}_2+2}{(2t-1)^2-s^2}-2\frac{(-2t)^n-1}{2t+1}\right).$$

Proof. The proof is easily seen using Lemma 2.12. □

Theorem 3.5. *Let $r\in\mathbb{C}$ and $\mathfrak{J}_{r^*}^{(k)} = C_{r^*}(\mathfrak{J}_0^{(k)}, \mathfrak{J}_1^{(k)}, \ldots, \mathfrak{J}_{n-1}^{(k)})$ be a $n\times n$ geometric circulant matrix with hyperharmonic Jacobsthal numbers, then the upper and lower bounds for spectral norm of C_{r^*} are computed as*

(i) For $|r|\ge 1$,

$$\frac{\mathfrak{J}_{n-1}^{(k+1)}}{\sqrt{n}} \le \|J_{r^*}^{(k)}\|_2 \le \mathfrak{J}_{n-1}^{(k+1)}\sqrt{\frac{|r|^2-|r|^{2n}}{1-|r|^2}}.$$

(ii) For $|r|\le 1$,

$$\frac{1}{n}\mathfrak{J}_{n-1}^{(k+1)} \le \|J_{r^*}^{(k)}\|_2 \le \sqrt{n-1}\,\mathfrak{J}_{n-1}^{(k+1)}.$$

Proof. The matrix $J_{r^*}^{(k)}$ is of the form

$$\mathfrak{J}_{r^*}^{(k)} = \begin{bmatrix} \mathfrak{J}_0^{(k)} & \mathfrak{J}_1^{(k)} & \mathfrak{J}_2^{(k)} & \cdots & \mathfrak{J}_{n-1}^{(k)} \\ r\mathfrak{J}_{n-1}^{(k)} & \mathfrak{J}_0^{(k)} & \mathfrak{J}_1^{(k)} & \cdots & \mathfrak{J}_{n-2}^{(k)} \\ r^2\mathfrak{J}_{n-2}^{(k)} & r\mathfrak{J}_{n-1}^{(k)} & \mathfrak{J}_0^{(k)} & \cdots & \mathfrak{J}_{n-3}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}\mathfrak{J}_1^{(k)} & r^{n-2}\mathfrak{J}_2^{(k)} & r^{n-3}\mathfrak{J}_3^{(k)} & \cdots & \mathfrak{J}_0^{(k)} \end{bmatrix}.$$

(i) For $|r| \geq 1$, we have from (2.8)

$$\begin{aligned}\|\mathfrak{I}_{r^*}^{(k)}\|_F^2 &= \sum_{i=0}^{n-1}(n-i)(\mathfrak{I}_i^{(k)})^2 + \sum_{i=1}^{n-1} i|r^{n-i}|^2(\mathfrak{I}_i^{(k)})^2 \\ &\geq \sum_{i=0}^{n-1}(n-i)(\mathfrak{I}_i^{(k)})^2 + \sum_{i=1}^{n-1} i(\mathfrak{I}_i^{(k)})^2 = n\sum_{i=0}^{n-1}(\mathfrak{I}_i^{(k)})^2.\end{aligned}$$

From the (2.8) and (2.11),

$$\frac{\mathfrak{I}_{n-1}^{(k+1)}}{\sqrt{n}} \leq \sqrt{\sum_{i=1}^{n-1}(\mathfrak{I}_i^{(k)})^2} \leq \frac{1}{\sqrt{n}}\|\mathfrak{I}_{r^*}^{(k)}\|_F \leq \|\mathfrak{I}_{r^*}^{(k)}\|_2.$$

Moreover, let $\mathfrak{I}_{r^*}^{(k)} = B^{(k)} \circ C^{(k)}$, where

$$B^{(k)} = \begin{bmatrix} \mathfrak{I}_0^{(k)} & 1 & 1 & \dots & 1 \\ r & \mathfrak{I}_0^{(k)} & 1 & \dots & 1 \\ r^2 & r & \mathfrak{I}_0^{(k)} & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & \mathfrak{I}_0^{(k)} \end{bmatrix}, \quad C^{(k)} = \begin{bmatrix} \mathfrak{I}_0^{(k)} & \mathfrak{I}_1^{(k)} & \mathfrak{I}_2^{(k)} & \dots & \mathfrak{I}_{n-1}^{(k)} \\ \mathfrak{I}_{n-1}^{(k)} & \mathfrak{I}_0^{(k)} & \mathfrak{I}_1^{(k)} & \dots & \mathfrak{I}_{n-2}^{(k)} \\ \mathfrak{I}_{n-2}^{(k)} & \mathfrak{I}_{n-1}^{(k)} & \mathfrak{I}_0^{(k)} & \dots & \mathfrak{I}_{n-3}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathfrak{I}_1^{(k)} & \mathfrak{I}_2^{(k)} & \mathfrak{I}_3^{(k)} & \dots & \mathfrak{I}_0^{(k)} \end{bmatrix}.$$

By the maximum row and column length norm of these matrices,

$$\begin{aligned} r_1(B^{(k)}) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|b_{nj}|^2} = \sqrt{(\mathfrak{I}_0^{(k)})^2 + |r|^2 + \ldots + |r^{n-1}|^2} = \sqrt{\frac{|r|^2 - |r|^{2n}}{1-|r|^2}} \\ c_1(C^{(k)} &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1}(\mathfrak{I}_i^{(k)})^2} \leq \mathfrak{I}_{n-1}^{(k+1)}.\end{aligned}$$

By (2.10) it is computed that

$$\|\mathfrak{I}_{r^*}^{(k)}\|_2 \leq r_1(B^{(k)})c_1(C^{(k)}) = \mathfrak{I}_{n-1}^{(k+1)}\sqrt{\frac{|r|^2 - |r|^{2n}}{1-|r|^2}}.$$

The proof is completed for the first part.

(ii) For $|r| \leq 1$, using (2.11) we have,

$$\begin{aligned}\|\mathfrak{I}_{r^*}^{(k)}\|_F^2 &= \sum_{i=0}^{n-1}(n-i)(\mathfrak{I}_i^{(k)})^2 + \sum_{i=1}^{n-1} i|r^{n-i}|^2(\mathfrak{I}_i^{(k)})^2 \\ &\geq \sum_{k=0}^{n-1}(n-i)(\mathfrak{I}_i^{(k)})^2 \geq \sum_{k=0}^{n-1}(\mathfrak{I}_i^{(k)})^2.\end{aligned}$$

From (2.8),

$$\frac{1}{\sqrt{n}}\mathfrak{I}_{n-1}^{(k+1)} \leq \sqrt{\sum_{i=0}^{n-1}(\mathfrak{I}_i^{(k)})^2} \leq \|\mathfrak{I}_{r^*}^{(k)}\|_F \leq \sqrt{n}\|\mathfrak{I}_{r^*}^{(k)}\|_2$$

$$\frac{1}{n}\mathfrak{I}_{n-1}^{(k+1)} \leq \|\mathfrak{I}_{r^*}^{(k)}\|_2.$$

Conversely, let $\mathfrak{I}_{r^*}^{(k)} = B^{(k)} \circ C^{(k)}$ where

$$B^{(k)} = \begin{bmatrix} \mathfrak{I}_0^{(k)} & 1 & 1 & \dots & 1 \\ r & \mathfrak{I}_0^{(k)} & 1 & \dots & 1 \\ r^2 & r & \mathfrak{I}_0^{(k)} & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & \mathfrak{I}_0^{(k)} \end{bmatrix}, \quad C^{(k)} = \begin{bmatrix} \mathfrak{I}_0^{(k)} & \mathfrak{I}_1^{(k)} & \mathfrak{I}_2^{(k)} & \dots & \mathfrak{I}_{n-1}^{(k)} \\ \mathfrak{I}_{n-1}^{(k)} & \mathfrak{I}_0^{(k)} & \mathfrak{I}_1^{(k)} & \dots & \mathfrak{I}_{n-2}^{(k)} \\ \mathfrak{I}_{n-2}^{(k)} & \mathfrak{I}_{n-1}^{(k)} & \mathfrak{I}_0^{(k)} & \dots & \mathfrak{I}_{n-3}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathfrak{I}_1^{(k)} & \mathfrak{I}_2^{(k)} & \mathfrak{I}_3^{(k)} & \dots & \mathfrak{I}_0^{(k)} \end{bmatrix}.$$

By the maximum row and column length norm of preceding matrices,

$$\begin{aligned} r_1(B^{(k)}) &= \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{(\mathfrak{I}_0^{(k)})^2 + n - 1} = \sqrt{n-1} \\ c_1(C^{(k)} &= \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1} (\mathfrak{I}_s^{(k)})^2} \le \mathfrak{I}_{n-1}^{(k+1)}. \end{aligned}$$

By (2.10), we see the desired result

$$\|\mathfrak{I}_{r^*}^{(k)}\|_2 \le r_1(B^{(k)}) c_1(C^{(k)}) = \sqrt{(n-1)}\mathfrak{I}_{n-1}^{(k+1)}.$$

□

4. Conclusion

In this study we compute upper and lower bounds of spectral norms of geometric circulant matrices with the (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas and hyperharmonic Jacobsthal numbers. The materials of this paper are geometric circulant matrices, a new different generalization of Jacobsthal and Jacobsthal Lucas sequences. If we take $s = t = 1$, we get upper and lower bounds of spectral norms of geometric circulant matrices with the Jacobsthal and Jacobsthal Lucas numbers.

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