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Invariant submanifold of generalized Sasakian space form with semi-symmetric metric connection

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Abstract

In this paper, we obtain necessary and sufficient condition for an Invariant submanifold of generalized Sasakian space form with semi-symmetric metric connections to be totally geodesic.

Keywords: Invariant submanifolds, generalized Sasakian space form, totally geodesic, semi-symmetric metric connection

2010 MSC: 53C15, 53C21, 53C25, 53C40

1. Introduction

In differential geometry, The theory of Invariant submanifold has been alluring field of research for a long time. The generalized Sasakian space forms (G.S.S.F.) have been investigated by numerous researchers like Alegre and Carriazo [1, 2, 3]. Thereafter generalized Sasakian spaceform have been study by many authors [4, 10, 15]. The conception of a semi-symmetric metric connection on a Riemannian manifold is introduced by H. A. Hayden [9] and studied by various authors [14, 17, 18] and [19]. Submanifolds of a Riemannian manifold with semi-symmetric metric connection was studied by Z. Nakao [16] and Invariants of a submanifold which was established by B. Y. Chen [6, 7, 8, 11, 12] and [13] .

In this paper, we obtain necessary and sufficient condition for an Invariant submanifold of generalized Sasakian space form with semi-symmetric metric connections to be totally geodesic. We have considered many geometrical conditions by using different curvature tensors such as pseudo projective, quasi conformal and *m*-projective curvature tensor on Invariant submanifold of generalized Sasakian space form (Invarnt.submanfold.of a G.S.S.F.) with semi-symmetric metric connection.

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An almost contact metric manifold $\overline{M}$ is called generalized Sasakian space form if there exist three functions f_1, f_2, f_3 on $\overline{M}$ such that the curvature tensor $\overline{R}$ is given by

$$\begin{aligned}\overline{R}(u_1,u_2)u_3 =& f_1\{g(u_2,u_3)u_1 - g(u_1,u_3)u_2\} + f_2\{g(u_1,\phi u_3)\phi u_2 \\ &- g(u_2,\phi u_3)\phi u_1 + 2g(u_1,\phi u_2)\phi u_3\} + f_3\{\eta(u_1)\eta(u_3)u_2 \\ &- \eta(u_2)\eta(u_3)u_1 + g(u_1,u_3)\eta(u_2)\xi - g(u_2,u_3)\eta(u_1)\xi\},\end{aligned} \tag{1.1}$$

for all vector fields u_1, u_2, u_3 on $\overline{M}$, where $\overline{R}$ is the curvature tensor of $\overline{M}$ of dimension $(2n+1)$. It is denoted as $\overline{M}(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$ then a generalized Sasakian space-form with Sasakian structure becomes Sasakian space-from.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)$ dimensional manifold equipped with almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\eta(\xi) = 1, \eta(u_1) = g(u_1,\xi), \phi^2 u_1 = -u_1 + \eta(u_1)\xi, \phi\xi = 0, \tag{2.1}$$

$$g(\phi u_1, \phi u_2) = g(u_1,u_2) - \eta(u_1)\eta(u_2), \tag{2.2}$$

$$g(\phi u_1, u_2) + g(u_1, \phi u_2) = 0, \eta(\phi u_1) = 0 \tag{2.3}$$

for all vector fields u_1, u_2.
In a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$, the following hold:

$$(\overline{\nabla}_{u_1}\phi)u_2 = (f_1 - f_3)[g(u_1,u_2)\xi - \eta(u_2)u_1], \tag{2.4}$$

$$\overline{\nabla}_{u_1}\xi = -(f_1 - f_3)\phi u_1, \tag{2.5}$$

$$\overline{S}(u_1,u_2) = (2nf_1 + 3f_2 - f_3)g(u_1,u_2) - \{3f_2 + (2n-1)f_3\}\eta(u_1)\eta(u_2), \tag{2.6}$$

for all u_1, u_2, u_3 on $\overline{M}^{2n+1}$ and $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ and $\overline{S}$ is the Ricci tensor and $\bar{r}$ is the scalar curvature of $\overline{M}$.
Let M be a submanifold immersed in a $(2n+1)$ dimensional contact metric manifold $\overline{M}$ induced with metric g. TM is the tangent bundle of the manifold M and $T^{\perp}M$ is the set of vector fields normal to M.
Gauss and Weingarten formula are given by,

$$\overline{\nabla}_{u_1}u_2 = \nabla_{u_1}u_2 + h(u_1,u_2), \quad \overline{\nabla}_{u_1}N = \nabla^{\perp}_{u_1}N - A_N u_1, \tag{2.7}$$

for any $u_1, u_2 \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle. The second fundamental form h and A_N are related by

$$g(A_N u_1, u_2) = g(h(u_1,u_2), N) \tag{2.8}$$

for any $u_1, u_2 \in \Gamma(TM)$, $N \in T^{\perp}M$.
If $h = 0$, then the submanifold is said to be totally geodesic which implies that the geodesics in M are also geodesics in $\overline{M}$.
Also we denote $Q(U,T)$ a $(0, k+2)$-type tensor field defined as follows

$$\begin{aligned}Q(U,T)(X_1, X_2, ...X_k; u_1, u_2) = -T((u_1 \wedge_U u_2)X_1, X_2, ...X_k) \\ - T(X_1, (u_1 \wedge_U u_2)X_2, ...X_k) - ... - T(X_1, X_2, ...X_{k-1}, (u_1 \wedge_U u_2)X_k)\end{aligned} \tag{2.9}$$

where $(u_1 \wedge_U u_2)u_3 = U(u_2,u_3)u_1 - U(u_1,u_3)u_2$.
A submanifold is said to be pseudo-parallel if

$$\overline{R}(u_1,u_2).h = fQ(g,h). \tag{2.10}$$

In an Invariant submanifold of a generalized Sasakian space form N is identically zero.

$$h(u_1, \xi) = 0 \tag{2.11}$$

for any vector field u_1 tangent to M.

Definition 2.1. In a $(2n + 1)$ dimensional Riemannian manifold,the pseudo projective curvature tensor $\overline{P}$, quasi conformal curvature tensor $\overline{C}$ and m-projective curvature tensor $\overline{W}$ are defined as follows:

$$\overline{P}(u_1, u_2)u_3 = a\overline{R}(u_1, u_2)u_3 + b[\overline{S}(u_2, u_3)u_1 - \overline{S}(u_1, u_3)u_2] - \left(\frac{\overline{r}}{2n+1}\right)\left(\left(\frac{a}{2n}\right) + b\right)[g(u_2, u_3)u_1 - g(u_1, u_3)u_2]. \tag{2.12}$$

$$\begin{aligned}\overline{C}(u_1, u_2)u_3 &= a\overline{R}(u_1, u_2)u_3 + b[\overline{S}(u_2, u_3)u_1 - \overline{S}(u_1, u_3)u_2 + g(u_2, Z)\overline{Q}u_1 \\ &- g(u_1, u_3)\overline{Q}u_2] - \left(\frac{\overline{r}}{2n+1}\right)\left(\left(\frac{a}{2n}\right) + 2b\right)[g(u_2, u_3)u_1 - g(u_1, u_3)u_2],\end{aligned} \tag{2.13}$$

$$\overline{W}(u_1, u_2)u_3 = \overline{R}(u_1, u_2)u_3 - \frac{1}{4n}[\overline{S}(u_2, u_3)u_1 - \overline{S}(u_1, u_3)u_2 + g(u_2, u_3)\overline{Q}u_1 - g(u_1, u_3)\overline{Q}u_2] \tag{2.14}$$

where a and b are constants.

A Semi-symmetric connection $\tilde{\overline{\nabla}}$ is called semi-symmetric metric connection if it satisfies $\tilde{\overline{\nabla}}g = 0$.

The connection among the Semi symmetric metric connection $\tilde{\overline{\nabla}}$ and the Riemannian connection $\overline{\nabla}$ of a G.S.S.F. $\overline{M}^{2n+1}(f_1, f_2, f_3)$ is given by

$$\tilde{\overline{\nabla}}_{u_1}u_2 = \overline{\nabla}_{u_1}u_2 + \eta(u_2)u_1 - g(u_1, u_2)\xi, \tag{2.15}$$

$$\tilde{\overline{R}}(u_1, u_2)u_3 = \overline{R}(u_1, u_2)u_3 - \alpha(u_2, u_3)u_1 + \alpha(u_1, u_3)u_2 + g(u_2, u_3)Ju_1 + g(u_1, u_3)Ju_2, \tag{2.16}$$

where, α is a $(0, 2)$ tensor field given by

$$\alpha(u_1, u_2) = (\tilde{\overline{\nabla}}_{u_1}\eta)u_2 + \frac{1}{2}g(u_1, u_2), \tag{2.17}$$

$$g(Ju_1, u_2) = g(\tilde{\overline{\nabla}}_{u_1}\xi, u_2) + \frac{1}{2}g(u_1, u_2) = \alpha(u_1, u_2) \tag{2.18}$$

$$\tilde{\overline{S}}(u_1, u_2) = \overline{S}(u_1, u_2) - (2n-1)\alpha(u_1, u_2) - cg(u_1, u_2), \tag{2.19}$$

where $c = trace(\alpha)$, $\tilde{\overline{R}}$, $\tilde{\overline{S}}$ and $\tilde{\overline{r}}$ are the curvature tensor, Ricci tensor and scalar curvature with respect to semi symmetric metric connection $\tilde{\overline{\nabla}}$. Further, $\overline{R}$,$\overline{S}$ and $\overline{r}$ are the Ricci tensor and scalar curvature of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to Levi-Civita connection respectively.

3. Invarnt.submanfold.of a G.S.S.F. satisfying $\overline{P}(u_1, u_2).h = fQ(g, h)$

Theorem 3.1. *Let M be an Invarnt.submanfold. of a G.S.S.F. $\overline{M}$ with semi-symmetric metric connection. Then M satisfies $\overline{P}(u_1, u_2).h = fQ(g, h)$ iff M is totally geodesic provided*

$$f \neq \left[a\left(f_1 - f_3 - \frac{1}{2}\right) + a\phi(f_1 - f_3) - \frac{3}{2}a + b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right]. \tag{3.1}$$

Proof. Let M be Invarnt.submanfold. of a G.S.S.F. $\overline{M}$ with semi-symmetric metric connection satisfying

$$\overline{P}(u_1, u_2).h = fQ(g, h), \tag{3.2}$$

$\forall u_1, u_2$ tangent to M, where f stands for the real valued function on M. Mathematical statement (3.2) can be drafted as,

$$R^{\perp}(u_1, u_2)h(v_1, v_2) - h(\overline{P}(u_1, u_2)v_1, v_2) - h(v_1, \overline{P}(u_1, u_2)v_2) = -f[h((u_1 \wedge_g u_2), v_2) + h(v_1, (u_1 \wedge_g u_2)v_2)]. \tag{3.3}$$

Put $u_1 = v_2 = \xi$ and using (2.9), (2.11) we get,

$$-h(\overline{P}(\xi, u_2)v_1, \xi) - h(v_1, \overline{P}(\xi, u_2)\xi) = f[h(v_1, u_2)]. \tag{3.4}$$

By virtue of (2.12), (2.11), (2.15), (2.16), (2.17), (2.18) and (2.19) we obtain

$$h(\overline{P}(\xi, u_2)v_1, \xi) = 0. \tag{3.5}$$

$$-h(v_1, \overline{P}(\xi, u_2)\xi) = \left[a\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2}a + b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right]h(v_1, u_2). \tag{3.6}$$

Substitute (3.5) and (3.6) in (3.4) we have

$$\left[a\left(f_1 - f_3 - \frac{1}{2}\right) + a\phi(f_1 - f_3) - \frac{3}{2}a + b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right]h(v_1, u_2) = f[h(v_1, u_2)]. \tag{3.7}$$

That is, $h(v_1, u_2) = 0$ implies M is totally geodesic provided,

$$f \neq \left[a\left(f_1 - f_3 - \frac{1}{2}\right) + a\phi(f_1 - f_3) - \frac{3}{2}a + b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right]. \tag{3.8}$$

Conversely, If M is totally geodesic, then we obtain M comply with $\overline{P}(u_1, u_2).h = fQ(g, h)$. □

4. Invarnt.submanfold. of a G.S.S.F. satisfying $\overline{C}(u_1, u_2).h = fQ(g, h)$

Theorem 4.1. *Let M be an Invarnt.submanfold. of a G.S.S.F. $\overline{M}$ with semi-symmetric metric connection. Then M satisfies $\overline{C}(u_1, u_2).h = fQ(g, h)$ iff M is totally geodesic provided*

$$f \neq \left[a\left(f_1 - f_3 - \frac{1}{2} + \phi(f_1 - f_3)\right) - \frac{3}{2}a + 2b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + 2b\right)\right]. \tag{4.1}$$

Proof. Let M be an Invarnt.submanfold. of a G.S.S.F. with semi-symmetric metric connection satisfying

$$\overline{C}(u_1, u_2).h = fQ(g, h), \tag{4.2}$$

equation (4.2) follows,

$$R^{\perp}(u_1, u_2)h(v_1, v_2) - h(\overline{C}(u_1, u_2)v_1, v_2) - h(v_1, \overline{C}(u_1, u_2)v_2) = -f[h((u_1 \wedge_g u_2), v_2) + h(v_1, (u_1 \wedge_g u_2)v_2)]. \tag{4.3}$$

By Taking $u_1 = v_2 = \xi$ and use (2.11), (2.9) we come by,

$$-h(\overline{C}(\xi, u_2)v_1, \xi) - h(v_1, \overline{C}(\xi, u_2)\xi) = f[h(v_1, u_2)]. \tag{4.4}$$

By virtue of (2.13), (2.11), (2.15), (2.16), (2.17), (2.18) and (2.19) we get

$$h(\overline{C}(\xi, u_2)v_1, \xi) = 0. \tag{4.5}$$

$$\begin{aligned}-h(v_1,\overline{C}(\xi,u_2)\xi) &= \Big[a\Big(f_1-f_3-\frac{1}{2}\Big)+\phi(f_1-f_3)a-\frac{3}{2}a\\ &+2b\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)-\Big(\frac{r}{2n+1}\Big)\Big(\frac{a}{2n}+2b\Big)\Big]h(v_1,u_2).\end{aligned} \tag{4.6}$$

Substitute (4.5) and (4.6) in (4.4) we get,

$$\Big[a\Big(f_1-f_3-\frac{1}{2}+\phi(f_1-f_3)\Big)-\frac{3}{2}a+2b\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)-\Big(\frac{r}{2n+1}\Big)\Big(\frac{a}{2n}+2b\Big)\Big]h(v_1,u_2)=f[h(v_1,u_2)]. \tag{4.7}$$

That is, $h(v_1,u_2)=0$ implies M is totally geodesic provided,

$$f\neq\Big[a\Big(f_1-f_3-\frac{1}{2}+\phi(f_1-f_3)\Big)-\frac{3}{2}a+2b\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)-\Big(\frac{r}{2n+1}\Big)\Big(\frac{a}{2n}+2b\Big)\Big]. \tag{4.8}$$

Conversely, if M be totally geodesic, then we obtain M comply with $\overline{C}(u_1,u_2).h=fQ(g,h)$. □

5. Invarnt.submanfold. of a G.S.S.F. satisfying $\overline{W}(u_1,u_2).h=fQ(g,h)$

Theorem 5.1. *Let M be an Invarnt.submanfold. of a G.S.S.F. $\overline{M}$. with semi-symmetric connection. Then M satisfies $\overline{W}(u_1,u_2).h=fQ(g,h)$ iff M is totally geodesic, provided,*

$$f\neq\Big[\Big(f_1-f_3-\frac{1}{2}\Big)+\phi(f_1-f_3)-\frac{3}{2}-\frac{1}{2n}\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)\Big]. \tag{5.1}$$

Proof. Let M be an Invarnt.submanfold. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{W}(u_1,u_2).h=fQ(g,h), \tag{5.2}$$

(5.2) come after,

$$R^{\perp}(u_1,u_2)h(v_1,v_2)-h(\overline{W}(u_1,u_2)v_1,v_2)-h(v_1,\overline{W}(u_1,u_2)v_2)=-f[h((u_1\wedge_g u_2),v_2)+h(v_1,(u_1\wedge_g u_2)v_2)]. \tag{5.3}$$

Put $u_1=v_2=\xi$ and using (2.9), (2.11) we obtain,

$$-h(\overline{W}(\xi,u_2)v_1,\xi)-h(v_1,\overline{W}(\xi,u_2)\xi)=f[h(v_1,u_2)]. \tag{5.4}$$

By virtue of (2.14), (2.11), (2.15), (2.16), (2.17), (2.18) and (2.19) we have

$$h(\overline{W}(\xi,u_2)v_1,\xi)=0. \tag{5.5}$$

$$-h(v_1,\overline{W}(\xi,u_2)\xi)=\Big[\Big(f_1-f_3-\frac{1}{2}\Big)+\phi(f_1-f_3)-\frac{3}{2}-\frac{1}{2n}\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)\Big]h(v_1,u_2). \tag{5.6}$$

Substitute (5.5) and (5.6) in (5.4) we get,

$$\Big[\Big(f_1-f_3-\frac{1}{2}\Big)+\phi(f_1-f_3)-\frac{3}{2}-\frac{1}{2n}\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)\Big]h(v_1,u_2=f[h(v_1,u_2)]. \tag{5.7}$$

That is, $h(v_1,u_2)=0$ implies M is totally geodesic provided,

$$f\neq\Big[\Big(f_1-f_3-\frac{1}{2}\Big)+\phi(f_1-f_3)-\frac{3}{2}-\frac{1}{2n}\Big(2n(f_1-f_3)-c-n+\frac{1}{2}\Big)\Big]. \tag{5.8}$$

Conversely, If M is totally geodesic, then we obtain M comply with $\overline{W}(u_1,u_2).h=fQ(g,h)$. □

6. Invarnt.submanfold. of a G.S.S.F. satisfying $\overline{P}(u_1, u_2).h = fQ(S, h)$

Theorem 6.1. *Let M be an Invarnt.submanfold. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{P}(u_1, u_2).h = fQ(S, h)$ iff M is totally geodesic provided,*

$$f \neq \frac{1}{2n(f_1 - f_3) - c - n + \frac{1}{2}}\left[a\left(f_1 - f_3 - \frac{1}{2}\right) + a\phi(f_1 - f_3) - \frac{3}{2}a - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right] + b. \tag{6.1}$$

Proof. Let M be an Invarnt.submanfold. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{P}(u_1, u_2).h = fQ(S, h), \tag{6.2}$$

For all vector fields u_1, u_2 tangent to M, where f stands for the real valued function on M^{2n+1}, (6.2) follows

$$R^{\perp}(u_1, u_2)h(v_1, v_2) - h(\overline{P}(u_1, u_2)v_1, v_2) - h(v_1, \overline{P}(u_1, u_2)v_2) = -f[h((u_1 \wedge_S u_2), v_2) + h(v_1, (u_1 \wedge_S u_2)v_2)]. \tag{6.3}$$

Using (2.9), (2.11), (2.19), (2.12) and also take $u_1 = v_2 = \xi$ we get,

$$\begin{aligned}\Bigg[&a\left(f_1 - f_3 - \frac{1}{2}\right) + a\phi(f_1 - f_3) - \frac{3}{2}a + b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)\\ &-\left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right) - f(\tilde{\overline{S}}(\xi, \xi))\Bigg]h(v_1, u_2) = 0.\end{aligned} \tag{6.4}$$

Substituting from (2.19) and on simplification we have $h(v_1, u_2) = 0$ implies M is totally geodesic provided,

$$f \neq \frac{1}{2n(f_1 - f_3) - c - n + \frac{1}{2}}\left[a\left(f_1 - f_3 - \frac{1}{2}\right) + a\phi(f_1 - f_3) - \frac{3}{2}a - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right] + b. \tag{6.5}$$

□

7. Invarnt.submanfold. of a G.S.S.F. satisfying $\overline{C}(u_1, u_2).h = fQ(S, h)$

Theorem 7.1. *Let M be an Invarnt.submanfold. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{C}(u_1, u_2).h = fQ(S, h)$ iff M is totally geodesic provided,*

$$f \neq \frac{1}{2n(f_1 - f_3) - c - n + \frac{1}{2}}\left[a\left(f_1 - f_3 - \frac{1}{2} + \phi(f_1 - f_3)\right) - \frac{3}{2}a - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right] + 2b. \tag{7.1}$$

Proof. Let M be an Invarnt.submanfold. of a G.S.S.F. with semi-symmetric satisfying

$$\overline{C}(u_1, u_2).h = fQ(S, h), \tag{7.2}$$

(7.2) can be written as

$$R^{\perp}(u_1, u_2)h(v_1, v_2) - h(\overline{C}(u_1, u_2)v_1, v_2) - h(v_1, \overline{C}(u_1, u_2)v_2) = -f[h((u_1 \wedge_S u_2), v_2) + h(v_1, (u_1 \wedge_S u_2)v_2)]. \tag{7.3}$$

Using (2.9), (2.11), (2.19), (2.13) and also put $u_1 = v_2 = \xi$, we come by

$$\left[a\left(f_1 - f_3 - \frac{1}{2} + \phi(f_1 - f_3)\right) - \frac{3}{2}a + 2b\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + 2b\right) - f(\tilde{\overline{S}}(\xi, \xi))\right]h(v_1, u_2) = 0. \tag{7.4}$$

That is, $h(v_1, u_2) = 0$ implies M is totally geodesic provided,

$$f \neq \frac{1}{2n(f_1 - f_3) - c - n + \frac{1}{2}}\left[a\left(f_1 - f_3 - \frac{1}{2} + \phi(f_1 - f_3)\right) - \frac{3}{2}a - \left(\frac{r}{2n+1}\right)\left(\frac{a}{2n} + b\right)\right] + 2b. \tag{7.5}$$

□

8. Invarnt.submanfold. of a G.S.S.F. satisfying $\overline{W}(u_1, u_2).h = fQ(S, h)$

Theorem 8.1. *Let M be an Invarnt.submanfold. of a G.S.S.F. $\overline{M}$ with semi-symmetric connection. Then M satisfies $\overline{W}(u_1, u_2).h = fQ(S, h)$ iff M is totally geodesic provided,*

$$f \neq \frac{1}{\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)}\left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2}\right] - \frac{1}{2n}. \tag{8.1}$$

Proof. Let M be an Invarnt.submanfold. of a G.S.S.F. with semi-symmetric connection satisfying

$$\overline{W}(u_1, u_2).h = fQ(S, h), \tag{8.2}$$

(8.2) can be written as

$$R^{\perp}(u_1, u_2)h(v_1, v_2) - h(\overline{W}(u_1, u_2)v_1, v_2) - h(v_1, \overline{W}(u_1, u_2)v_2) = -f[h((u_1 \wedge_S u_2), v_2) + h(v_1, (u_1 \wedge_S u_2)v_2)]. \tag{8.3}$$

By Putting $u_1 = v_2 = \xi$ and use (2.9), (2.11), (2.19), (2.14), we acquire,

$$\left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2} - \frac{1}{2n}\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right) - f(\tilde{\overline{S}}(\xi, \xi))\right]h(v_1, u_2) = 0. \tag{8.4}$$

On substituting (2.19) we have $h(v_1, u_2) = 0$ implies M^{2n+1} is totally geodesic provided,

$$f \neq \frac{1}{\left(2n(f_1 - f_3) - c - n + \frac{1}{2}\right)}\left[\left(f_1 - f_3 - \frac{1}{2}\right) + \phi(f_1 - f_3) - \frac{3}{2}\right] - \frac{1}{2n}. \tag{8.5}$$

□

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