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On generalizations of Tribonacci numbers

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Abstract

In this article, some generalizations of the results in the literature are obtained by using the sequences over an integral domain with the help of matrix method. Then some generalizations for the d'Ocagne identity, the Honsberger's formula, the Cassini's identity, the Catalan's identity are given. Finally, Binet formulas of sequences in the literature are unified in a theorem.

Keywords: Integral domain, Tribonacci numbers, Tribonacci sequences

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1. Introduction

After being defined the Fibonacci numbers in 13th century, it has been in the center of interest of most of researchers. Then a lot of researchers have generalized the Fibonacci numbers to the Fibonacci polynomials, the Pell numbers and polynomials, the Lucas numbers and polynomials, the Jacobstal numbers and polynomials, the Pell-Lucas numbers and polynomials, the Jacobstal-Lucas numbers and polynomials, the Chebyshev numbers and polynomials, which are derived from the Fibonacci numbers ([2, 3, 5, 7, 16, 17, 19, 24, 25, 27]).

In the ring theory, it is well-known that the polynomial rings with finite indeterminates over integral domains are an integral domain. Thus in this article, we deal with the generalizations of the Tribonacci sequences over an integral domain and examine them with the help of the matrix method.

In 1965, Horadam defined the sequences of numbers, which are called as Horodam numbers, on rational numbers in [7] as follows:

$$h_n = ph_{n-1} - qh_{n-2} \ (n \geq 3)$$

with the initial conditions $h_1 = a$ and $h_2 = bx$.

The Tribonacci numbers in [12] was defined as $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$ with the initial conditions $T_1 = T_2 = 1$ and $T_3 = 2$ and the Tribonacci numbers were generalized to the Tribonacci polynomials in [22].

Definition 1.1 (*cf.* [18])**.** Let p, q, r, a and b be elements of an integral domain R. Then, for a positive integer $n \geqslant 2$, the recurrence relations are defined as

$$w_{n+1} = pw_n + qw_{n-1} + rw_{n-2},$$

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where $w_0 = a, w_1 = b, w_2 = pw_1 + qw_0$ and it is said that

$$w = \{w_{n+1} = pw_n + qw_{n-1} + rw_{n-2} : n \in \mathbb{Z}^+, n \geqslant 2\}$$

is the sequence of p, q, r with a, b on R denoted by $w_n(a, b, p, q, r)$ or briefly $\{w_n\}$.

It is clear that Definition 1.1 generalizes not only the Horadam numbers and polynomials in [7], but also the Tribonacci numbers and polynomials, the Tricobstal numbers and polynomials. Moreover, when $p \neq 0$, $q \neq 0$ and $r = 0$, then Definition 1.1 is equivalent to the definition in [1].

Now, we fix the notations in Definition 1.1.

Remark 1.2. Let $R = \mathbb{Z}[x, y]$. If $p = x, q = y$, $r = 0$, $a = 0$ and $b = 1$, then $w_n(x, y)$ is the n-th Fibonacci polynomial with two variables for positive integer n.

Fibonacci numbers	Fibonacci polynomials with two variables
0	$w_0(x, y) = 0$
1	$w_1(x, y) = 1$
1	$w_2(x, y) = x$
2	$w_3(x, y) = x^2 + y$
3	$w_4(x, y) = x^3 + 2xy$
5	$w_5(x, y) = x^4 + 3x^2y + y^2$
8	$w_6(x, y) = x^5 + 4x^3y + 3xy^2$
13	$w_7(x, y) = x^6 + 5x^4y + 6x^2y^2 + y^3$

Example 1.3. Let $p = x^2, q = x$ and $r = 1$ be elements of an integral domain $R = \mathbb{Z}[x]$. Then we recall that $w_n(x)$ is called Tribonacci polynomials when $a = 0$ and $b = 1$.

Tribonacci numbers	Tribonacci polynomials
0	$w_0(x) = 0$
1	$w_1(x) = 1$
1	$w_2(x) = x^2$
2	$w_3(x) = x^4 + x$
4	$w_4(x) = x^6 + 2x^3 + 1$
7	$w_5(x) = x^8 + 3x^5 + 3x^2$
13	$w_6(x) = x^{10} + 4x^7 + 6x^4 + 2x$
24	$w_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1$

Remark 1.4 (*cf.* [18]). Let R be the ring of integers ($R = \mathbb{Z}$). Then we have

p	q	r	a	b	w_n is sequence of numbers
1	1	1	0	1	the Tribonacci number
1	1	1	1	1	the Tricobsthal number

Remark 1.5 (*cf.* [18]). Let R be the polynomial ring of integers ($R = \mathbb{Z}[x]$). Then we have

p	q	r	a	b	w_n is sequence of polynomials
x^2	x	1	0	1	the Tribonacci polynomial
1	x	x^2	1	1	the Tricobsthal polynomial

Let R be an integral domain. Let M be an $n \times n$ matrix with entries from R and let $\mathrm{Adj}(M)$ be the adjugate of M. Then it is well-known that $M\mathrm{Adj}(M) = \det(M)I$ and $\det(M)$ is invertible in R iff M is an invertible matrix (*cf.* [21]).

To benefit from the methods in linear algebra, we use the matrix $W(n)$ on sequences in [18] as follows:

$$W(n) = \begin{bmatrix} w_{n+1} & w_n & w_{n-1} \\ w_n & w_{n-1} & w_{n-2} \\ w_{n-1} & w_{n-2} & w_{n-3} \end{bmatrix}.$$

Lemma 1.6 (*cf.* [18])**.** *For any integer n, we have the following relation*

$$K.W(n-1) = W(n),$$

where $K = \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

In this article, some generalizations of the results in the literature are obtained by using the sequences over an integral domain with the help of matrix method. Then the generalizations for the d'Ocagne identity, the Honsberger's formula, the Cassini's identity and the Catalan's identity are obtained with the help of them.

2. Tribonacci sequences by matrix method

Theorem 2.1. *For integers n and m, the following hold:*
i) $W(n) = pW(n-1) + qW(n-2) + rW(n-3)$.
ii) $W(n+m) = K^n.W(m)$.

Proof. *i)* By Definition 1.1, it is clear.
ii) By Lemma 1.6, we have that $W(n) = K^n.W(0)$ for an integer n. Then it follows

$$W(n+m) = K^{n+m}.W(0) = K^n.K^m.W(0) = K^n.W(m)$$

□

When $w_0 = 0$ and $w_1 = 1$, we have a specific sequence (matrix) and so use the notation k_n ($K(n)$) instead of w_n ($W(n)$), respectively, i.e.

$$K(n) = \begin{bmatrix} k_{n+1} & k_n & k_{n-1} \\ k_n & k_{n-1} & k_{n-2} \\ k_{n-1} & k_{n-2} & k_{n-3} \end{bmatrix} \quad \text{and} \quad K(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} \\ 0 & \frac{1}{r} & -\frac{q}{r^2} \end{bmatrix}.$$

Let us define the matrix $K_A(n)$ as $\begin{bmatrix} k_{n+1} & qk_n & 0 \\ k_n & qk_{n-1} & 0 \\ k_{n-1} & qk_{n-2} & 0 \end{bmatrix}$ and the matrix $K_B(n)$ as $\begin{bmatrix} 0 & rk_{n-1} & rk_n \\ 0 & rk_{n-2} & rk_{n-1} \\ 0 & rk_{n-3} & rk_{n-2} \end{bmatrix}$.

Theorem 2.2. *For an integer n, we have the following:*
i) $K_A(n+1) = K.(K_A(n))$.
ii) $K_B(n+1) = K.(K_B(n))$.
iii) $K^n = K_A(n) + K_B(n)$.

Proof. *i)*

$$\begin{aligned} K.(K_A(n)) &= \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_{n+1} & qk_n & 0 \\ k_n & qk_{n-1} & 0 \\ k_{n-1} & qk_{n-2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} k_{n+2} & qk_{n+1} & 0 \\ k_{n+1} & qk_n & 0 \\ k_n & qk_{n-1} & 0 \end{bmatrix} \\ &= K_A(n+1). \end{aligned}$$

ii)

$$\begin{aligned} K.(K_B(n)) &= \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & rk_{n-1} & rk_n \\ 0 & rk_{n-2} & rk_{n-1} \\ 0 & rk_{n-3} & rk_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & rk_n & rk_{n+1} \\ 0 & rk_{n-1} & rk_n \\ 0 & rk_{n-2} & rk_{n-1} \end{bmatrix} \\ &= K_B(n+1). \end{aligned}$$

iii) By the induction method, we have completed the proof. □

Theorem 2.3. *For integers n and m, we have the following for the sequence* $\{w_n\}$.

1) Honsberger's formula:

$$w_{m+n+1} = k_{n+1}w_{m+1} + w_m(qk_n + rk_{n-1}) + rk_n w_{m-1}.$$

2) Catalan's identity: $\det(W(n)) = r^m \det(W(n-m))$.

3) Cassini's or Simson's identity:

$$2w_n w_{n-1} w_{n-2} + w_{n+1} w_{n-1} w_{n-3} - w_{n-1}^3 - w_{n+1} w_{n-2}^2 - w_{n-3} w_n^2 = r^n\left(-w_{-3}w_0^2 - w_1 w_{-2}^2\right).$$

4)

$$2k_n k_{n-1} k_{n-2} + k_{n+1} k_{n-1} k_{n-3} - k_{n-1}^3 - k_{n+1} k_{n-2}^2 - k_{n-3} k_n^2 = -r^{n-2}.$$

5)

$$w_{n+1} = k_{n+1}b + a(qk_n + rk_{n-1}).$$

Proof. By Theorem 2.1, we prove the following.

1) We have

$$W(n+m) = K^n.W(m).$$

By the matrix equality, we get that

$$w_{m+n+1} = k_{n+1}w_{m+1} + w_m(qk_n + rk_{n-1}) + rk_n w_{m-1}.$$

2) We have $W(n) = K^m.W(n-m)$, for any integers m and n. Then, by the determinant, it follows that

$$\det(W(n)) = r^m \det(W(n-m)).$$

3) We have $W(n) = K^n.W(0)$. Then, by the determinant, it follows that

$$\begin{aligned} & 2w_n w_{n-1} w_{n-2} + w_{n+1} w_{n-1} w_{n-3} - w_{n-1}^3 - w_{n+1} w_{n-2}^2 - w_{n-3} w_n^2 \\ = \ & r^n\left(-w_{-3}w_0^2 + 2w_0 w_{-1} w_{-2} - w_{-1}^3 + w_1 w_{-3} w_{-1} - w_1 w_{-2}^2\right). \end{aligned}$$

4) We have

$$\begin{aligned} 2k_n k_{n-1} k_{n-2} + k_{n+1} k_{n-1} k_{n-3} - k_{n-1}^3 - k_{n+1} k_{n-2}^2 - k_{n-3} k_n^2 &= r^n\left(-k_{-2}^2\right) \\ &= -r^{n-2}. \end{aligned}$$

5) It is clear from (i). □

Corollary 2.4. *For any integers n and m, we have the following equalities for the n-th Tribonnaci numbers T_n and the n-th Tricobsthal numbers J_n.*

1)

$$\begin{aligned} T_{m+n} &= T_{n+1}T_m + T_{m-1}(T_n + T_{n-1}) + T_nT_{m-2}. \\ J_{m+n} &= T_{n+1}J_m + J_{m-1}(T_n + T_{n-1}) + T_nJ_{m-2}. \end{aligned}$$

2)

$$\begin{aligned} -1 &= 2J_nJ_{n-1}J_{n-2} + J_{n+1}J_{n-1}J_{n-3} - J_{n-1}^3 - J_{n+1}J_{n-2}^2 - J_{n-3}J_n^2 \\ &= 2T_nT_{n-1}T_{n-2} + T_{n+1}T_{n-1}T_{n-3} - T_{n-1}^3 - T_{n+1}T_{n-2}^2 - T_{n-3}T_n^2 \end{aligned}$$

3)

$$J_{n+1} = T_{n+1} + T_n + T_{n-1}.$$

The sequence in Definition 1.1 equals the sequence defined in [1] when $r = 0$ and so the following equalities are obtained from [1].

Corollary 2.5 (*cf.* [1])**.** *For any integers n and m, we have the following equalities.*

1) Catalan's identity:

$$(-q)^m w_{n-m}^2 - w_n^2 = (-q)^m w_{n-m-1}w_{n-m+1} - w_{n-1}w_{n+1}.$$

2) Cassini's or Simson's identity:

$$w_{n-1}w_{n+1} - w_n^2 = (-q)^n\left(w_1^2 - qw_0^2 - pw_0w_1\right).$$

3) $k_{n-1}k_{n+1} - k_n^2 = (-q)^{n-1}$.

4) Honsberger's formula: $w_{m+n} = k_nw_{m+1} + qk_{n-1}w_m$.

5) d'Ocagne's identity: $(-q)^n w_m = k_{n+1}w_{m+n} - k_nw_{m+n+1}$.

Theorem 2.6. *For any integer n, we have*

$$w_{-n} = \frac{1}{r^{n-1}}\left(\begin{vmatrix} k_{n+1} & k_n \\ k_{n-1} & k_{n-2} \end{vmatrix} a + (-q)^{n-2}b\right).$$

Proof. To obtain w_{-n}, we compute the $(2,1)th$ entry in the matrix $W(-n)$ by Theorem 2.1 as follows:

$$\begin{aligned} w_{-n} &= \frac{1}{r^n}\begin{bmatrix} -\begin{vmatrix} k_n & rk_{n-1} \\ k_{n-1} & rk_{n-2} \end{vmatrix} & \begin{vmatrix} k_{n+1} & rk_n \\ k_{n-1} & rk_{n-2} \end{vmatrix} & -\begin{vmatrix} k_{n+1} & rk_n \\ k_n & rk_{n-1} \end{vmatrix} \end{bmatrix}\begin{bmatrix} w_1 \\ w_0 \\ w_{-1} \end{bmatrix} \\ &= \frac{1}{r^{n-1}}\left(-\begin{vmatrix} k_n & k_{n-1} \\ k_{n-1} & k_{n-2} \end{vmatrix} b + \begin{vmatrix} k_{n+1} & k_n \\ k_{n-1} & k_{n-2} \end{vmatrix} a\right) \end{aligned}$$

Then we have $w_{-n} = \frac{1}{r^{n-1}}\left(\begin{vmatrix} k_{n+1} & k_n \\ k_{n-1} & k_{n-2} \end{vmatrix} a - \begin{vmatrix} k_n & k_{n-1} \\ k_{n-1} & k_{n-2} \end{vmatrix} b\right)$.

From (3) in Corollary 2.5, we have that $k_{n-1}k_{n+1} - k_n^2 = (-q)^{n-1}$. Thus

$$w_{-n} = \frac{1}{r^{n-1}}\left(\begin{vmatrix} k_{n+1} & k_n \\ k_{n-1} & k_{n-2} \end{vmatrix} a + (-q)^{n-2}b\right).$$

□

Corollary 2.7. *Let n and m be integers. Then we have*

$$w_{m-n} = \frac{1}{r^{n-1}}\left(\begin{vmatrix} k_{n+1} & k_n \\ k_{n-1} & k_{n-2} \end{vmatrix} w_m - (-q)^{n-2}(w_{m+1} + qw_{m-1})\right).$$

Let us compute w_{-2} by using Theorem 2.6.

Example 2.8. Since $k_0 = 0, k_1 = 1, k_2 = p, k_3 = p^2 + q$, w_{-2} can be computed as follows:

$$\begin{aligned} w_{-2} &= \frac{1}{r}\left(\begin{vmatrix} k_3 & k_2 \\ k_1 & k_0 \end{vmatrix} a + (-q)^0 b\right) \\ &= \frac{b - pa}{r}. \end{aligned}$$

Theorem 2.9. *Let* $t^3 + pt^2 + qt + r = (t-\alpha)(t-\beta)(t-\delta)$ *for some* $\alpha, \beta, \gamma \in R$ *, which are different from each others. Then for any integer n, we have*

$$k_n = \frac{\alpha^{n+1}}{(\alpha-\gamma)(\alpha-\beta)} - \frac{\beta^{n+1}}{(\beta-\gamma)(\alpha-\beta)} + \frac{\gamma^{n+1}}{(\beta-\gamma)(\alpha-\gamma)}$$

$$w_n = w_1 k_n - w_0\left(\frac{\alpha^{n+1}(\beta+\gamma)}{(\alpha-\gamma)(\alpha-\beta)} - \frac{\beta^{n+1}(\alpha+\gamma)}{(\beta-\gamma)(\alpha-\beta)} + \frac{\gamma^{n+1}(\alpha+\beta)}{(\beta-\gamma)(\alpha-\gamma)}\right).$$

Proof. Let α, β, γ be elements of R , which are different from each others and $t^3 + pt^2 + qt + r = (t-\alpha)(t-\beta)(t-\delta)$. Let P and D be the matrices as follows:

$$P = \begin{bmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}.$$

Then we get that $\det(P) = (\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)$.

Moreover, it is clear that $\det(P)K = (PD)(\mathrm{Adj}(P))$ and thus we have $\det(P)K^n = (PD^n)(\mathrm{Adj}(P))$. It follows that

$$\det(P)W(n) = \det(P)K^n W(0) = (PD^n)(\mathrm{Adj}(P))W(0).$$

By the matrix equality, we get that

$$\begin{aligned} \det(P)w_{n+1} = \; & w_1\left(\alpha^2\alpha^n(\beta-\gamma) - \beta^2\beta^n(\alpha-\gamma) + \gamma^2\gamma^n(\alpha-\beta)\right) \\ & -w_0\left(\alpha^2\alpha^n\left(\beta^2-\gamma^2\right) - \beta^2\beta^n\left(\alpha^2-\gamma^2\right) + \gamma^2\gamma^n\left(\alpha^2-\beta^2\right)\right) \\ & -w_{-1}\left(\alpha^2\alpha^n\left(\beta\gamma^2-\beta^2\gamma\right) - \beta^2\beta^n\left(\alpha\gamma^2-\alpha^2\gamma\right) + \gamma^2\gamma^n\left(\alpha\beta^2-\alpha^2\beta\right)\right). \end{aligned}$$

By the definition of the sequence k_n and since $k_0 = 0, k_1 = 1, k_{-1} = w_{-1} = 0$, we have that

$$\det(P)k_{n+1} = \left(\alpha^2\alpha^n(\beta-\gamma) - \beta^2\beta^n(\alpha-\gamma) + \gamma^2\gamma^n(\alpha-\beta)\right)$$

and so

$$\begin{aligned} k_{n+1} &= \frac{\left(\alpha^2\alpha^n(\beta-\gamma) - \beta^2\beta^n(\alpha-\gamma) + \gamma^2\gamma^n(\alpha-\beta)\right)}{(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)} \\ &= \frac{\alpha^{n+2}}{(\alpha-\gamma)(\alpha-\beta)} - \frac{\beta^{n+2}}{(\beta-\gamma)(\alpha-\beta)} + \frac{\gamma^{n+2}}{(\beta-\gamma)(\alpha-\gamma)}. \end{aligned}$$

Again we focus on the above equation

$$\begin{aligned} w_{n+1} &= w_1\frac{\left(\alpha^2\alpha^n(\beta-\gamma) - \beta^2\beta^n(\alpha-\gamma) + \gamma^2\gamma^n(\alpha-\beta)\right)}{(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)} \\ &\quad -w_0\frac{\left(\alpha^2\alpha^n\left(\beta^2-\gamma^2\right) - \beta^2\beta^n\left(\alpha^2-\gamma^2\right) + \gamma^2\gamma^n\left(\alpha^2-\beta^2\right)\right)}{(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)} \\ &\quad -w_{-1}\frac{\left(\alpha^2\alpha^n\left(\beta\gamma^2-\beta^2\gamma\right) - \beta^2\beta^n\left(\alpha\gamma^2-\alpha^2\gamma\right) + \gamma^2\gamma^n\left(\alpha\beta^2-\alpha^2\beta\right)\right)}{(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)} \\ &= w_1k_{n+1} - w_0\left(\frac{\alpha^{n+2}(\beta+\gamma)}{(\alpha-\gamma)(\alpha-\beta)} - \frac{\beta^{n+2}(\alpha+\gamma)}{(\beta-\gamma)(\alpha-\beta)} + \frac{\gamma^{n+2}(\alpha+\beta)}{(\beta-\gamma)(\alpha-\gamma)}\right) - \alpha\beta\gamma w_{-1}k_n \\ &= ak_{n+1} - b\left(\frac{\alpha^{n+2}(\beta+\gamma)}{(\alpha-\gamma)(\alpha-\beta)} - \frac{\beta^{n+2}(\alpha+\gamma)}{(\beta-\gamma)(\alpha-\beta)} + \frac{\gamma^{n+2}(\alpha+\beta)}{(\beta-\gamma)(\alpha-\gamma)}\right). \end{aligned}$$

since $w_{-1} = 0$. Then the proof is completed. □

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