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Arithmetic of Sheffer sequences arising from Riemann, Volkenborn and Kim integrals

Dae San Kim [a], Taekyun Kim [b], Lee-Chae Jang [c], Hye Kyung Kim [d], Jongkyum Kwon [e]

[a]Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
[b]Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
[c]Graduate School of Education, Konkuk University, Seoul 05029, Republic of Korea
[d]Department of Mathematics Education, Daegu Catholic University, Gyeongsan 38430, Republic of Korea
[e]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Republic of Korea

Abstract

Let $\{s_n(x)\}$ be any sequence of polynomials with rational coefficients which is Sheffer for some Sheffer pair. Then we consider the Riemann integral from 0 to 1, the Volkenborn integral on $\mathbb{Z}_p$ and the Kim integral on $\mathbb{Z}_p$ of $s_n(x+y)$ with respect to y. They all give rise to some different Sheffer polynomials. The aim of this paper is to derive some properties of those polynomials, especially their convolution identities, and to illustrate our results with some examples.

Keywords: Riemann integral, Volkenborn integral, Kim integral, Umbral calculus.

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1. Introduction

Let p be any fixed prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ be the normalized nonarchimedean absolute value of $\mathbb{C}_p$, such that $|p|_p = \frac{1}{p}$. For a uniformly differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic Volkenborn integral of f is defined by (see [5, 15])

$$\int_{\mathbb{Z}_p} f(z)d\mu(z) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j). \tag{1.1}$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1)d\mu(z) = \int_{\mathbb{Z}_p} f(z)d\mu(z) + f'(z). \tag{1.2}$$

†*Article ID:* MTJPAM-D-21-00078
Email addresses: dskim@sogang.ac.kr (Dae San Kim), tkkim@kw.ac.kr (Taekyun Kim), Lcjang@konkuk.ac.kr (Lee-Chae Jang), hkkim@cu.ac.kr (Hye Kyung Kim), mathkjk26@gnu.ac.kr (Jongkyum Kwon)
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⋆*Corresponding Author:* Jongkyum Kwon

Let p be any fixed odd prime. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the Kim integral of f (also called the fermionic p-adic integral of f) is defined by (see [3, 5, 6])

$$\int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z) = \lim_{N\to\infty} \sum_{j=0}^{p^N-1} (-1)^j f(j). \tag{1.3}$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1)d\mu_{-1}(z) + \int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z) = 2f(0). \tag{1.4}$$

Assume that $s_n(x)$ is a sequence of polynomials with rational coefficients for the Sheffer pair $(g(t), f(t))$. Then we have

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)}, \tag{1.5}$$

where $\bar{f}(t)$ is the compositional inverse of $\bar{f}(t)$ satisfying $f(\bar{f}(t)) = t = \bar{f}(f(t))$.

We consider the following three types of integrals, namely Riemann integral, Volkenborn integral and Kim integral (also called the fermionic p-adic integral) respectively given by

$$\alpha_n(x) = \int_0^1 s_n(x+y)dy, \tag{1.6}$$

$$\beta_n(x) = \int_{\mathbb{Z}_p} s_n(x+y)d\mu(y), \tag{1.7}$$

$$\gamma_n(x) = \int_{\mathbb{Z}_p} s_n(x+y)d\mu_{-1}(y). \tag{1.8}$$

Then $\alpha_n(x)$, $\beta_n(x)$, and $\gamma_n(x)$ are Sheffer respectively for $(g(t)\frac{t}{e^t-1}, f(t))$, $(g(t)\frac{e^t-1}{t}, f(t))$, and $(g(t)\frac{e^t+1}{2}, f(t))$. Thus, starting from any Sheffer sequence and by considering the Riemann integral from 0 to 1, Volkenborn integral and Kim integral of the given Sheffer polynomial, we obtain three different Sheffer polynomials. Then we consider the convolution of any two of them and the convolution of all three of them as well. From these considerations, we deduce some interesting identities. In addition, we deduce Sheffer type identities, recurrence relations and some identities not coming from convolution considerations. Further, we illustrate our results with some choices of Sheffer sequences. These are done by using umbral calculus (see [1, 2, 8, 9, 13, 14]) and p-adic integrations (see [5, 6, 15]). Here we remark that the Kim integrals in (1.3) and (1.4) have been called the fermionic p-adic integrals. But they were introduced for the first time by T. Kim in [3, 5, 6].

More generally, for any positive integer r, we may consider r times iterated integrals in (1.6), (1.7) and (1.8) and get $\alpha_n^{(r)}(x)$, $\beta_n^{(r)}(x)$, and $\gamma_n^{(r)}(x)$, which are Sheffer respectively for $(g(t)(\frac{t}{e^t-1})^r, f(t))$, $(g(t)(\frac{e^t-1}{t})^r, f(t))$, and $(g(t)(\frac{e^t+1}{2})^r, f(t))$. The outline of this paper is as follows. After this introduction, we will briefly review umbral calculus in Chapter 2. In Chapter 3, we will consider arithmetic of associated sequences related to Riemann, Volkenborn and Kim integrals. Then we will illustrate the results with some examples in Chapter 4. In Chapter 5, we will investigate the general case of arithmetic of Sheffer sequences in connection with Riemann, Volkenborn and Kim integrals. Finally, we illustrate the results with some examples in Chapter 6.

2. Review of umbral calculus

Here we will briefly go over very basic facts about umbral calculus. For more details on this, we recommend the reader to refer to [1, 2, 13, 14]. We remark that recently umbral calculus has been extended to the case of λ-umbral calculus in order to treat degenerate special polynomials and numbers, which involves λ-exponentials (see [8, 9]). Let $\mathbb{C}$ be the field of complex numbers. Then $\mathcal{F}$ denotes the algebra of formal power series in t over $\mathbb{C}$, given by

$$\mathcal{F} = \Big\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\Big|\, a_k \in \mathbb{C} \Big\},$$

and $\mathbb{P} = \mathbb{C}[x]$ indicates the algebra of polynomials in x with coefficients in $\mathbb{C}$.

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. If $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, then the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \quad \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex number.

For $f(t) \in \mathcal{F}$ with $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, we define the linear functional on $\mathbb{P}$ by

$$\langle f(t)|x^k\rangle = a_k. \tag{2.1}$$

From (2.1), we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad (n,k \geq 0),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Some remarkable linear functionals are as follows:

$$\begin{aligned} &\langle e^{yt}|p(x)\rangle = p(y), \\ &\langle e^{yt}-1|p(x)\rangle = p(y)-p(0), \\ &\left\langle \frac{e^{yt}-1}{t}\middle|p(x)\right\rangle = \int_0^y p(u)du. \end{aligned} \tag{2.2}$$

Let

$$f_L(t) = \sum_{k=0}^{\infty} \langle L|x^k\rangle \frac{t^k}{k!}. \tag{2.3}$$

Then, by (2.1) and (2.3), we get

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle.$$

That is, $f_L(t) = L$, as linear functionals on $\mathbb{P}$. In fact, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$.

Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra and the umbral calculus is the study of umbral algebra. For each nonnegative integer k, the differential operator t^k on $\mathbb{P}$ is defined by

$$t^k x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.4}$$

Extending (2.4) linearly, any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

gives the differential operator on $\mathbb{P}$ defined by

$$f(t)x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}, \quad (n \geq 0). \tag{2.5}$$

It should be observed that, for any formal power series $f(t)$ and any polynomial $p(x)$, we have

$$\langle f(t)|p(x)\rangle = \langle 1|f(t)p(x)\rangle = f(t)p(x)|_{x=0}. \tag{2.6}$$

Here we note that an element $f(t)$ of $\mathcal{F}$ is a formal power series, a linear functional and a differential operator. Some notable differential operators are as follows:

$$\begin{aligned} & e^{yt}p(x) = p(x+y), \\ & (e^{yt}-1)p(x) = p(x+y) - p(x), \\ & \frac{e^{yt}-1}{t}p(x) = \int_x^{x+y} p(u)du. \end{aligned} \tag{2.7}$$

The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then $f(t)$ is called a delta series.

For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) of polynomials such that

$$\langle g(t)f(t)^k | s_n(x) \rangle = n!\delta_{n,k}, \quad (n, k \geq 0). \tag{2.8}$$

The sequence $s_n(x)$ is said to be the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. We observe from (2.8) that

$$s_n(x) = \frac{1}{g(t)}p_n(x), \tag{2.9}$$

where $p_n(x) = g(t)s_n(x) \sim (1, f(t))$.

In particular, if $s_n(x) \sim (g(t), t)$, then $p_n(x) = x^n$, and hence

$$s_n(x) = \frac{1}{g(t)}x^n. \tag{2.10}$$

It is well known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!}t^k, \tag{2.11}$$

for all $x \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ such that $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

The following equations (2.12), (2.13), and (2.14) are equivalent to the fact that $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$:

$$f(t)\, s_n(x) = n s_{n-1}(x), \quad (n \geq 0), \tag{2.12}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x)\, p_{n-j}(y), \tag{2.13}$$

with $p_n(x) = g(t)\, s_n(x)$,

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g\left(\bar{f}(t)\right)^{-1} \bar{f}(t)^j \big| x^n \rangle x^j. \tag{2.14}$$

If $s_n(x) \sim (g(t), f(t))$, then the following recurrence relation holds:

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x). \tag{2.15}$$

Let $\{\alpha_n\}, \{\beta_n\}$ be sequences of complex numbers, and let $\{\alpha_n(x)\}, \{\beta_n(x)\}$ be sequences of polynomials with complex coefficients. For brevity, the following notations will be used throughout this paper:

$$(\alpha * \beta)_n = \sum_{i=0}^{n} \binom{n}{i} \alpha_i \beta_{n-i}, \tag{2.16}$$

$$(\alpha * \beta)_n(x) = \sum_{i=0}^{n} \binom{n}{i} \alpha_i(x) \beta_{n-i}(x). \tag{2.17}$$

These notations can be defined in an obvious way for more than two sequences of numbers and of polynomials. Finally, we recall the definitions for the following special polynomials.
The polynomials $A_n(x)$, which does not have any name, are defined by

$$\frac{e^t-1}{t}e^{xt}=\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!}. \tag{2.18}$$

Then we see that

$$A_n(x)=\frac{1}{n+1}\left\{(x+1)^{n+1}-x^{n+1}\right\}=\frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}x^i.$$

The Bernoulli polynomials $B_n(x)$ are given by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{2.19}$$

The Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{2.20}$$

Let r be any positive integer. More generally, $A_n^{(r)}(x)$, $B_n^{(r)}$, and $E_n^{(r)}$ are given by

$$\left(\frac{e^t-1}{t}\right)^r e^{xt}=\sum_{n=0}^{\infty}A_n^{(r)}(x)\frac{t^n}{n!}, \tag{2.21}$$

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}, \tag{2.22}$$

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}. \tag{2.23}$$

3. Arithmetic of associated sequences

Assume that $s_n(x)$ is a sequence of polynomials with rational coefficients associated with the delta series $f(t)$ so that $s_n(x) \sim (1, f(t))$. Then we have

$$\sum_{n=0}^{\infty}s_n(x)\frac{t^n}{n!}=e^{x\bar{f}(t)}. \tag{3.1}$$

Then we consider the following three types of integrals, namely Riemann, Volkenborn and Kim integrals respectively given by

$$\int_0^1 s_n(x)dx=\alpha_n, \tag{3.2}$$

$$\int_{\mathbb{Z}_p} s_n(x)d\mu(x)=\beta_n, \tag{3.3}$$

$$\int_{\mathbb{Z}_p} s_n(x)d\mu_{-1}(x)=\gamma_n. \tag{3.4}$$

Then we see that the generating functions of $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are given by

$$\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!}=\frac{e^{\bar{f}(t)}-1}{\bar{f}(t)}, \tag{3.5}$$

$$\sum_{n=0}^{\infty}\beta_n\frac{t^n}{n!}=\frac{\bar{f}(t)}{e^{\bar{f}(t)}-1}, \tag{3.6}$$

$$\sum_{n=0}^{\infty}\gamma_n\frac{t^n}{n!}=\frac{2}{e^{\bar{f}(t)}+1}. \tag{3.7}$$

For example, we see (3.6) from the following:

$$\sum_{n=0}^{\infty}\beta_n\frac{t^n}{n!}=\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}s_n(x)\frac{t^n}{n!}d\mu(x)=\int_{\mathbb{Z}_p}e^{x\bar{f}(t)}d\mu(x).$$

As (3.5) and (3.6) are reciprocals to each other, we have

$$1=\sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!}\sum_{j=0}^{\infty}\beta_j\frac{t^j}{j!}=\sum_{n=0}^{\infty}(\alpha*\beta)_n\frac{t^n}{n!}.$$

This entails that

$$\sum_{i=0}^{n}\binom{n}{i}\alpha_i\beta_{n-i}=0,\quad (n\geq 1). \tag{3.8}$$

From (3.6) and (3.7), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(\beta*\gamma)_n\frac{t^n}{n!}&=\sum_{i=0}^{\infty}\beta_i\frac{t^i}{i!}\sum_{j=0}^{\infty}\gamma_j\frac{t^j}{j!} \\ &=\frac{2\bar{f}(t)}{e^{2\bar{f}(t)}-1}=\frac{\overline{f(\frac{t}{2})}}{e^{\overline{f(\frac{t}{2})}}-1}.\end{aligned} \tag{3.9}$$

Here $2\bar{f}(t)$ is the compositional inverse of $f(\frac{t}{2})$, so that $2\bar{f}(t)=\overline{f(\frac{t}{2})}$. We should observe that $\overline{f(\frac{t}{2})}$ is not equal to $\bar{f}(\frac{t}{2})$. For example, if $f(t)=e^t-1$, then $\overline{f(\frac{t}{2})}=2\log(1+t)$, but $\bar{f}(\frac{t}{2})=\log(1+\frac{t}{2})$.
From (3.5) and (3.7), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\gamma)_n\frac{t^n}{n!}&=\sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!}\sum_{j=0}^{\infty}\gamma_j\frac{t^j}{j!} \\ &=\frac{2(e^{\bar{f}(t)}-1)}{\bar{f}(t)(e^{\bar{f}(t)}+1)}=\frac{4(e^{\frac{1}{2}\overline{f(\frac{t}{2})}}-1)}{\overline{f(\frac{t}{2})}(e^{\frac{1}{2}\overline{f(\frac{t}{2})}}+1)}.\end{aligned} \tag{3.10}$$

Next, we put

$$\alpha_n(x)=\int_0^1 s_n(x+y)dy=\int_x^{x+1}s_n(y)dy=\frac{e^t-1}{t}s_n(x), \tag{3.11}$$

$$\beta_n(x)=\int_{\mathbb{Z}_p}s_n(x+y)d\mu(y), \tag{3.12}$$

$$\gamma_n(x)=\int_{\mathbb{Z}_p}s_n(x+y)d\mu_{-1}(y). \tag{3.13}$$

Then it is immediate to see that

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!}=\frac{e^{\bar{f}(t)}-1}{\bar{f}(t)}e^{x\bar{f}(t)},\quad \alpha_n(x)\sim\left(\frac{t}{e^t-1},f(t)\right),\tag{3.14}$$

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}=\frac{\bar{f}(t)}{e^{\bar{f}(t)}-1}e^{x\bar{f}(t)},\quad \beta_n(x)\sim\left(\frac{e^t-1}{t},f(t)\right),\tag{3.15}$$

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!}=\frac{2}{e^{\bar{f}(t)}+1}e^{x\bar{f}(t)},\quad \gamma_n(x)\sim\left(\frac{e^t+1}{2},f(t)\right).\tag{3.16}$$

For example, we see (3.15) from the following:

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}s_n(x+y)\frac{t^n}{n!}d\mu(y)=e^{x\bar{f}(t)}\int_{\mathbb{Z}_p}e^{y\bar{f}(t)}d\mu(y).$$

Remark 3.1. Replacing t by $f(t)$ respectively in (3.14), (3.15), and (3.16) gives us the following:

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{f(t)^n}{n!}=\frac{e^t-1}{t}e^{xt}=\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{f(t)^n}{n!}=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{f(t)^n}{n!}=\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

Remark 3.2. We note first that

$$f(t)\alpha_n(x)=n\alpha_{n-1}(x),f(t)\beta_n(x)=n\beta_{n-1}(x),f(t)\gamma_n(x)=n\gamma_{n-1}(x).$$

Applying Remark 3.1 with $x=y$ to $s_n(x),\alpha_n(x),\beta_n(x)$, and to $\gamma_n(x)$, we obtain

$$\int_{x+y}^{x+y+1}s_n(u)du=\sum_{k=0}^{n}\binom{n}{k}\alpha_k(y)s_{n-k}(x)=\sum_{k=0}^{n}A_k(y)\frac{(s_n(x))^{(k)}}{k!}=\alpha_n(x+y),\tag{3.17}$$

$$\int_{x+y}^{x+y+1}\alpha_n(u)du=\sum_{k=0}^{n}\binom{n}{k}\alpha_k(y)\alpha_{n-k}(x)=\sum_{k=0}^{n}A_k(y)\frac{(\alpha_n(x))^{(k)}}{k!},$$

$$\int_{x+y}^{x+y+1}\beta_n(u)du=\sum_{k=0}^{n}\binom{n}{k}\alpha_k(y)\beta_{n-k}(x)=\sum_{k=0}^{n}A_k(y)\frac{(\beta_n(x))^{(k)}}{k!},$$

$$\int_{x+y}^{x+y+1}\gamma_n(u)du=\sum_{k=0}^{n}\binom{n}{k}\alpha_k(y)\gamma_{n-k}(x)=\sum_{k=0}^{n}A_k(y)\frac{(\gamma_n(x))^{(k)}}{k!},$$

$$\frac{t}{e^t-1}e^{yt}s_n(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_k(y)s_{n-k}(x)=\sum_{k=0}^{n}B_k(y)\frac{(s_n(x))^{(k)}}{k!}=\beta_n(x+y),\tag{3.18}$$

$$\frac{t}{e^t-1}e^{yt}\alpha_n(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_k(y)\alpha_{n-k}(x)=\sum_{k=0}^{n}B_k(y)\frac{(\alpha_n(x))^{(k)}}{k!},$$

$$\frac{t}{e^t-1}e^{yt}\beta_n(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_k(y)\beta_{n-k}(x)=\sum_{k=0}^{n}B_k(y)\frac{(\beta_n(x))^{(k)}}{k!},$$

$$\begin{aligned}
&\frac{t}{e^t-1}e^{yt}\gamma_n(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_k(y)\gamma_{n-k}(x)=\sum_{k=0}^{n}B_k(y)\frac{(\gamma_n(x))^{(k)}}{k!},\\
&\frac{2}{e^t+1}e^{yt}s_n(x)=\sum_{k=0}^{n}\binom{n}{k}\gamma_k(y)s_{n-k}(x)=\sum_{k=0}^{n}E_k(y)\frac{(s_n(x))^{(k)}}{k!}=\gamma_n(x+y),\\
&\frac{2}{e^t+1}e^{yt}\alpha_n(x)=\sum_{k=0}^{n}\binom{n}{k}\gamma_k(y)\alpha_{n-k}(x)=\sum_{k=0}^{n}E_k(y)\frac{(\alpha_n(x))^{(k)}}{k!},\\
&\frac{2}{e^t+1}e^{yt}\beta_n(x)=\sum_{k=0}^{n}\binom{n}{k}\gamma_k(y)\beta_{n-k}(x)=\sum_{k=0}^{n}E_k(y)\frac{(\beta_n(x))^{(k)}}{k!},\\
&\frac{2}{e^t+1}e^{yt}\gamma_n(x)=\sum_{k=0}^{n}\binom{n}{k}\gamma_k(y)\gamma_{n-k}(x)=\sum_{k=0}^{n}E_k(y)\frac{(\gamma_n(x))^{(k)}}{k!},
\end{aligned}\tag{3.19}$$

where $(s_n(x))^{(k)}=(\frac{d}{dx})^k s_n(x)$, and the same is true for the others.

We note that

$$s_n(x)=\frac{t}{e^t-1}\alpha_n(x)=\frac{e^t-1}{t}\beta_n(x)=\frac{e^t+1}{2}\gamma_n(x)\sim(1,f(t)).\tag{3.20}$$

Acting e^{yt} on (3.20), we have

$$s_n(x+y)=\frac{t}{e^t-1}e^{yt}\alpha_n(x)=\frac{e^t-1}{t}e^{yt}\beta_n(x)=\frac{e^t+1}{2}e^{yt}\gamma_n(x).\tag{3.21}$$

By using Remark 3.1, (3.21), we have

$$\begin{aligned}
s_n(x+y)&=\sum_{k=0}^{n}\binom{n}{k}\beta_k(y)\alpha_{n-k}(x)=\sum_{k=0}^{n}\binom{n}{k}\alpha_k(y)\beta_{n-k}(x)\\
&=\int_{x+y}^{x+y+1}\beta_n(u)du=\frac{1}{2}(\gamma_n(x+y+1)+\gamma_n(x+y)).
\end{aligned}\tag{3.22}$$

We note from (3.21) that $\frac{2}{e^t+1}e^{yt}\alpha_n(x)=\frac{e^t-1}{t}e^{yt}\gamma_n(x)$ holds and hence we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\gamma_k(y)\alpha_{n-k}(x)=\sum_{k=0}^{n}\binom{n}{k}\alpha_k(y)\gamma_{n-k}(x)=\int_{x+y}^{x+y+1}\gamma_n(u)du.\tag{3.23}$$

We see from (3.21) that $\frac{t}{e^t-1}e^{yt}\gamma_n(x)=\frac{2}{e^t+1}e^{yt}\beta_n(x)$ holds and thus we get

$$\sum_{k=0}^{n}\binom{n}{k}\gamma_k(y)\beta_{n-k}(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_k(y)\gamma_{n-k}(x).\tag{3.24}$$

We observe from (3.21) that $\frac{e^t-1}{t}\frac{e^t-1}{t}\beta_n(x)=\alpha_n(x)$ holds and therefore we deduce

$$\int_x^{x+1}\int_v^{v+1}\beta_n(u)dudv=\alpha_n(x).\tag{3.25}$$

From (3.21) and Remark 3.1, $\frac{2(e^t-1)}{t(e^t+1)}e^{2yt}=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\binom{k}{i}\alpha_i(y)\gamma_{k-i}(y)\frac{f(t)^k}{k!}$,
and $\frac{2(e^t-1)}{t(e^t+1)}e^{2yt}\beta_n(x)=e^{2yt}\gamma_n(x)$. By using these, we have

$$\begin{aligned}
\gamma_n(x+2y)&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}\alpha_i(y)\gamma_{k-i}(y)\beta_{n-k}(x)\\
&=\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n}{i,k-i,n-k}\alpha_i(y)\gamma_{k-i}(y)\beta_{n-k}(x).
\end{aligned}\tag{3.26}$$

Remark 3.3. (a) From (3.14) and (3.15), we have

$$\sum_{n=0}^{\infty}(\alpha*\beta)_n(x)\frac{t^n}{n!} = e^{2x\bar{f}(t)} = e^{x\overline{f(\frac{t}{2})}},$$

so that $s_n(2x) = (\alpha*\beta)_n(x) \sim (1, f(\frac{t}{2}))$.
(b) From (3.15) and (3.16), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!} &= \frac{2\bar{f}(t)}{e^{2\bar{f}(t)}-1}e^{2x\bar{f}(t)}\\ &= \frac{\overline{f(\frac{t}{2})}}{e^{\overline{f(\frac{t}{2})}}-1}e^{x\overline{f(\frac{t}{2})}},\end{aligned}$$

so that $(\beta*\gamma)_n(x) \sim (\frac{e^t-1}{t}, f(\frac{t}{2}))$.
(c) From (3.14) and (3.16), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!} &= \frac{2(e^{\bar{f}(t)}-1)}{\bar{f}(t)(e^{\bar{f}(t)}+1)}e^{2x\bar{f}(t)}\\ &= \frac{4(e^{\frac{1}{2}\overline{f(\frac{t}{2})}}-1)}{\overline{f(\frac{t}{2})}(e^{\frac{1}{2}\overline{f(\frac{t}{2})}}+1)}e^{x\overline{f(\frac{t}{2})}},\end{aligned}$$

so that $\sum_{i=0}^{n}(\alpha*\gamma)_n(x) \sim \left(\frac{t(e^{\frac{1}{2}t}+1)}{4(e^{\frac{1}{2}t}-1)}, f(\frac{t}{2})\right)$.
(d) From (3.14), (3.15) and (3.16), we get

$$\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!} = \frac{2}{e^{\bar{f}(t)}+1}e^{3x\bar{f}(t)} = \frac{2}{e^{\frac{1}{3}\overline{f(\frac{t}{3})}}+1}e^{x\overline{f(\frac{t}{3})}},$$

so that $\gamma_n(3x) = (\alpha*\beta*\gamma)_n(x) \sim (\frac{e^{\frac{1}{3}t}+1}{2}, f(\frac{t}{3}))$.

4. Examples on associated sequences

(a) Let $s_n(x) = x^n \sim (1, t)$. Then from (3.14)-(3.16) and Remark 3.1 we have

$$\begin{aligned}\alpha_n(x) &= \int_0^1 (x+y)^n dy = A_n(x),\\ \beta_n(x) &= \int_{\mathbb{Z}_p}(x+y)^n d\mu(y) = B_n(x),\\ \gamma_n(x) &= \int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y) = E_n(x).\end{aligned}$$

From Remark 3.3 (a), we obtain the following:

$$(2x)^n = (\alpha*\beta)_n(x) = \sum_{i=0}^{n}\binom{n}{i}\frac{1}{i+1}\left\{(x+1)^{i+1}-x^{i+1}\right\}B_{n-i}(x),$$

which implies

$$\sum_{i=0}^{n}\binom{n}{i}\frac{1}{i+1}B_{n-i} = 0, \quad \text{for } n \geq 1.$$

From Remark 3.3 (b), we have

$$\sum_{n=0}^{\infty}(\beta * \gamma)_n(x)\frac{t^n}{n!} = \frac{2t}{e^{2t}-1}e^{2xt} = \sum_{n=0}^{\infty}2^n B_n(x)\frac{t^n}{n!},$$

so that this yields

$$\sum_{i=0}^{n}\binom{n}{i}B_i(x)E_{n-i}(x) = 2^n B_n(x).$$

From Remark 3.3 (c), we observe

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha * \gamma)_n(x)\frac{t^n}{n!} &= \frac{e^t-1}{t}\frac{2}{e^t+1}e^{2xt}\\ &= \sum_{i=0}^{\infty}\frac{t^i}{(i+1)!}\sum_{j=0}^{\infty}E_j(2x)\frac{t^j}{j!}\\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i+1}E_{n-i}(2x)\frac{t^n}{n!}.\end{aligned}$$

This yields that

$$\sum_{i=0}^{n}\binom{n}{i}\frac{1}{i+1}\left\{(x+1)^{i+1}-x^{i+1}\right\}E_{n-i}(x) = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i+1}E_{n-i}(2x),$$

$$\sum_{i=0}^{n}\binom{n}{i}\frac{1}{i+1}E_{n-i} = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i+1}E_{n-i}.$$

From (3.22), we get the following:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n}\binom{n}{k}B_k(y)\frac{1}{n-k+1}\left\{(x+1)^{n-k+1}-x^{n-k+1}\right\}\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\left\{(y+1)^{k+1}-y^{k+1}\right\}B_{n-k}(x)\\ &= \int_{x+y}^{x+y+1}B_n(u)du = \frac{1}{2}(E_n(x+y+1)+E_n(x+y)).\end{aligned}$$

From (3.23), we obtain the following:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}E_k(y)\frac{1}{n-k+1}\left\{(x+1)^{n-k+1}-x^{n-k+1}\right\}\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\left\{(y+1)^{k+1}-y^{k+1}\right\}E_{n-k}(x) = \int_{x+y}^{x+y+1}E_n(u)du.\end{aligned}$$

From (3.24), we have the following:

$$\sum_{k=0}^{n}\binom{n}{k}E_k(y)B_{n-k}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k(y)E_{n-k}(x).$$

From (3.25), we note the following:

$$\int_{x}^{x+1}\int_{v}^{v+1}B_n(u)dudv = \frac{1}{n+1}\left\{(x+1)^{n+1}-x^{n+1}\right\}.$$

Lastly, we observe from (3.26) that

$$E_n(x+2y)=\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n}{i,k-i,n-k}\frac{1}{i+1}\left\{(y+1)^{i+1}-y^{i+1}\right\}E_{k-i}(y)B_{n-k}(x).$$

In particular, we have

$$\begin{aligned}E_n(3x)&=\sum_{i+j+k=n}\binom{n}{i,j,k}\frac{1}{i+1}\left\{(x+1)^{i+1}-x^{i+1}\right\}B_j(x)E_k(x),\\ E_n&=\sum_{i+j+k=n}\binom{n}{i,j,k}\frac{1}{i+1}B_jE_k.\end{aligned}$$

(b) Let $s_n(x)=(x)_n\sim(1,e^t-1)$. Then from (3.14)-(3.16) we have

$$\begin{aligned}\alpha_n(x)&=\int_0^1(x+y)_n dy=b_n(x),\ \sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!}=\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},\\ \beta_n(x)&=\int_{\mathbb{Z}_p}(x+y)_n d\mu(y)=D_n(x),\ \sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}=\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!},\\ \gamma_n(x)&=\int_{\mathbb{Z}_p}(x+y)_n d\mu_{-1}(y)=Ch_n(x),\ \sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!}=\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!},\end{aligned}$$

where $b_n(x), D_n(x), Ch_n(x)$ are respectively called Bernoulli polynomials of the second kind, Daehee polynomials (see [4, 7, 10, 16]) and Changhee polynomials (see [11, 12, 16]).
From Remark 3.3 (a), we have

$$(2x)_n=(\alpha*\beta)_n(x)=\sum_{i=0}^{n}\binom{n}{i}b_i(x)D_{n-i}(x),$$

which implies

$$\sum_{i=0}^{n}\binom{n}{i}b_iD_{n-i}=0,\quad \text{for } n\geq 1.$$

From Remark 3.3 (b), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!}&=\frac{2\log(1+t)}{e^{2\log(1+t)}-1}e^{2x\log(1+t)}=\sum_{l=0}^{\infty}2^lB_l(x)\frac{((\log(1+t))^l}{l!}\\ &=\sum_{l=0}^{\infty}2^lB_l(x)\sum_{n=l}^{\infty}S_1(n,l)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}2^lS_1(n,l)B_l(x)\frac{t^n}{n!}.\end{aligned}$$

Hence we get

$$\sum_{i=0}^{n}\binom{n}{i}D_i(x)Ch_{n-i}(x)=\sum_{l=0}^{n}2^lS_1(n,l)B_l(x).$$

From Remark 3.3 (c), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!} &= \frac{e^{\log(1+t)}-1}{\log(1+t)}\frac{2}{e^{\log(1+t)}+1}e^{2x\log(1+t)}\\ &= \sum_{l=0}^{\infty}\frac{1}{l+1}\sum_{i=0}^{l}\binom{l+1}{i+1}E_{l-i}(2x)\frac{\big(\log(1+t)\big)^l}{l!}\\ &= \sum_{l=0}^{\infty}\frac{1}{l+1}\sum_{i=0}^{l}\binom{l+1}{i+1}E_{l-i}(2x)\sum_{n=l}^{\infty}S_1(n,l)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\sum_{i=0}^{l}\frac{1}{l+1}\binom{l+1}{i+1}S_1(n,l)E_{l-i}(2x)\Big)\frac{t^n}{n!}.\end{aligned}$$

Thus we obtain

$$\sum_{i=0}^{n}\binom{n}{i}b_i(x)Ch_{n-i}(x) = \sum_{l=0}^{n}\frac{S_1(n,l)}{l+1}\sum_{i=0}^{l}\binom{l+1}{i+1}E_{l-i}(2x).$$

Now, from (3.17), we obtain

$$\begin{aligned}b_n(x+y) &= \sum_{k=0}^{n}\binom{n}{k}b_k(y)(x)_{n-k} = \sum_{k=0}^{n}A_k(y)\frac{1}{k!}((x)_n)^{(k)}\\ &= \sum_{k=0}^{n}A_k(y)\frac{1}{k!}\Big(\sum_{l=0}^{n}S_1(n,l)x^l\Big)^{(k)}\\ &= \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{l}{k}A_k(y)x^{l-k}S_1(n,l)\\ &= \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}A_k(y)x^{l-k}S_1(n,l)\\ &= \sum_{l=0}^{n}S_1(n,l)A_l(x+y).\end{aligned}$$

In summary, we have

$$b_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}b_k(y)(x)_{n-k} = \sum_{l=0}^{n}S_1(n,l)A_l(x+y).$$

Similarly, from (3.18) and (3.19) we see that

$$\begin{aligned}D_n(x+y) &= \sum_{k=0}^{n}\binom{n}{k}D_k(y)(x)_{n-k} = \sum_{l=0}^{n}S_1(n,l)B_l(x+y),\\ Ch_n(x+y) &= \sum_{k=0}^{n}\binom{n}{k}Ch_k(y)(x)_{n-k} = \sum_{l=0}^{n}S_1(n,l)E_l(x+y).\end{aligned}$$

Finally, from Remark 3.3 (d) we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!} &= \sum_{k=0}^{\infty}E_k(3x)\frac{(\log(1+t))^k}{k!}\\ &= \sum_{k=0}^{\infty}E_k(3x)\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}S_1(n,k)E_k(3x)\frac{t^n}{n!},\end{aligned}$$

which entails that

$$\sum_{k+l+m=n}\binom{n}{k,l,m}b_k(x)D_l(x)Ch_m(x) = \sum_{k=0}^{n}S_1(n,k)E_k(3x).$$

(c) Let $s_n(x) = \mathrm{Bel}_n(x) \sim (1, \log(1+t))$. Then we have

$$\alpha_n(x) = \int_0^1 \mathrm{Bel}_n(x+y)dy = \sum_{k=0}^{n}S_2(n,k)A_k(x),$$

since

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!} = \sum_{k=0}^{\infty}A_k(x)\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}S_2(n,k)A_k(x).$$

Similarly, we can show that

$$\begin{aligned}\beta_n(x) &= \int_{\mathbb{Z}_p}\mathrm{Bel}_n(x+y)d\mu(y) = \sum_{k=0}^{n}S_2(n,k)B_k(x),\\ \gamma_n(x) &= \int_{\mathbb{Z}_p}\mathrm{Bel}_n(x+y)d\mu_{-1}(y) = \sum_{k=0}^{n}S_2(n,k)E_k(x).\end{aligned}$$

From Remark 3.3 (a), we get

$$\mathrm{Bel}_n(2x) = (\alpha*\beta)_n(x) = \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{l=0}^{n-i}\binom{n}{i}S_2(i,k)S_2(n-i,l)A_k(x)B_l(x).$$

From Remark 3.3 (b), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}2^lB_l(x)\frac{(e^t-1)^l}{l!} = \sum_{l=0}^{\infty}2^lB_l(x)\sum_{n=l}^{\infty}S_2(n,l)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}2^lS_2(n,l)B_l(x)\frac{t^n}{n!},\end{aligned}$$

which yields that

$$\sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{l=0}^{n-i}\binom{n}{i}S_2(i,k)S_2(n-i,l)B_k(x)E_l(x) = \sum_{l=0}^{n}2^lS_2(n,l)B_l(x).$$

From Remark 3.3 (c), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!} &= \frac{e^{e^t-1}-1}{e^t-1}\frac{2}{e^{e^t-1}+1}e^{2x(e^t-1)}\\ &= \sum_{l=0}^{\infty}\frac{1}{l+1}\sum_{i=0}^{l}\binom{l+1}{i+1}E_{l-i}(2x)\frac{(e^t-1)^l}{l!}\\ &= \sum_{l=0}^{\infty}\frac{1}{l+1}\sum_{i=0}^{l}\binom{l+1}{i+1}E_{l-i}(2x)\sum_{n=l}^{\infty}S_2(n,l)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}(\sum_{l=0}^{n}\sum_{i=0}^{l}\frac{1}{l+1}\binom{l+1}{i+1}S_2(n,l)E_{l-i}(2x))\frac{t^n}{n!},\end{aligned}$$

which implies that

$$\sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{l=0}^{n-i}\binom{n}{i}S_2(i,k)S_2(n-i,l)A_k(x)E_l(x) = \sum_{l=0}^{n}\sum_{i=0}^{l}\frac{1}{l+1}\binom{l+1}{i+1}S_2(n,l)E_{l-i}(2x).$$

Finally, from Remark 3.3 (d) we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!} &= \sum_{k=0}^{\infty}E_k(3x)\frac{(e^t-1)^k}{k!}\\ &= \sum_{k=0}^{\infty}E_k(3x)\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}S_2(n,k)E_k(3x)\frac{t^n}{n!},\end{aligned}$$

which entails that

$$\sum_{k=0}^{n}S_2(n,k)E_k(3x) = \sum_{k+l+m=n}\sum_{a=0}^{k}\sum_{b=0}^{l}\sum_{c=0}^{m}\binom{n}{k,l,m}S_2(k,a)S_2(l,b)S_2(m,c)A_a(x)B_b(x)E_c(x).$$

5. Arithmetic of Sheffer sequences

Assume that $s_n(x)$ is a sequence of polynomials with rational coefficients for the Sheffer pair $(g(t), f(t))$, so that $s_n(x) \sim (g(t), f(t))$. Then we have

$$\sum_{n=0}^{\infty}s_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)}, \tag{5.1}$$

where we put $s_n = s_n(0)$. Then we consider the following three types of integrals, namely Riemann, Volkenborn and Kim integrals respectively given by

$$\alpha_n(x) = \int_0^1 s_n(x+y)dy, \tag{5.2}$$

$$\beta_n(x) = \int_{\mathbb{Z}_p} s_n(x+y)d\mu(y), \tag{5.3}$$

$$\gamma_n(x) = \int_{\mathbb{Z}_p} s_n(x+y)d\mu_{-1}(y). \tag{5.4}$$

Then we see that the generating functions of $\{\alpha_n(x)\}$, $\{\beta_n(x)\}$, and $\{\gamma_n(x)\}$ are given by

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}\frac{e^{\bar{f}(t)}-1}{\bar{f}(t)}e^{x\bar{f}(t)}, \quad \alpha_n(x) \sim (g(t)\frac{t}{e^t-1}, f(t)), \tag{5.5}$$

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}\frac{\bar{f}(t)}{e^{\bar{f}(t)}-1}e^{x\bar{f}(t)}, \quad \beta_n(x) \sim (g(t)\frac{e^t-1}{t}, f(t)), \tag{5.6}$$

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}\frac{2}{e^{\bar{f}(t)}+1}e^{x\bar{f}(t)}, \quad \gamma_n(x) \sim (g(t)\frac{e^t+1}{2}, f(t)), \tag{5.7}$$

where we let $\alpha_n = \alpha_n(0), \beta_n = \beta_n(0), \gamma_n = \gamma_n(0)$.

Indeed, for instance we see (5.7) from the following:

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}s_n(x+y)\frac{t^n}{n!}d\mu_{-1}(y) = \frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)}\int_{\mathbb{Z}_p}e^{y\bar{f}(t)}d\mu_{-1}(y).$$

Remark 5.1. Replacing t by $f(t)$ in (5.5), (5.6), and (5.7), we obtain

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{f(t)^n}{n!} = \frac{1}{g(t)}\frac{e^t-1}{t}e^{xt} = \frac{1}{g(t)}\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!},$$
$$\sum_{n=0}^{\infty}\beta_n(x)\frac{f(t)^n}{n!} = \frac{1}{g(t)}\frac{t}{e^t-1}e^{xt} = \frac{1}{g(t)}\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},$$
$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{f(t)^n}{n!} = \frac{1}{g(t)}\frac{2}{e^t+1}e^{xt} = \frac{1}{g(t)}\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

Remark 5.2. (a) From (5.5) and (5.6), we obtain

$$\sum_{n=0}^{\infty}(\alpha*\beta)_n(x)\frac{t^n}{n!} = \frac{1}{(g(\bar{f}(t)))^2}e^{2x\bar{f}(t)} = \frac{1}{(g(\frac{1}{2}\overline{f(\frac{t}{2})}))^2}e^{x\overline{f(\frac{t}{2})}},$$

so that $(\alpha*\beta)_n(x) \sim ((g(\frac{t}{2}))^2, f(\frac{t}{2}))$.
(b) From (5.6) and (5.7), we get

$$\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!} = \frac{1}{(g(\bar{f}(t)))^2}\frac{2\bar{f}(t)}{e^{2\bar{f}(t)}-1}e^{2x\bar{f}(t)}$$
$$= \frac{1}{(g(\frac{1}{2}\overline{f(\frac{t}{2})}))^2}\frac{\overline{f(\frac{t}{2})}}{e^{\overline{f(\frac{t}{2})}}-1}e^{x\overline{f(\frac{t}{2})}},$$

so that $(\beta*\gamma)_n(x) \sim ((g(\frac{t}{2}))^2\frac{e^t-1}{t}, f(\frac{t}{2}))$.
(c) From (5.5) and (5.7), we see that

$$\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!} = \frac{1}{(g(\bar{f}(t)))^2}\frac{2(e^{\bar{f}(t)}-1)}{\bar{f}(t)(e^{\bar{f}(t)}+1)}e^{2x\bar{f}(t)}$$
$$= \frac{1}{(g(\frac{1}{2}\overline{f(\frac{t}{2})}))^2}\frac{4(e^{\frac{1}{2}\overline{f(\frac{t}{2})}}-1)}{\overline{f(\frac{t}{2})}(e^{\frac{1}{2}\overline{f(\frac{t}{2})}}+1)}e^{x\overline{f(\frac{t}{2})}},$$

so that $(\alpha*\gamma)_n(x)\sim\left((g(\frac{t}{2}))^2\frac{t(e^{\frac{t}{2}}+1)}{4(e^{\frac{t}{2}}-1)},f(\frac{t}{2})\right)$.
(d) From (5.5), (5.6), and (5.7), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!}&=\frac{1}{(g(\bar{f}(t)))^3}\frac{2}{e^{\bar{f}(t)}+1}e^{3x\bar{f}(t)}\\&=\frac{1}{\left(g(\frac{1}{3}\overline{f(\frac{t}{3})}\right)^3}\frac{2}{e^{\frac{1}{3}\overline{f(\frac{t}{3})}}+1}e^{x\overline{f(\frac{t}{3})}},\end{aligned}$$

so that $(\alpha*\beta*\gamma)_n(x)\sim\left((g(\frac{t}{3}))^3\frac{e^{\frac{t}{3}}+1}{2},f(\frac{t}{3})\right)$.

Using (2.15), (5.5)-(5.7), and Remark 5.2, we obtain the following recurrence relations:

$$\begin{aligned}\alpha_{n+1}(x)&=\left(x-\frac{g'(t)}{g(t)}-\frac{e^t-1-te^t}{t(e^t-1)}\right)\frac{1}{f'(t)}\alpha_n(x),\\\beta_{n+1}(x)&=\left(x-\frac{g'(t)}{g(t)}-\frac{1+te^t-e^t}{t(e^t-1)}\right)\frac{1}{f'(t)}\beta_n(x),\\\gamma_{n+1}(x)&=\left(x-\frac{g'(t)}{g(t)}-\frac{e^t}{e^t+1}\right)\frac{1}{f'(t)}\gamma_n(x),\\(\alpha*\beta)_{n+1}(x)&=2\left(x-\frac{g'(\frac{t}{2})}{g(\frac{t}{2})}\right)\frac{1}{f'(\frac{t}{2})}(\alpha*\beta)_n(x),\\(\beta*\gamma)_{n+1}(x)&=2\left(x-\frac{g'(\frac{t}{2})}{g(\frac{t}{2})}-\frac{1+te^t-e^t}{t(e^t-1)}\right)\frac{1}{f'(\frac{t}{2})}(\beta*\gamma)_n(x),\\(\alpha*\gamma)_{n+1}(x)&=2\left(x-\frac{g'(\frac{t}{2})}{g(\frac{t}{2})}-\frac{e^t-1-te^{\frac{t}{2}}}{t(e^t-1)}\right)\frac{1}{f'(\frac{t}{2})}(\alpha*\gamma)_n(x),\\(\alpha*\beta*\gamma)_{n+1}(x)&=3\left(x-\frac{g'(\frac{t}{3})}{g(\frac{t}{3})}-\frac{e^{\frac{t}{3}}}{3(e^{\frac{t}{3}}+1)}\right)\frac{1}{f'(\frac{t}{3})}(\alpha*\beta*\gamma)_n(x).\end{aligned}$$

Next, for any positive integer r, we let

$$\alpha_n^{(r)}(x)=\int_0^1\cdots\int_0^1 s_n(x+y_1+\cdots+y_r)dy_1\cdots dy_r, \tag{5.8}$$

$$\beta_n^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} s_n(x+y_1+\cdots+y_r)d\mu(y_1)\cdots d\mu(y_r), \tag{5.9}$$

$$\gamma_n^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} s_n(x+y_1+\cdots+y_r)d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r), \tag{5.10}$$

where the integrals are iterated r times. Then we see that the generating functions of $\left\{\alpha_n^{(r)}(x)\right\}$, $\left\{\beta_n^{(r)}(x)\right\}$, and $\left\{\gamma_n^{(r)}(x)\right\}$ are given by

$$\sum_{n=0}^{\infty}\alpha_n^{(r)}(x)\frac{t^n}{n!}=\frac{1}{g(\bar{f}(t))}\left(\frac{e^{\bar{f}(t)}-1}{\bar{f}(t)}\right)^r e^{x\bar{f}(t)},\quad \alpha_n^{(r)}(x)\sim(g(t)(\frac{t}{e^t-1})^r,f(t)), \tag{5.11}$$

$$\sum_{n=0}^{\infty}\beta_n^{(r)}(x)\frac{t^n}{n!}=\frac{1}{g(\bar{f}(t))}\left(\frac{\bar{f}(t)}{e^{\bar{f}(t)}-1}\right)^r e^{x\bar{f}(t)},\quad \alpha_n^{(r)}(x)\sim(g(t)(\frac{e^t-1}{t})^r,f(t)), \tag{5.12}$$

$$\sum_{n=0}^{\infty}\gamma_n^{(r)}(x)\frac{t^n}{n!}=\frac{1}{g(\bar{f}(t))}\left(\frac{2}{e^{\bar{f}(t)}+1}\right)^r e^{x\bar{f}(t)},\quad \gamma_n^{(r)}(x)\sim(g(t)(\frac{e^t+1}{2})^r,f(t)). \tag{5.13}$$

Remark 5.3. Replacing t by $f(t)$ respectively in (5.11), (5.12), and (5.13) gives us the following:

$$\sum_{n=0}^{\infty}\alpha_n^{(r)}(x)\frac{f(t)^n}{n!}=\frac{1}{g(t)}\left(\frac{e^t-1}{t}\right)^r e^{xt}=\frac{1}{g(t)}\sum_{n=0}^{\infty}A_n^{(r)}(x)\frac{t^n}{n!}, \tag{5.14}$$

$$\sum_{n=0}^{\infty}\beta_n^{(r)}(x)\frac{f(t)^n}{n!}=\frac{1}{g(t)}\left(\frac{t}{e^t-1}\right)^r e^{xt}=\frac{1}{g(t)}\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}, \tag{5.15}$$

$$\sum_{n=0}^{\infty}\gamma_n^{(r)}(x)\frac{f(t)^n}{n!}=\frac{1}{g(t)}\left(\frac{2}{e^t+1}\right)^r e^{xt}=\frac{1}{g(t)}\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}. \tag{5.16}$$

6. Examples on Sheffer sequences

(a) Let $s_n(x)=B_n(x)\sim(\frac{e^t-1}{t},t)$. Then we have

$$\alpha_n(x)=\int_0^1 B_n(x+y)dy=x^n, \tag{6.1}$$

$$\beta_n(x)=\int_{\mathbb{Z}_p} B_n(x+y)d\mu(y)=B_n^{(2)}(x), \tag{6.2}$$

$$\gamma_n(x)=\int_{\mathbb{Z}_p} B_n(x+y)d\mu_{-1}(y)=2^nB_n(\frac{x}{2}), \tag{6.3}$$

as we have from (5.5)-(5.7) the following:

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}\frac{e^t-1}{t}e^{xt}=e^{xt}, \tag{6.4}$$

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}\frac{t}{e^t-1}e^{xt}=(\frac{t}{e^t-1})^2e^{xt}, \tag{6.5}$$

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}\frac{2}{e^t+1}e^{xt}=\frac{2t}{e^{2t}-1}e^{xt}. \tag{6.6}$$

Here we remark that (6.1) is nothing but the well known identity:

$$\frac{1}{n+1}\{B_{n+1}(x+1)-B_{n+1}(x)\}=x^n, \quad (n\geq 0).$$

Note that the following holds:

$$\sum_{n=0}^{\infty}(\alpha*\beta)_n(x)\frac{t^n}{n!}=\left(\frac{t}{e^t-1}e^{xt}\right)^2=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(x)\frac{t^n}{n!}, \tag{6.7}$$

$$\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!}=(\frac{t}{e^t-1})^3e^{xt}\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_k^{(3)}(x)E_{n-k}(x)\frac{t^n}{n!}, \tag{6.8}$$

$$\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt}\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{n}\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(x)\frac{t^n}{n!}, \tag{6.9}$$

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!}&=\frac{t}{e^t-1}e^{xt}(\frac{t}{e^t-1})^2e^{xt}\frac{2}{e^t+1}e^{xt}\\&=\sum_{n=0}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}B_i(x)B_j^{(2)}(x)E_k(x)\frac{t^n}{n!}.\end{aligned} \tag{6.10}$$

Thus from (6.1)-(6.3) and (6.7)-(6.10) we get:

$$\sum_{k=0}^{n}\binom{n}{k}x^k B_{n-k}^{(2)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(x),$$

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}B_k^{(2)}(x)B_{n-k}(\frac{x}{2})=\sum_{k=0}^{n}\binom{n}{k}B_k^{(3)}(x)E_{n-k}(x),$$

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}x^k B_{n-k}(\frac{x}{2})=\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(x),$$

$$\sum_{i+j+k}\binom{n}{i,j,k}2^k x^i B_j^{(2)}(x)B_k(\frac{x}{2})=\sum_{i+j+k=n}\binom{n}{i,j,k}B_i(x)B_j^{(2)}(x)E_k(x).$$

(b) Let $s_n(x)=b_n(x)\sim(\frac{t}{e^t-1},e^t-1)$. Then we have

$$\alpha_n(x)=\int_0^1 b_n(x+y)dy=b_n^{(2)}(x), \tag{6.11}$$

$$\beta_n(x)=\int_{\mathbb{Z}_p} b_n(x+y)d\mu(y)=(x)_n, \tag{6.12}$$

$$\gamma_n(x)=\int_{\mathbb{Z}_p} b_n(x+y)d\mu_{-1}(y)=\sum_{k=0}^{n}\binom{n}{k}b_k Ch_{n-k}(x), \tag{6.13}$$

as we have from (5.5)-(5.7) the following:

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!}=(\frac{t}{\log(1+t)})^2(1+t)^x=\sum_{n=0}^{\infty}b_n^{(2)}(x)\frac{t^n}{n!}, \tag{6.14}$$

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}=(1+t)^x=\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}, \tag{6.15}$$

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!}=\frac{t}{\log(1+t)}\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k Ch_{n-k}(x)\frac{t^n}{n!}. \tag{6.16}$$

Note that the following holds:

$$\sum_{n=0}^{\infty}(\alpha*\beta)_n(x)\frac{t^n}{n!}=\left(\frac{t}{\log(1+t)}(1+t)^x\right)^2=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k(x)b_{n-k}(x)\frac{t^n}{n!}, \tag{6.17}$$

$$\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!}=\frac{t}{\log(1+t)}(1+t)^x\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k(x)Ch_{n-k}(x)\frac{t^n}{n!}, \tag{6.18}$$

$$\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!}=\left(\frac{t}{\log(1+t)}\right)^3(1+t)^x\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k^{(3)}(x)Ch_{n-k}(x)\frac{t^n}{n!}, \tag{6.19}$$

$$\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!}=\frac{t}{\log(1+t)}(1+t)^x(\frac{t}{\log(1+t)})^2(1+t)^x\frac{2}{2+t}(1+t)^x \tag{6.20}$$

$$=\sum_{n=0}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}b_i(x)b_j^{(2)}(x)Ch_k(x)\frac{t^n}{n!}.$$

Now, from (6.14)-(6.20) we obtain:

$$\sum_{k=0}^{n}\binom{n}{k}b_k^{(2)}(x)(x)_{n-k} = \sum_{k=0}^{n}\binom{n}{k}b_k(x)b_{n-k}(x),$$

$$\sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{n}{k,i,n-k-i}b_i(x)_k Ch_{n-k-i}(x) = \sum_{k=0}^{n}\binom{n}{k}b_k(x)Ch_{n-k}(x),$$

$$\sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{n}{k,i,n-k-i}b_i b_k^{(2)}(x)Ch_{n-k-i}(x) = \sum_{k=0}^{n}\binom{n}{k}b_k^{(3)}(x)Ch_{n-k}(x),$$

$$\sum_{i+j+k=n}\sum_{l=0}^{k}\binom{n}{i,j,l,k-l}b_l b_i^{(2)}(x)(x)_j Ch_{k-l}(x) = \sum_{i+j+k=n}\binom{n}{i,j,k}b_i(x)b_j^{(2)}(x)Ch_k(x).$$

(c) Let $s_n(x) = D_n(x) \sim (\frac{e^t-1}{t}, e^t - 1)$. Then we have

$$\alpha_n(x) = \int_0^1 D_n(x+y)dy = (x)_n, \tag{6.21}$$

$$\beta_n(x) = \int_{\mathbb{Z}_p} D_n(x+y)d\mu(y) = D_n^{(2)}(x), \tag{6.22}$$

$$\gamma_n(x) = \int_{\mathbb{Z}_p} D_n(x+y)d\mu_{-1}(y) = \sum_{m=0}^{n}\binom{n}{m}(m)_{n-m}2^{2m-n}D_m(\frac{x}{2}), \tag{6.23}$$

as we have from (5.5)-(5.7) the following:

$$\sum_{n=0}^{\infty}\alpha_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}\frac{t}{\log(1+t)}(1+t)^x \tag{6.24}$$

$$= (1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}\frac{\log(1+t)}{t}(1+t)^x \tag{6.25}$$

$$= \left(\frac{\log(1+t)}{t}\right)^2(1+t)^x = \sum_{n=0}^{\infty}D_n^{(2)}(x)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}D_k Ch_{n-k}(x)\frac{t^n}{n!}. \tag{6.26}$$

Note that the following holds:

$$\sum_{n=0}^{\infty}(\alpha*\beta)_n(x)\frac{t^n}{n!} = \left(\frac{\log(1+t)}{t}(1+t)^x\right)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}D_k(x)D_{n-k}(x)\frac{t^n}{n!}, \tag{6.27}$$

$$\sum_{n=0}^{\infty}(\beta*\gamma)_n(x)\frac{t^n}{n!} = \left(\frac{\log(1+t)}{t}\right)^3(1+t)^x\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}D_k^{(3)}(x)Ch_{n-k}(x)\frac{t^n}{n!}, \tag{6.28}$$

$$\sum_{n=0}^{\infty}(\alpha*\gamma)_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}(1+t)^x\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}D_k(x)Ch_{n-k}(x)\frac{t^n}{n!}, \tag{6.29}$$

$$\sum_{n=0}^{\infty}(\alpha*\beta*\gamma)_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}(1+t)^x\Big(\frac{\log(1+t)}{t}\Big)^2(1+t)^x\frac{2}{2+t}(1+t)^x \tag{6.30}$$
$$= \sum_{n=0}^{\infty}\sum_{i+j+k=n}\binom{n}{i,j,k}D_i(x)D_j^{(2)}(x)Ch_k(x)\frac{t^n}{n!}.$$

Now, from (6.24)-(6.30) we obtain:

$$\sum_{k=0}^{n}\binom{n}{k}D_k^{(2)}(x)(x)_{n-k} = \sum_{k=0}^{n}\binom{n}{k}D_k(x)D_{n-k}(x),$$
$$\sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{n}{k,i,n-k-i}D_iD_k^{(2)}(x)Ch_{n-k-i}(x) = \sum_{k=0}^{n}\binom{n}{k}D_k^{(3)}(x)Ch_{n-k}(x),$$
$$\sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{n}{k,i,n-k-i}D_i(x)_kCh_{n-k-i}(x) = \sum_{k=0}^{n}\binom{n}{k}D_k(x)Ch_{n-k}(x),$$
$$\sum_{i+j+k=n}\sum_{l=0}^{k}\binom{n}{i,j,l,k-l}D_lD_i^{(2)}(x)(x)_jCh_{k-l}(x) = \sum_{i+j+k=n}\binom{n}{i,j,k}D_i(x)D_j^{(2)}(x)Ch_k(x).$$

Remark 6.1. We observe that

$$\sum_{n=0}^{\infty}\gamma_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}\frac{2}{2+t}(1+t)^x = \frac{\log(1+2t+t^2)}{2t+t^2}(1+2t+t^2)^{\frac{x}{2}} \tag{6.31}$$
$$= \sum_{m=0}^{\infty}D_m(\frac{x}{2})\frac{t^m}{m!}\sum_{k=0}^{\infty}(m)_k2^{m-k}\frac{t^k}{k!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(m)_{n-m}2^{2m-n}D_m(\frac{x}{2})\frac{t^n}{n!}.$$

Thus, from (6.26) and (6.31), we obtain

$$\gamma_n(x) = \sum_{k=0}^{n}\binom{n}{k}D_kCh_{n-k}(x) = \sum_{m=0}^{n}\binom{n}{m}(m)_{n-m}2^{2m-n}D_m(\frac{x}{2}).$$

7. Conclusion

For any given Sheffer sequence $s_n(x)$ with rational coefficients for the pair $(g(t), f(t))$, we considered the Riemann integral from 0 to 1, Volkenborn integral on $\mathbb{Z}_p$ and Kim integral on $\mathbb{Z}_p$ of $s_n(x+y)$ with respect to y. In this way, we obtained three different Sheffer sequences $\alpha_n(x)$,$\beta_n(x)$, and $\gamma_n(x)$ which are respectively Sheffer for $(g(t)\frac{t}{e^t-1}, f(t))$, $(g(t)\frac{e^t-1}{t}, f(t))$, and $(g(t)\frac{e^t+1}{2}, f(t))$. This provides us with means of constructing new Sheffer sequences out of any given Sheffer sequence. For the new Sheffer sequences, among other things we investigated their convolutions and derived some interesting identities. More generally, we constructed 'Sheffer sequences of order r,' namely $\alpha_n^{(r)}(x)$,$\beta_n^{(r)}(x)$, and $\gamma_n^{(r)}(x)$ arising from the r times iterated integrals of each of the aforementioned integrals of $s_n(x+y_1+\cdots+y_r)$ with respect to $y_1,\dots y_r$. Furthermore, we illustrated our results with some examples in either case.
We think that this idea of constructing new Sheffer sequences out of any given one by using Riemann, Volkenborn and Kim integrals has possible applications in physics, science and engineering as well as in mathematics. As one of our future research projects, we would like to continue and extend this idea to various directions.

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References

[1] S. Araci, *Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus*, Appl. Math. Comput. **233**, 599–607, 2014.
[2] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **22 (3)**, 433–438, 2012.
[3] B. Diarra, *The p-adic q-distribution,* Advances in Ultrametric Analysis: 12th International Conference on *p*-adic Functional Analysis, 45–62.
[4] B. S. El-Desouky and A. Mustafa, *New results on higher-order Daehee and Bernoulli numbers and polynomials*, Adv. Difference Equ. **2016** **(32)**, 2016.
[5] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **9 (3)**, 288–299, 2002.
[6] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys. **16**, 93–96, 2009.
[7] D. S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. **7 (120)**, 5969–5976, 2013.
[8] D. S. Kim and T. Kim, *Degenerate Sheffer sequences and* λ*-Sheffer sequences*, J. Math. Anal. Appl. **493 (1)**, 1–21 Paper No. 124521, 2021.
[9] D. S. Kim, T. Kim, J. Kwon, S.-H. Lee and S. Park, *On* λ*-linear functionals arising from p-adic integrals on* $\mathbb{Z}_p$, Adv. Difference Equ. **2021** **(479)**, 2021.
[10] D. S. Kim, T. Kim, S.-H. Lee and J.-J. Seo, *Higher-order Daehee numbers and polynomials*, Int. J. Math. Anal. **8 (6)**, 273–283, 2014.
[11] D. S. Kim, T. Kim and J. J. Seo, *A note on Changhee polynomials and numbers*, Adv. Stud. Theor. Phys. **7**, 1–10, 2013.
[12] D. S. Kim, T. Kim, J.-J. Seo and S.-H. Lee, *Higher-order Changhee numbers and polynomials*, Adv. Stud. Theor. Phys. **8 (8)**, 365–373, 2014.
[13] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[14] S. Roman, P. De Land, R. Shiflett and H. Shultz, *The umbral calculus and the solution to certain recurrence relations*, J. Comb. Inf. Syst. Sci. **8 (4)**, 235–240, 1983.
[15] W. H. Schikhof, *Ultrametric calculus*, An introduction to *p*-adic analysis, Cambridge University Press, Cambridge, 2010.
[16] Y. Simsek, *Identities on the Changhee numbers and Apostol-type Daehee polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **27 (2)**, 199–212, 2017.