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Stability of Semigroups Defined on Tensor Products of Hilbert Spaces

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Abstract

The paper deals with a class of strongly continuous semigroups generated by operators defined on the tensor product of Hilbert spaces. Explicit exponential stability conditions for the considered semigroups are derived. Applications of the obtained conditions to semigroups generated by matrix differential operators and integro-differential operators are also discussed.

Keywords: Hilbert space, Semigroup, Tensor product, Stability, Matrix differential operator, Integro-differential equation

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1. Introduction and notations

This paper is devoted to stability conditions for a class of semigroups generated by operators defined on the tensor product of Hilbert spaces. Such semigroups arise in various problems of pure and applied mathematics (*cf.* [9, 10]) and the references given therein. The literature on the asymptotic behavior of semigroups is very rich (*cf.* [5, 7, 8]) and many other works, but to the best of our knowledge, the stability conditions for semigroups on the tensor product of Hilbert spaces have not been investigated yet. In the present paper we derive explicit exponential stability conditions for the considered semigroups. We also discuss applications of our results to semigroups generated by matrix differential and integro-differential operators. Our main tool is the recent norm estimates for operator functions.

Introduce the notations. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be complex separable Hilbert spaces with scalar products $< .,. >_1$ and $< .,. >_2$, respectively. $\|.\|_j = \sqrt{< .,. >_j}$ is the norm and $I = I_j$ is the unit operator in $\mathcal{E}_j$ $(j = 1, 2)$. Recall that the tensor product $\mathcal{H} = \mathcal{E}_1 \otimes \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is defined in the following way (*cf.* [4, 6]). Consider the collection of all formal finite sums of the form

$$u = \sum_j y_j \otimes h_j \ \ (y_j \in \mathcal{E}_1, h_j \in \mathcal{E}_2)$$

with the understanding that

$$\lambda(y \otimes h) = (\lambda y) \otimes h = y \otimes (\lambda h), (y + y_1) \otimes h = y \otimes h + y_1 \otimes h,$$

$$y \otimes (h + h_1) = y \otimes h + y \otimes h_1 \ \ (y, y_1 \in \mathcal{E}_1;\ h, h_1 \in \mathcal{E}_2;\ \lambda \in \mathbb{C}).$$

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On that collection define the scalar product as

$$< y\otimes h, y_1\otimes h_1 >_{\mathcal{H}} =< y, y_1 >_1 < h, h_1 >_2 \quad (y, y_1 \in \mathcal{E}_1, h, h_1 \in \mathcal{E}_2)$$

and take the norm $\|.\|_{\mathcal{H}} = \sqrt{< .,. >_{\mathcal{H}}}$. Then $\mathcal{H}$ is the completion of the considered collection in the norm $\|.\|_{\mathcal{H}}$. Besides $I_{\mathcal{H}} = I$ denotes the unit operator in $\mathcal{H}$.

For a linear operator A, $\sigma(A)$ is the spectrum, $D(A)$ is the domain, A^{-1} is the inverse operator, $R_\lambda(A) := (A - I\lambda)^{-1}$ ($\lambda \notin \sigma(A)$) is the resolvent, $\lambda_k(A)$ ($k = 1, 2, \cdots$) are the eigenvalues taken with their multiplicities, A^* is the adjoint operator, $\alpha(A) = \sup \mathrm{Re}\,(\sigma(A))$ and $\beta(A) = \inf \mathrm{Re}\,(\sigma(A))$. $L(\mathcal{E})$ denotes the set of all bounded operators in a space $\mathcal{E}$ and $\|A\| = \|A\|_{\mathcal{E}}$ means the operator norm of $A \in L(\mathcal{E})$.

Recall that Kronecker's product of $A \in L(\mathcal{E}_1)$ and $B \in L(\mathcal{E}_2)$ denoted by $A \otimes B$ is defined by

$$(A \otimes B)(f_1 \otimes f_2) = (Af_1) \otimes (Bf_2) \quad (f_l \in \mathcal{E}_l, l = 1, 2),$$

(*cf.* [4]). In the next section we recall some properties of the Kronecker product of operators.

Below we consider perturbations of the semigroup generated by of the operator $S + I_1 \otimes T$, where $T \in L(\mathcal{E}_2)$ and S is an unbounded operator on $\mathcal{H}$ commuting with $I_1 \otimes T$.

2. Properties of the Kronecker products

The results of this section are taken from [4].

Lemma 2.1. *With $A, A_1 \in \mathcal{E}_1$, $B, B_1 \in \mathcal{E}_2$ we have*

(a) $(A + A_1) \otimes B = A \otimes B + A_1 \otimes B$; $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$;

$(\lambda A \otimes B) = \lambda(A \otimes B)$; $(A \otimes \lambda B) = \lambda(A \otimes B)$ $(\lambda \in \mathbb{C})$.

(b) $I_1 \otimes I_2 = I_{\mathcal{H}}$.

(c) $(A\otimes B)(A_1\otimes B_1) = (AA_1)\otimes(BB_1)$.

(d) $(A\otimes B)^* = A^*\otimes B^*$.

(e) $\|A\otimes B\|_{\mathcal{H}} = \|A\|_1\|B\|_2$.

(f) *$A\otimes B$ is invertible if and only if A and B are both invertible,*

in which case $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Theorem 2.2. *If $A \in L(\mathcal{E}_1)$ and $B \in L(\mathcal{E}_2)$, then the spectrum, of $A \otimes B$ is*

$$\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B) = \{ts : t \in \sigma(A), s \in \sigma(B)\}.$$

About the recent results on operators on tensor products of Hilbert spaces see for instance, in [12, 14, 15].

3. Lyapunov's equation

A C_0-semigroup e^{Bt} generated by an operator B is said to be (uniformly) exponentially stable, if there are constants $m_0 \geq 1$ and $\delta_0 > 0$, such that $\|e^{Bt}\| \leq m_0 e^{-\delta_0 t}$ $(t \geq 0)$.

We need the following well-known theorem, *cf.* [5, Theorem 5.1.3, p.217].

Theorem 3.1. *Suppose that B is the infinitesimal generator of a C_0-semigroup e^{Bt} on a Hilbert space $\mathcal{H}$. Then e^{Bt} is exponentially stable if and only if there exists a bounded positive definite operator P, such that*

$$< Bz, Pz >_{\mathcal{H}} + < Pz, Bz >_{\mathcal{H}} = - < z, z >_{\mathcal{H}} \quad (z \in D(B)). \tag{3.1}$$

Moreover, if B is the infinitesimal generator of an exponentially stable C_0-semigroup then due to [5, Section 5.5.3a, equality (5.62)], for any $Q \in \mathcal{L}(\mathcal{H})$ the equation

$$< Bz, Pz >_{\mathcal{H}} + < Pz, Bz >_{\mathcal{H}} = - < z, Qz >_{\mathcal{H}} \ \mathcal{H} \ (z \in D(B)) \tag{3.2}$$

has a solution $P \in \mathcal{L}(\mathcal{H})$, which due to [5, Section to 5.5.3a] is representable as

$$P = \int_0^\infty e^{B^* t} Q e^{Bt} dt. \tag{3.3}$$

For a self-adjoint operator R we write $R > 0$ ($R < 0$), if it is positive (negative) definite. Similarly, we write $R \geq 0$ ($R \leq 0$), if it is positive (negative) semidefinite. Besides for two self-adjoint operators R and R_1 we write $R \geq R_1$ if $R - R_1 \geq 0$. We also need the following well-known result (*cf.* [7, Theorem III.7.1v, p.129]).

Theorem 3.2. *Suppose that B is the infinitesimal generator of a C_0-semigroup e^{Bt} on a Hilbert space $\mathcal{H}$. Then e^{Bt} is exponentially stable if and only if there exists a positive definite operator $P \in \mathcal{L}(\mathcal{H})$ with $PD(B) \subseteq D(B^*)$, such that*

$$B^*P + PB \leq -I \text{ on } D(B).$$

Taking $\hat{P} = d_0 P$ ($d_0 = const > 0$), from this theorem we get

Corollary 3.3. *Suppose that B is the infinitesimal generator of a C_0-semigroup e^{Bt} on a Hilbert space $\mathcal{H}$. Then e^{Bt} is exponentially stable if and only if there exists a positive definite operator $\hat{P} \in \mathcal{L}(\mathcal{H})$ with $\hat{P}D(B) \subseteq D(B^*)$, and a positive constant d_0, such that*

$$B^*\hat{P} + \hat{P}B \leq -d_0 I \text{ on } D(B).$$

Let us investigate perturbations. To this end consider on $\mathcal{H}$ an operator $\tilde{B}$, satisfying the conditions

$$D(\tilde{B}) = D(B) \text{ and } \|\tilde{B} - B\| < \infty. \tag{3.4}$$

Remark 3.4. Since $\tilde{B} - B$ is bounded, due to [8, Corollary III.1.5, p.119] $\tilde{B}$ generates a C_0-semigroup if B generates a C_0-semigroup and $\tilde{B} - B$ maps $D(B)$ into itself. In addition, due to [8, Proposition III.1.12, p.122] $\tilde{B}$ generates a an analytic semigroup if B generates an analytic semigroup.

Lemma 3.5. *Suppose that operators B and $\tilde{B}$ defined on $\mathcal{H}$ generate the C_0-semigroups e^{Bt} and $e^{\tilde{B}t}$, respectively, and e^{Bt} is exponentially stable and condition by* (3.4) *hold. Let X be a solution of*

$$B^*X + XB = -2I \tag{3.5}$$

and

$$\|\tilde{B} - B\|\|X\| < 1. \tag{3.6}$$

Then $e^{\tilde{B}t}$ is also exponentially stable.

Proof. Put $E = \tilde{B} - B$. Then from (3.5) it follows

$$\begin{aligned} \tilde{B}^*X + X\tilde{B} = (B + E)^*X + X(B + E) &= B^*X + XB + E^*X + XE \\ &= -2I + E^*X + XE \end{aligned}$$

If condition (3.6) holds, then $\tilde{B}^*X + X\tilde{B} < -d_0 I$ with $d_0 = 2(1 - \|E\|\|X\|)$. Now Corollary 3.3 implies the required result. □

4. The main result

Let S be a generator of a C_0-semigroup e^{St} on $\mathcal{H}$, satisfying

$$\|e^{St}\|_{\mathcal{H}} \le c_0\, e^{\nu t} \ \ (\nu, c_0 = const;\ t \ge 0). \tag{4.1}$$

In addition, $T \in L(\mathcal{E}_2)$, and $I_1 \otimes T$ commutes with S. Take

$$A = S + I_1 \otimes T. \tag{4.2}$$

To investigate perturbations of A consider the operator $\tilde{A}$ satisfying

$$D(\tilde{A}) = D(A) \text{ and } q := \|\tilde{A} - A\|_{\mathcal{H}} < \infty. \tag{4.3}$$

Since $I_1 \otimes T$ and S commute, we have $e^{tA} = e^{tS}\, e^{t(I_1 \otimes T)}$, and therefore,

$$\|e^{tA}\|_{\mathcal{H}} \le \|e^{tS}\|_{\mathcal{H}} \|e^{tT}\|_2 \le c_0 e^{\nu t} \|e^{tT}\|_2 \ \ (t \ge 0). \tag{4.4}$$

Note that $(I_1 \otimes T)^k = I_1 \otimes T^k$ $(k = 1, 2, ...)$ and consequently, $e^{t(I_1 \otimes T)} = I_1 \otimes e^{Tt}$.

In addition, since T is bounded, by the Dunford integral,

$$e^{tT} = \frac{1}{2\pi} \int_l e^{tz} (T - zI_2)^{-1} dz,$$

where l is a Jordan contour surrounding $\sigma(T)$. Hence, for any $\epsilon > 0$, taking a fitting contour, we can write

$$\|e^{tT}\|_2 \le b_\epsilon e^{(\alpha(T)+\epsilon)t} \ \ (t \ge 0;\ b_\epsilon = const). \tag{4.5}$$

Assume that

$$\nu + \alpha(T) < 0. \tag{4.6}$$

Then, according to (4.1)

$$\int_0^\infty \|e^{At}\|^2 dt \le c_0^2 \int_0^\infty e^{2\nu t} \|e^{Tt}\|^2 dt < \infty.$$

So the operator

$$Y := 2 \int_0^\infty e^{A^* t} e^{At} dt \tag{4.7}$$

is bounded and

$$\|Y\| \le 2 \int_0^\infty c_0^2 e^{2\nu t} \|e^{tT}\|^2 dt. \tag{4.8}$$

According to (3.3)

$$2\mathrm{Re}\ (AY) = A^* Y + YA = -2I. \tag{4.9}$$

Hence, making use of Lemma 3.5 and (4.5), we get the main result of the paper.

Theorem 4.1. *Let $\tilde{A}$ be the infinitesimal generator of a C_0-semigroup $e^{\tilde{A}t}$. Let the conditions* (4.1), (4.3), (4.6) *and*

$$q c_0^2 \int_0^\infty e^{2\nu t} \|e^{tT}\|^2 dt < 1 \tag{4.10}$$

hold. Then $e^{\tilde{A}t}$ is exponentially stable.

Now put

$$w(T) = \frac{c_0^2}{\pi}\int_{-\infty}^{\infty} \|(T + (\nu - is)I_2)^{-1}\|_2^2 ds,$$

assuming that the integral converges. Due to (4.1),

$$< Yx, x >_{\mathcal{H}} = 2\int_0^{\infty} \|e^{St}e^{t(I_1\otimes T)}x\|_{\mathcal{H}}^2 dt \le 2c_0^2\int_0^{\infty} \|e^{\nu t}e^{t(I\otimes T)}x\|_{\mathcal{H}}^2 dt \quad (x \in \mathcal{H}).$$

Under (4.6) the Laplace original of $(I_1 \otimes T + (\nu - zI_{\mathcal{H}}))^{-1}$ (Re $z \ge 0$) is $e^{\nu t}I_1 \otimes e^{tT}$. By the Plancherel equality [2, Theorem 5.2.1, p. 351], for any $x \in \mathcal{H}$ we have

$$\int_0^{\infty} e^{2\nu t}\|I_1 \otimes e^{tT}x\|_{\mathcal{H}}^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|(I_1 \otimes (T + (\nu - is)I_2)^{-1}x\|_{\mathcal{H}}^2 ds.$$

Hence,

$$\|Y\|_{\mathcal{H}} \le w(T). \tag{4.11}$$

Now Theorem 4.1 implies

Corollary 4.2. *Let $\tilde{A}$ generate a C_0-semigroup $e^{\tilde{A}t}$. Let the conditions* (4.1), (4.3), (4.6) *and $qw(T) < 1$ hold. Then $e^{\tilde{A}t}$ is exponentially stable.*

Furthermore, suppose that S is normal with $\alpha(S) < \infty$, $M \in L(\mathcal{H})$ and $\tilde{A} = S + M$. Then $q = \|\tilde{A} - A\|_{\mathcal{H}} = \|M - I_1 \otimes T\|_{\mathcal{H}} < \infty$, S generates an analytic semigroup, $\|e^{St}\| = e^{t\alpha(S)}$ $(t \ge 0)$ and according to Remark 3.4. $e^{\tilde{A}t}$ generates an analytic semigroup, since M is bounded. Assume that

$$\alpha(S) + \alpha(T) < 0. \tag{4.12}$$

Then according to (4.8) we have

$$\|Y\|_{\mathcal{H}} \le \int_0^{\infty} e^{2\alpha t}\|e^{tT}\|_2^2 dt.$$

Now Theorem 4.1 implies

Corollary 4.3. *Let S be a normal operator, and $\tilde{A} = S + M$ with $M \in \mathcal{L}(\mathcal{H})$. If the conditions* (4.12) *and*

$$q\int_0^{\infty} e^{2\alpha t}\|e^{tT}\|_2^2 dt < 1$$

hold, then $e^{\tilde{A}t}$ is exponentially stable.

5. Norm estimates for perturbed semigroups

Let $\tilde{A}$ generate a C_0-semigroup $e^{\tilde{A}t}$ and Y be defined by (4.7), and

$$\alpha_0 := 1 - q\|Y\| > 0. \tag{5.1}$$

Then

$$2\text{Re}\,(Y\tilde{A}) = Y\tilde{A} + \tilde{A}^*Y \le -2\alpha_0 I,$$

and, letting $x(t) = e^{\tilde{A}t}x_0$ $(x_0 \in D(\tilde{A}))$, we obtain $\frac{dx}{dt} = \tilde{A}x$ and

$$\frac{d}{dt} < Yx, x >_{\mathcal{H}} = 2\text{Re}\ < Y\tilde{A}x, x >_{\mathcal{H}} = -2\alpha_0 < x, x >_{\mathcal{H}} \quad (x = x(t), t \ge 0). \tag{5.2}$$

But

$$< x, x >_{\mathcal{H}} = < Y^{-1}Yx, x >_{\mathcal{H}} \ge \beta(Y^{-1}) < Yx, x >_{\mathcal{H}}$$

$$= \frac{1}{\alpha(Y)} < Yx, x >_{\mathcal{H}} \leq \frac{1}{\|Y\|_{\mathcal{H}}} < Yx, x >_{\mathcal{H}},$$

where $\beta(Y^{-1}) = \inf \sigma(Y^{-1}), \alpha(Y) = \sup \sigma(Y)$. Therefore,

$$\frac{d}{dt} < Yx, x >_{\mathcal{H}} \leq -\frac{2\alpha_0}{\|Y\|_{\mathcal{H}}} < Yx, x >_{\mathcal{H}} .$$

Solving this inequality, we obtain

$$< Yx(t), x(t) >_{\mathcal{H}} \leq \exp[-\frac{2\alpha_0 t}{\|Y\|_{\mathcal{H}}}] < Yx(0), x(0) >_{\mathcal{H}} .$$

Introducing the norm

$$\|x\|_Y = \sqrt{< Yx, x >_{\mathcal{H}}},$$

we arrive at the inequality

$$\|e^{\tilde{A}t} x_0\|_Y \leq \exp[-\frac{\alpha_0 t}{\|Y\|_{\mathcal{H}}}]\|x_0\|_Y. \tag{5.3}$$

Furthermore, from (5.2) we have

$$\int_t^\infty \frac{d}{ds} < Yx(s), x(s) >_{\mathcal{H}} ds \leq -2\alpha_0 \int_t^\infty < x(s), x(s) >_{\mathcal{H}} ds \ \ (t \geq 0).$$

Hence, taking into account that $x(\infty) = 0$, we obtain

$$< Yx(t), x(t) >_{\mathcal{H}} \geq 2\alpha_0 \int_t^\infty < x(s), x(s) >_{\mathcal{H}} ds \ \ (t \geq 0). \tag{5.4}$$

We thus have prowed the following result.

Lemma 5.1. *Let $\tilde{A}$ generate a C_0 semigroup $e^{\tilde{A}t}$ and condition* (5.1) *hold. Then the inequalities* (5.3) *and*

$$2\alpha_0 \int_0^\infty \|e^{\tilde{A}s} x_0\|_{\mathcal{H}}^2 ds \leq < Yx_0, x_0 >_{\mathcal{H}} \leq \|Y\|_{\mathcal{H}} \|x_0\|_{\mathcal{H}}^2 \ \ (x_0 \in D(\tilde{A}))$$

are valid.

Now we can apply (4.8) and (4.11).

The recent norm estimates for solutions of various differential equations in a Hilbert space can be found, for example in the books [1, 13].

6. The case $\dim \mathcal{E}_2 < \infty$

In this section $\dim \mathcal{E}_2 = n < \infty$. So $\mathcal{E}_2 = \mathbb{C}^n$ the Euclidean space and T is an $n \times n$-matrix. Introduce the quantity (the departure from normality) of T:

$$g(T) := [N_2^2(T) - \sum_{k=1}^n |\lambda_k(T)|^2]^{1/2},$$

where $N_2(T) = (\text{Trace } T^*T)^{1/2}$ is the Hilbert-Schmidt (Frobenius) norm of T; $\lambda_j(T)$ $(j = 1, ..., n)$ are the eigenvalues of T taken with their multiplicities. $g(T)$ enjoys the following properties, which are checked in [11, Section 3.1].

$$g^2(T) \leq N_2^2(T) - |\text{trace } T^2| \text{ and } g^2(T) \leq 2N_2^2(T_I),$$

where $T_I = (T - T^*)/2i$. If T is a normal matrix: $TT^* = T^*T$, then $g(T) = 0$. If T_1 and T are commuting matrices, then

$$g(T_1 + T) \leq g(T_1) + g(T).$$

By the inequality between geometric and arithmetic mean values,

$$(\frac{1}{n}\sum_{k=1}^{n}|\lambda_k(T)|^2)^n \geq (\prod_{k=1}^{n}|\lambda_k(T)|)^2.$$

So $g^2(T) \leq N_2^2(T) - n(\det\ T)^{2/n}$.

Lemma 6.1. *Let T be an $n\times n$-matrix, the conditions* (4.1), (4.6) *hold and Y be defined by* (4.7). *Then $\|Y\|_{\mathcal{H}} \leq \hat{J}(Y)$, where*

$$\hat{J}(Y) := 2c_0^2\sum_{j,k=0}^{n-1}\frac{g^{j+k}(T)(k+j)!}{(2|\nu+\alpha(T)|)^{j+k+1}(j!\ k!)^{3/2}}.$$

Proof. By virtue of Example 3.2 from [11], we have

$$\|e^{Tt}\|_2 \leq \exp[\alpha(T)t]\sum_{k=0}^{n-1}\frac{g^k(T)t^k}{(k!)^{3/2}}\ (t\geq 0).$$

Now (4.8) implies

$$\begin{aligned}
\|Y\|_{\mathcal{H}} &\leq 2c_0^2\int_0^\infty e^{2\nu t}\|e^{Tt}\|^2 dt\\
&\leq 2c_0^2\int_0^\infty \exp[2(\nu+\alpha(T))t]\left(\sum_{k=0}^{n-1}\frac{g^k(T)t^k}{(k!)^{3/2}}\right)^2 dt\\
&= 2c_0^2\int_0^\infty \exp[2(\nu+\alpha(T))t]\sum_{j,k=0}^{n-1}\frac{g^{j+k}(T)t^{j+k}}{(j!k!)^{3/2}}dt.
\end{aligned}$$

Since

$$\int_0^\infty \exp[2(\nu+\alpha(T))t]\frac{t^{j+k}}{(j!k!)^{3/2}}dt = \frac{(j+k)!}{(2|\nu+\alpha(T)|)^{j+k+1}},$$

we get the required result.

□

If T is normal, then $g(T) = 0$ and, taking $0^0 = 1$ we have

$$\hat{J}(Y) = \frac{c_0^2}{|\nu+\alpha(T)|}.$$

The latter lemma and Theorem 4.1 imply

Corollary 6.2. *Let T be an $n\times n$ matrix and the conditions* (4.1), (4.3), (4.6) *and $q\hat{J}(T) < 1$ hold. Then $e^{\tilde{A}t}$ is exponentially stable.*

According to Corollary 4.2, instead of a norm estimate for the exponential function one can apply a norm estimate for the resolvent. About norm estimates for resolvents of matrices see for example [11, Section 3.2].

7. Operators with Hilbert-Schmidt Hermitian components

Denote by $\mathcal{S}_2$ the Hilbert-Schmidt ideal of operators K in $\mathcal{E}_2$ with the finite norm $N_2(K) := [\text{trace}\,(KK^*)]^{1/2}$.
In this section we obtain estimates for Y, assuming that $T \in L(\mathcal{E}_2)$ and

$$T_I := \frac{1}{2i}(T - T^*) \in \mathcal{S}_2. \tag{7.1}$$

To this end put

$$g_I(T) := [2N_2^2(T_I) - 2\sum_{k=1}^{\infty} |\text{Im } \lambda_k(T)|^2 \ ]^{1/2} \leq \sqrt{2}N_2(T_I),$$

where $\lambda_k(T)$ $(k = 1, 2, ...)$ are the eigenvalues of T taken with their multiplicities and ordered as $|\text{Im } \lambda_{k+1}(T)| \leq |\text{Im } \lambda_k(T)|$ $(k = 1, 2, ...)$.

If T is normal, then $g_I(T) = 0$, *cf.* [11, Lemma 9.3]. Some properties of $g_I(.)$ are checked in [11, Chapter 9].

Lemma 7.1. *Let the conditions* (4.1), (4.6) *and* (7.1) *hold. Then* $\|Y\| \leq m_I(Y)$, *where*

$$m_I(Y) := 2c_0^2 \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(T)(k+j)!}{(2|\nu + \alpha(T)|^{j+k+1}(j!\ k!)^{3/2}}.$$

Proof. By Theorem 10.1 from [11], for any $B \in \mathcal{B}(\mathcal{H})$ with the property: $B_I = (B - B^*)/(2i)$ is a Hilbert-Schmidt operator, we have

$$\|e^{Bt}\| \leq \exp[\alpha(B)t] \sum_{k=0}^{\infty} \frac{g_I^k(B)t^k}{(k!)^{3/2}} \ (t \geq 0).$$

Hence, according to (4.8) it follows that

$$\begin{aligned}
\|Y\|_{\mathcal{H}} &\leq 2c_0^2 \int_0^{\infty} e^{2(\nu+\alpha(T))t} \left(\sum_{k=0}^{\infty} \frac{g_I^k(T)t^k}{(k!)^{3/2}} \right)^2 dt \\
&= 2c_0^2 \int_0^{\infty} \exp[2(\nu + \alpha(T))t] \left(\sum_{j,k=0}^{\infty} \frac{g_I^{k+j}(T)t^{k+j}}{(j!k!)^{3/2}} \right) dt \\
&= 2c_0^2 \sum_{j,k=0}^{\infty} \frac{(k+j)!g_I^{j+k}(T)}{(2|\nu + \alpha(T)|)^{j+k+1}(j!\ k!)^{3/2}},
\end{aligned}$$

as claimed. □

If T is normal, then $g_I(T) = 0$ and with $0^0 = 1$ we have

$$m_I(Y) = \frac{c_0^2}{|\nu + \alpha(T)|}.$$

The latter lemma and Theorem 4.1 imply

Corollary 7.2. *Let the conditions* (4.1), (4.3), (4.6), (7.1) *and* $qm_I(Y) < 1$ *hold. Then* $e^{\tilde{A}t}$ *is exponentially stable.*

If S is normal with $\alpha(S) < \infty$, $M \in L(\mathcal{H})$, $\tilde{A} = S + M$ and condition (4.12) holds, then $q = \|M - I_1 \otimes T\|_{\mathcal{H}} < \infty$, $\|e^{St}\| = e^{t\alpha(S)}$ $(t \geq 0)$. So $\nu = \alpha(S)$, $c_0 = 1$, $\alpha(S) + \alpha(T) = \alpha(A)$ and

$$m_I(Y) = 2\sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(T)(k+j)!}{(2|\alpha(A)|)^{j+k+1}(j!\ k!)^{3/2}}. \tag{7.2}$$

About the norm estimates for exponential functions and resolvents of some other operators see [11, Chapter 10].

8. Regular differential operators with matrix coefficients

In this section we apply our results to matrix differential operators. Besides, $\mathcal{E}_1 = L^2(0,1)$, $\mathcal{E}_2 = \mathbb{C}^n$ and $\mathcal{H} = \mathcal{E}_1 \otimes \mathcal{E}_2 = L^2([0,1], \mathbb{C}^n)$. On the domain

$$D(A) = \{u \in L^2([0,1], \mathbb{C}^n) : u'' \in L^2([0,1], \mathbb{C}^n);\ u(0) = u(1) = 0\}$$

consider the operator

$$\tilde{A} = \frac{d}{dx}a(x)\frac{d}{dx} + C(x) \quad (x \in (0,1)), \tag{8.1}$$

where $a(x)$ is a scalar positive continuously differentiable function and $C(x)$ is a piece-wise continuous $n \times n$-matrix defined on $[0,1]$. For the simplicity we have restricted ourselves by the Dirichlet boundary condition, although our reasonings are valid for arbitrary self-adjoint boundary conditions.

We will consider $\tilde{A}$ as a perturbation of the operator

$$A = \frac{d}{dx}a(x)\frac{d}{dx} + C_0, \tag{8.2}$$

defined on $D(A)$, with a constant $n \times n$-matrix C_0. One can take for instance

$$C_0 = C(0) \text{ or } C_0 = \int_0^1 C(s)ds.$$

To apply our results take

$$S = \frac{d}{dx}a(x)\frac{d}{dx} \quad (D(S) = D(A)) \tag{8.3}$$

and $T = C_0$. Obviously, S is selfadjoint and

$$< Sf, f >_{\mathcal{H}} =< \frac{d}{dx}a(x)\frac{df}{dx}, f >_{\mathcal{H}} \leq -a_0 < f', f' >_{\mathcal{H}} = a_0 < S_0 f, f >_{\mathcal{H}} \quad (f \in D(A)),$$

where $a_0 := \inf_x a(x)$ and $S_0 = \frac{d^2}{dx^2}$, $D(S_0) = D(A)$. But

$$\sigma(S_0) = \{-\pi^2 k^2 : k = 1, 2, \dots\},$$

and thus

$$\alpha(S) = -a_0 \pi^2.$$

In addition, $q = \|A - \tilde{A}\| = \sup_x \|C(x) - C_0\|_{C^n}$. Here $\|.\|_{C^n} = \|.\|_2$ means the spectral norm (the operator norm with respect to the Euclidean vector norm). Assume that

$$\alpha(A) = \alpha(C_0) - a_0\pi^2 < 0, \tag{8.4}$$

then by Lemma 6.1 $\|Y\|_{\mathcal{H}} \leq J(A)$, where

$$J(A) := 2 \sum_{j,k=0}^{n-1} \frac{g^{j+k}(C_0)(k+j)!}{(2|\alpha(A)|)^{j+k+1}(j!\,k!)^{3/2}}.$$

Now Corollary 6.1 implies

Corollary 8.1. *Let $\tilde{A}$ be defined by* (8.1)*, and the conditions* (8.4) *and* $qJ(A) < 1$ *hold. Then $e^{\tilde{A}t}$ is exponentially stable.*

Similarly, instead of the matrix differential operator we can consider matrix elliptic operators with self-adjoint boundary conditions.

9. Partial integro-differential operators

Now take $\mathcal{E}_1 = L^2(0,1)$, $\mathcal{E}_2 = L^2(a,b)$ and $\mathcal{H} = \mathcal{E}_1 \otimes \mathcal{E}_2 = L^2(\Omega)$, where $[a,b]$ is a finite real segment and $\Omega = [0,1] \times [a,b]$. Consider the operator $\tilde{A}$ defined on the domain

$$D(A) = \{u(x,y) \in L^2(\Omega) : \frac{\partial^2 u(x,y)}{\partial x^2} \in L^2(\Omega);\ u(0,y) = u(1,y) = 0, 0 < x < 1; a \leq y \leq b\}$$

by the expression

$$(\tilde{A}f)(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \int_a^b K(y,s)f(x,s)ds \ \ (f \in D(A); x \in (0,1); y \in [a,b]), \tag{9.1}$$

where $K(x,s)$ is a complex function defined on $[a,b]^2$ and satisfying

$$\int_a^b \int_a^b |K(y,s)|^2 ds\, dy < \infty. \tag{9.2}$$

We will consider $\tilde{A}$ as a perturbation of the operator A defined on $D(A)$ by

$$(Af)(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \int_a^y K(y,s)f(x,s)ds \ \ (f \in D(A); x \in (0,1); y \in [a,b]).$$

Clearly, the operator V defined by

$$(Vv)(y) = \int_a^y K(y,s)v(s)ds \ \ (v \in L^2(a,b); y \in [a,b])$$

is a Hilbert-Schmidt Volterra operator with the Hilbert-Schmidt norm

$$N_2(V) = (\int_a^b \int_a^y |K(y,s)|^2 ds\, dy)^{1/2}.$$

To apply Theorem 4.1 take $S = \frac{\partial^2}{\partial x^2}$ with $D(S) = D(A)$ and $T = V$. Then $A = S + I_1 \otimes V$ and

$$q = \|\tilde{A} - A\|_{\mathcal{H}} \le q_1 := (\int_a^b \int_y^b |K(y,s)|^2 ds\, dy.$$

Simple calculations show that V is quasi-nilpotent, i.e. $\sigma(V) = \{0\}$. Since S commute with $I_1 \otimes V$, we have

$$\sigma(A) = \sigma(S) \cup \sigma(V) = \sigma(S).$$

But $\sigma(S)$ consists of the eigenvalues $\lambda_k(S) = -\pi^2 k^2$ $(k = 1,2,...)$. Hence,

$$\alpha(A) = -\pi^2.$$

Furthermore, under consideration, $T = V$ and therefore $g_I(T) = g_I(V) = \sqrt{2}N_2(V_I)$. Here $V_I = (V - V^*)/2i$. But by Lemma 9.2 from [11] we have $2N_2^2(V_I) = N_2^2(V)$. Thus,

$$g_I(V) = \sqrt{2}N_2(V_I) = N_2(V).$$

Hence,

$$m_I(Y) \le m(V) := 2\sum_{j,k=0}^{\infty} \frac{N_2^{j+k}(V)(k+j)!}{(2\pi^2)^{j+k+1}(j!\,k!)^{3/2}}.$$

Since upper bounded selfadjoint operators generate analytic semigroups, according to Remark 3.4 $\tilde{A}$ generates an analytic semigroup. Now Corollary 4.2 implies

Corollary 9.1. *Let $\tilde{A}$ be defined by* (9.1). *Let the conditions* (9.2) *and* $q_1 m(V) < 1$ *hold. Then $e^{\tilde{A}t}$ is e xponentially stable.*

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