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Some classes of higher order general convex functions and variational inequalities

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Abstract

Some new classes the higher order convex functions with respect to an arbitrary function are introduced and studied. Properties of the general functions are investigated. Higher order general variational inequalities are considered. Several important problems such as are deduced as special cases. Several iterative schemes are proposed. Convergence of the proposed methods are analyzed. Parallelogram laws are derived as applications. Results obtained can be viewed as important refinement of the known results.

Keywords: General convex functions, variational inequalities, parallelogram laws

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1. Introduction

The techniques of the convexity theory is being used to consider unrelated problem in a unified frame work. Convexity theory is being extended in various directions to tackle complicated and complex problems, which occur in different branches of engineering, economics, regional, physics, finance, mathematical science, management sciences and optimization. Strongly convex functions were introduced by Polyak [43] in optimization, which played the crucial role complementarity problems [11]. Lin et al. [13] studied the applications of the higher order convex functions in equilibrium problems. Mohsen et al. [14] introduced the higher order convex functions (HOCF) involving an arbitrary bifunctions, which presented the refinement of the previous known results of Lin et al. [13] and Alabdali et al. [2]. See also [1]-[4], [11], [13]-[16], [25]-[29], [32, 37, 43, 48]. For differentiable convex functions, the optimality conditions are represented by the variational inequalities [44]. For more details, see [10], [17]-[21], [23]-[41], [48] and the references therein.

If the set is not convex set, then it can be made convex set involving an arbitrary function. Noor [23] considered the general convex sets (GCS), which are called Noor convex set. For the applications, see Cristescu et al. [8, 9]. Continuing these research activities in the convexity theory, we introduce some new classes of the higher order general convex functions (HOGCF) with respect to an arbitrary function. We would like to point out that HOGCF are quite and distinct from the one considered in [28, 31, 36]. This paper is continuous of our recent research in this area. Several new concepts of monotonicity are introduced. We have shown that the minimum of a differentiable HOGCF can be characterized by a class of variational inequality. This results inspired us to consider higher order general variational

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inequalities (HOGVI). The auxiliary principle approach is applied to suggest some new implicit iterative schemes for solving HOGVI. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. As novel applications of the HOGCF, We derive various parallelogram laws for Banach spaces as applications of HOGCF. Some new special cases are discussed as applications of the HOGCF. It is an open problem to explore new of HOGCF and HOGVI in various fields.

2. Formulations and basic concepts

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively.

Definition 2.1 (*cf.* [9, 15, 42])**.** A set Ω in $\mathcal{H}$ is said to be a convex set, if

$$\mu + \lambda(\nu - \mu) \in \Omega, \quad \forall \mu, \nu \in \Omega,\ \lambda \in [0,1].$$

Definition 2.2 (*cf.* [9, 15, 42])**.** Φ is said to be a convex function, if

$$\phi((1-\lambda)\mu + \lambda\nu) \leq (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu), \quad \forall \mu, \nu \in \Omega,\ \lambda \in [0,1]. \tag{2.1}$$

One can show that $\mu \in \Omega$ is the minimum of a differential function Φ is equivalent to finding $\mu \in \Omega$ such that

$$\langle \Phi'(\mu), \nu - \mu \rangle \geq 0, \quad \nu \in \Omega, \tag{2.2}$$

which is called the variational inequality, introduced by Stampacchia [44].

In many important applications the underlying the set may not be a convex set. This fact inspired us to consider convex set with respect to an arbitrary function.

Definition 2.3. The set Ω_h in H is said to be a general convex set (GCS), if there exists an arbitrary function ξ such that

$$(1-\lambda)\xi(\mu) + \lambda\nu \in \Omega_\xi, \quad \forall \mu, \nu \in \Omega_\xi,\ \lambda \in [0,1].$$

We would like to emphasize that GCS set Ω_ξ is not equal to the GCS set Ω_g which was introduced by Noor and Noor [28, 31, 36].

For example, $[\xi(\mu), \nu] \neq [\mu, g(\nu)], \quad \forall \mu, \nu \in \mathcal{H}.$

We now discuss some special cases of the GCS Ω_ξ.

(I). If $\xi = I$, then ξ-convex set reduces to the classical convex set. Clearly every convex set is a ξ-convex set, but the converse is not true.

(II). If $\xi(\mu) = m\mu,\ m \in (0,1)$, then Ω_h becomes the m-convex set, which is mainly due to Toader [45].

Definition 2.4 (*cf.* [45])**.** The set Ω_m is said to be m-convex set Ω_m, if

$$(1-\lambda)m\mu + \lambda\nu \in \Omega_\xi, \quad \forall \mu, \nu \in \Omega_m,\ \lambda \in [0,1].$$

For $\lambda = 0$, we have $m\mu \in \Omega_m$. This shows that $m\mu$ is the initial end point of the set Ω_m. This applies that every convex set Ω can be made m-convex set. This property of the m-convex plays an important role in deriving the results for m-convex sets.

For the sake of simplicity, we always assume that Ω_ξ is a general convex set, unless otherwise specified.

Definition 2.5. A function Φ is said to be a general convex function involving arbitrary non-negative function ξ, such that

$$\Phi((1-\lambda)\xi(\mu) + \lambda\nu) \leq (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu), \quad \forall \mu, \nu \in \Omega_\xi,\ \lambda \in [0,1]. \tag{2.3}$$

Using the technique of Noor [23], we can easily prove that the minimum $\mu \in \Omega_\xi$ of the differentiable GCF Φ satisfies the inequality:

$$\langle \Phi'(\xi(\mu)), \nu - h(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega_\xi \tag{2.4}$$

which is known as general variational inequality (GVI), see Noor [19].

We now define the GCF on the interval $\Omega_\xi = I_\xi = [\eta_1, \eta_2]$.

Definition 2.6. Let $I_\xi = [\xi(\eta_1), \eta_2]$. Then Φ is a GVI, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ \xi(\eta_1) & x & \eta_2 \\ \Phi(\xi(\eta_1)) & \Phi(x) & \Phi(\eta_2) \end{vmatrix} \geq 0; \quad \xi(\eta_1) \leq x \leq \eta_2.$$

One can easily show that the following are equivalent:

1. Φ is a GCF.
2. $\Phi(x) \leq \Phi(\xi(\eta_1)) + \frac{\Phi(\eta_2)-\Phi(\xi(\eta_1))}{\eta 2-\xi(\eta_1)}(x - \xi(\eta_1))$.
3. $\frac{\Phi(x)-\Phi(\xi(\eta_1))}{x-\xi(\eta_1)} \leq \frac{\Phi(\eta_2)-\Phi(\xi(\eta_1))}{\beta(\eta_2-\xi(\eta_1))}$.
4. $\frac{\Phi(\xi(\eta_1))}{(\eta_2-\xi(\eta_1)))(\xi(\eta_1)-x)} + \frac{\Phi(x)}{(x-\eta_2)(\xi(\eta_1)-x)} + \frac{\Phi(\eta_2)}{\beta(\eta_2-\xi(\eta_1))(x-\eta_2)} \leq 0$,

where $x = \xi(\eta_1) + \lambda(\eta_2 - \xi(\eta_1)) \in [\xi(\eta_1), \eta_2]$.

Definition 2.7. A function Φ on Ω_ξ is said to be higher order general convex (HOGCF) with respect to an arbitrary function ξ, if there exists a constant $\mu > 0$, such that

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \leq (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu) - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \\ \forall \mu, \nu \in \Omega_\xi, \lambda \in [0,1], \; p \geq 1. \tag{2.5}$$

A function Φ is said to higher order general concave with respect to an arbitrary function ξ, if and only if, $-\Phi$ is HOGCF with respect to an arbitrary function ξ.
If $\lambda = \frac{1}{2}$, then

$$\Phi\Big(\frac{\xi(\mu)+\nu}{2}\Big) \leq \frac{\Phi(\xi(\mu)) + \Phi(\nu)}{2} - \mu\frac{1}{2^p}\|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi, \lambda \in [0,1], \; p \geq 1. \tag{2.6}$$

The function Φ is said to be HOG J-convex function.

We now discuss some important and interesting special cases.

(I). For $\xi = I$, Definition 2.7 reduces to:

Definition 2.8. A function Φ on Ω_h is said to be HOGCF, if there exists a constant $\mu > 0$, such that

$$\Phi(\mu + \lambda(\nu - \mu)) \leq (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu) - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \mu\|^p, \\ \forall u, v \in \Omega_\xi, \; \lambda \in [0,1], \; p \geq 1,$$

which are mainly due to Mohsen et al. [14]. It has been shown that HOGCF can be viewed as a significant refinement of the results [13].
(II). For $\xi(\mu) = m\mu, \; m \in (0,1)$, Definition 2.7 reduces to:

Definition 2.9. A function Φ is said to be higher order m-convex function, if there exists a constant $\mu_1 > 0$, such that

$$\Phi((1-\lambda)m\mu + \lambda\nu) \leq (1-\lambda)\Phi(m\mu) + \lambda F(\nu) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - m\mu\|^p, \quad \forall \mu,\nu \in \Omega_m, \lambda \in [0,1],\ p \geq 1,\ m \in [0.1]. \tag{2.7}$$

(III). If $\xi(m\mu) = m\mu, \quad m \in (0,1)$ and $\Phi(m\mu) = m\Phi(\mu)$, then Definition 2.9 reduces to:

Definition 2.10. A function Φ is said to be HOGCF m-convex function in the sense of Toader [25], if there exists a constant $\mu > 0$, such that

$$\Phi((1-\lambda)m\mu + \lambda\nu) \leq (1-\lambda)m\Phi(\mu) + \lambda\Phi(\nu) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - m\mu\|^p, \quad \forall \nu.\mu \in \Omega_m,\ \lambda \in [0,1],\ p \geq 1. \tag{2.8}$$

(IV). If $p = 2$, then the HOGCF becomes:

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \leq (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu) - \mu\lambda(1-\lambda)\|\nu - \xi(\mu)\|^2, \quad \forall \mu,\nu \in \Omega_\xi,\ \lambda \in [0,1].$$

(V). If $p = 1$, then HOGCF is called approximate general convex functions, that is,

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \quad \leq \quad (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu) - \mu_1\lambda(1-\lambda)\|\nu - \xi(\mu)\|, \quad \forall \mu,\nu \in \Omega_\xi,\ \lambda \in [0,1].$$

Definition 2.11. A function Φ on the general convex set Ω_ξ is said to be a higher order affine general convex function with respect to an arbitrary function ξ, if there exists a constant $\mu_1 > 0$, such that

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) = (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu)) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \quad \forall \mu,\nu \in \Omega_\xi,\ \lambda \in [0,1],\ p \geq 1.$$

Note that, if a functions is both HOG convex and HOG concave, then it is higher order affine general convex function.

Definition 2.12. A function Φ is called higher order quadratic equation with respect to an arbitrary function ξ, if there exists a constant $\mu_1 > 0$, such that

$$\Phi\Big(\frac{\xi(\mu) + \nu}{2}\Big) = \frac{\Phi(\xi(\mu)) + \Phi(\nu)}{2} - \mu_1\frac{1}{2^p}\|\nu - \xi(\mu)\|^p, \quad \forall \mu,\nu \in \Omega_\xi,\ \lambda \in [0,1]. \tag{2.9}$$

This function Φ is also called higher order affine J-general convex function.

Definition 2.13. A function Φ on the convex set Ω_ξ is said to be HOGCF with respect to an arbitrary function ξ, if there exists a constant $\mu_1 > 0$ such that

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \leq \max\{\Phi(\xi(\mu)), \Phi(\nu)\} - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \quad \forall \mu,\nu \in \Omega_\xi,\ \lambda \in [0,1],\ p \geq 1.$$

Definition 2.14. A function Φ on the general convex set Ω_ξ is said to be higher order general log-convex with respect to an arbitrary function h, if there exists a constant $\mu_1 > 0$ such that

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \leq (\Phi(\xi(\mu)))^{1-\lambda}(\Phi(\nu))^\lambda - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \quad \forall \mu,\nu \in \Omega_\xi,\ \lambda \in [0,1],\ p \geq 1,$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned}
\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) &\leq (\Phi(\xi(u)))^{1-\lambda}(\Phi(\nu)^\lambda - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - h(\mu)\|^p \\
&\leq (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p \\
&\leq \max\{\Phi(\xi(\mu)), \Phi(\nu)\} - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \quad p \geq 1.
\end{aligned}$$

Clearly higher order general log-convex function $\Rightarrow$ HOGCF and every HOGCF $\Rightarrow$ higher order general quasi-convex function. The converse is not true.

Definition 2.15. An operator $T:\Omega_\xi\rightarrow\mathcal{H}$ is said to be:

(i). higher order general monotone, if there exists a constant $\alpha>0$ such that

$$\langle\mathcal{T}(\xi(\mu))-\mathcal{T}\nu,\xi(\mu)-\nu\rangle\geq\alpha\|\nu-\xi(\mu)\|^p,\quad\forall\mu,\nu\in\Omega_\xi.$$

(ii). higher order general pseudomonotone, if there exists a constant $\eta>0$ such that

$$\langle\mathcal{T}(h(\mu)),\nu-\xi(\mu)\rangle+\eta\|\nu-\xi(\mu)\|^p\geq0\Rightarrow\langle\mathcal{T}(\xi(\mu)),\nu-\xi(\mu)\rangle-\eta\|\nu-\xi(\mu)\|^p\geq0,\quad\forall\mu,\nu\in\Omega_\xi.$$

(iii). higher order general relaxed pseudomonotone, if there exists a constant $\mu_1>0$ such that

$$\langle\mathcal{T}(\xi(\mu)),\nu-\xi(\mu)\rangle\geq0\Rightarrow-\langle\mathcal{T}(\xi(\nu)),\xi(\mu)-\nu\rangle+\mu_1\|\nu-\xi(\mu)\|^p\geq0,\quad\forall\mu,\nu\in\Omega_\xi.$$

Definition 2.16. A differentiable function Φ on the GC set Ω_h is said to be higher order general pseudo convex function, if and only if, if there exists a constant $\mu_1>0$ such that

$$\langle\Phi'(\xi(\mu)),\nu-\xi(\mu)\rangle+\mu_1\|\nu-\xi(\mu)\|^p\geq0\Rightarrow\Phi(\nu)\geq\Phi(\mu),\quad\forall\mu,\nu\in\Omega_\xi.$$

3. Properties

In this section, we discuss some basic properties of HOGCF.

Theorem 3.1. *Let Φ be a differentiable function on Ω_ξ. Then the function Φ is HOGCF, if and only if,*

$$\Phi(\nu)-\Phi(\xi(\mu))\geq\langle\Phi'(\xi(\mu)),\nu-\xi(\mu)\rangle+\mu_1\|\nu-\xi(\mu)\|^p,\quad\forall\mu.\nu\in\Omega_\xi.\tag{3.1}$$

Proof. Let Φ be a HOGCF on Ω_h. Then

$$\Omega(\xi(\mu)+\lambda(\nu-\xi(\mu)))\quad\leq\quad(1-\lambda)\Phi(\xi(\mu))+\lambda\Phi(\nu)-\mu_1\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\nu-\xi(\mu)\|^p,\quad\forall\mu,\nu\in\Omega_\xi,\ p\geq1.$$

From which, we have

$$\Phi(\nu)-\Phi(\xi(\mu))\geq\left\{\frac{\Phi(\xi(\mu)+\lambda(\nu-\xi(\mu)))-\Phi(\xi(\mu))}{\lambda}\right\}+\mu_1\left\{\lambda^{p-1}(1-\lambda)+(1-\lambda)^p\right\}\|\nu-\xi(\mu)\|^p.$$

Letting $\lambda\rightarrow0$, we have

$$\Phi(\nu)-\Phi(\xi(\mu))\geq\langle\Phi'(\xi(\mu)),\nu-\xi(\mu))\rangle+\mu_1\|\nu-\xi(\mu)\|^p,\quad\forall\mu,\nu\in\Omega_\xi,$$

which is (3.1).

Conversely, let (3.1) hold. Then, $\forall\mu,\nu\in\Omega_\xi,\lambda\in[0,1]$, $\nu_\lambda=\xi(\mu)+\lambda(\nu-\xi(\mu))\in\Omega_\xi$, we have

$$\begin{aligned}\Phi(\nu)-\Phi(\nu_\lambda)\quad&\geq\quad\langle\Phi'(\nu_\lambda),\nu-\nu_\lambda)\rangle+\mu_1\|\nu-\nu_\lambda\|^p\\&=\quad(1-\lambda)\langle\Phi'(\nu_\lambda),\nu-\xi(\mu)\rangle+\mu(1-\lambda)^p\|\nu-\xi(\mu)\|^p,\quad\forall\mu,\nu\in\Omega_\xi.\end{aligned}\tag{3.2}$$

Similarly

$$\begin{aligned}\Phi(h(\mu))-\Phi(\nu_\lambda)\quad&\geq\quad\langle\Phi'(\nu_\lambda),\xi(\mu)-\nu_\lambda)\rangle+\mu_1\|\xi(\mu)-\nu_\lambda\|^p\\&=\quad-\lambda\langle F'(\nu_t),\nu-\xi(\mu)\rangle+\mu_1\lambda^p\|\nu-\xi(\mu)\|^p.\end{aligned}\tag{3.3}$$

Multiplying (3.2) by λ and (3.3) by $(1-\lambda)$ and adding, we obtain

$$\Phi(\xi(\mu)+\lambda(\nu-\xi(\mu)))\leq(1-\lambda)\Phi(\xi(\mu))+\lambda\Phi(\nu)-\mu_1\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\nu-\xi(\mu)\|^p,\quad\forall\mu,\nu\in\Omega_h.$$

Thus Φ is a HOGCF. □

Theorem 3.2. *Let Φ be a differentiable HOGCF on Ω_ξ. Then $\Phi'(.)$ is a higher order general monotone operator.*

Proof. Let Φ be a HOGCF. Then Theorem 3.1 implies that,

$$\Phi(\nu) - \Phi(\xi(\mu)) \geq \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle + \mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu.\nu \in \Omega_\xi. \tag{3.4}$$

Similarly

$$\Phi(\xi(\mu)) - \Phi(\nu) \geq \langle \Phi'(\nu), \xi(\mu) - \nu)\rangle + \mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\langle \Phi'(\xi(\mu)) - \Phi'(\nu), \xi(\mu) - \nu\rangle \geq 2\mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi, \tag{3.6}$$

which is the required result. □

Theorem 3.3. *If the differential operator $\Phi'(.)$ satisfies* (3.6)*, then*

$$\Phi(\nu) - \Phi(\xi(\mu)) \geq \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle + 2\mu_1 \frac{1}{p} \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi. \tag{3.7}$$

Proof. Let (3.6) hold, Then

$$\langle \Phi'(\nu), \xi(\mu) - \nu\rangle \geq \langle \Phi'(\xi(\mu)), \xi(\mu) - g(\nu))\rangle + 2\mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi. \tag{3.8}$$

$\forall u, v \in \Omega_h$, $\lambda \in [0, 1]$, $\nu_\lambda = \xi(\mu) + \lambda(\nu - \xi(\mu)) \in \Omega_\xi$. Setting $\nu = \nu_\lambda$ in (3.8), we have

$$\begin{aligned} \langle \Phi'(\nu_\lambda), \xi(\mu) - \nu_\lambda\rangle &\leq \langle \Phi'(\xi(\mu)), \xi(\mu) - \nu_\lambda\rangle - 2\mu_1 \|\nu - \xi(\mu)\|^p \\ &= -\lambda \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle - 2\mu_1 \lambda^p \|\nu - \xi(\mu)\|^p, \end{aligned}$$

from which, it follows that

$$\langle \Phi'(\nu_\lambda), \nu - \xi(\mu)\rangle \geq \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle + 2\mu_1 \lambda^{p-1} \|\nu - \xi(\mu)\|^P. \tag{3.9}$$

Consider

$$\zeta(\lambda) = \Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))), \quad \forall \mu, \nu \in \Omega_\xi.$$

Now, from (3.9), we have

$$\zeta(1) = \Phi(\nu), \quad \zeta(0) = \Phi(\xi(\mu))$$

and

$$\zeta'(\lambda) = \langle \Phi'(\nu_\lambda), \nu - \xi(\mu)\rangle \geq \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle + 2\mu_1 \lambda^{p-1} \|\nu - \xi(\mu)\|^p. \tag{3.10}$$

Integrating (3.10), we obtain

$$\zeta(1) - \zeta(0) = \int_0^1 \zeta'(\lambda) d\lambda \geq \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle + 2\mu_1 \frac{1}{p} \|\nu - \xi(\mu)\|^p.$$

Consequently,

$$\Phi(\nu) - \Phi(\xi(\mu)) \geq \langle \Phi'(\xi(\mu)), \nu - \xi(\mu)\rangle + 2\mu_1 \frac{1}{p} \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi,$$

which is the required (3.7). □

Theorem 3.4. *If $\Phi'(.)$ is a higher order general relaxed pseudomonotone operator, then Φ is a higher order general pseudo-convex function.*

Proof. Let Φ' be a higher order general relaxed pseudomonotone operator. Then,

$$\langle \Phi'(\xi(\mu)), \nu - \xi(\mu) \rangle \geq 0, \quad \forall \mu, \nu \in \Omega_{\xi},$$

from which, we have

$$\langle \Phi'(\nu), \nu - \xi(\mu) \rangle \geq \mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_{\xi}. \tag{3.11}$$

$\forall \mu, \nu \in \Omega_h,\ \lambda \in [0,1],\ \nu_\lambda = \mu + \lambda(\nu - \xi(\mu)) \in \Omega_{\xi}.$
Setting $\nu = \nu_\lambda$ in (3.11), we have

$$\langle \Phi'(\nu_\lambda), \nu - \xi(\mu) \rangle \geq \mu_1 \lambda^{p-1} \|\nu - \xi(\mu)\|^p. \tag{3.12}$$

Let

$$\zeta(\lambda) = \Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) = \Phi(\nu_\lambda), \quad \forall \mu, \nu \in \Omega_{\xi},\ \lambda \in [0,1].$$

Using (3.12), we have

$$\zeta'(\lambda) = \langle \Phi'(\nu_\lambda), \nu - \xi(\mu)) \rangle \geq \mu_1 \lambda^{p-1} \|\nu - \xi(\mu)\|^p.$$

Integrating, we have

$$\zeta(1) - \zeta(0) = \int_0^1 \zeta'(\lambda) d\lambda \geq \frac{\mu_1}{p} \|\nu - \xi(\mu)\|^p.$$

Thus

$$\Phi(\nu) - \Phi(\xi(\mu)) \geq \frac{\mu_1}{p} \|\nu - \xi(\mu)\|^p), \quad \forall \mu, \nu \in \Omega_{\xi}.$$

□

Definition 3.5. A function Φ is said to be sharply higher order general pseudo convex, if there exists a constant $\mu_1 > 0$ such that

$$\begin{aligned} &\langle \Phi'(\xi(\mu)), \nu - \xi(\mu) \rangle \geq 0 \\ &\Rightarrow \\ &\Phi(\nu) \geq \Phi(\nu + \lambda(\xi(\mu) - \nu)) + \mu_1 \{\lambda^p (1-\lambda) + \lambda(1-\lambda)^p\} \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_{\xi}. \end{aligned}$$

Theorem 3.6. *If Φ is a sharply higher order general pseudo convex function with a constant $\mu_1 > 0$, then*

$$\langle \Phi'(\nu), \nu - \xi(\mu) \rangle \geq \mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_{\xi},\ p \geq 1.$$

Proof. Let Φ be a sharply higher order general pseudo convex function. Then

$$\Phi(\nu) \geq \Phi(\nu + t(\xi(\mu) - \nu)) + \mu_1 \{\lambda^p (1-\lambda) + \lambda(1-\lambda)^p\} \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_{\xi},\ \lambda \in [0,1].$$

That means

$$\left\{ \frac{\Phi(\nu + \lambda(\xi(\mu) - \nu) - \Phi(\nu)}{\lambda} \right\} + \mu_1 \{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\} \|\nu - \xi(\mu)\|^p \geq 0.$$

Letting $\lambda \to 0$, we have

$$\langle \Phi'(\nu), \nu - \xi(\mu) \rangle \geq \mu_1 \|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_{\xi}.$$

□

Definition 3.7. A function Φ is said to be a general pseudo convex function, if there exists a strictly positive bifunction $\zeta_1(.,.)$, such that

$$\Phi(\nu) < \Phi(\xi(\mu)) \Rightarrow \Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) < \Phi(\xi(\mu)) + \lambda(\lambda - 1)\zeta_1(\nu, \xi(\mu)), \quad \forall \mu, \nu \in \Omega_\xi, \ \lambda \in [0, 1].$$

Theorem 3.8. *If the function $\Phi(\nu) < \Phi(\xi(\mu))$, then Φ is higher order general pseudo convex function.*

Proof. Let $\Phi(\nu) < \Phi(\xi(\mu))$ and F be HOGCF. Then, $\forall \mu, \nu \in \Omega_\xi, \ \lambda \in [0, 1]$, we have

$$\begin{aligned}
& \Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \\
\leq \ & \Phi(\xi(\mu)) + \lambda(\Phi(\nu) - \Phi(\xi(\mu))) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p \\
< \ & \Phi(\xi(\mu)) + \lambda(1-\lambda)(\Phi(\nu) - \Phi(\xi(\mu))) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p \\
= \ & \Phi(\mu) + \lambda(\lambda-1)(\Phi(\xi(\mu)) - \phi(\nu)) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p \\
< \ & \Phi(\mu) + \lambda(\lambda-1)\zeta_1(\xi(\mu), \nu) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi,
\end{aligned}$$

where $\zeta_1(\xi(\mu), \nu) = \Phi(\xi(\mu)) - \Phi(\nu) > 0$. Consequently the function Φ is higher order general pseudo convex. □

Theorem 3.9. *If ϕ is a higher order general affine function (HOGAF). Then Φ is a HOGCF, if and only if, $\mathcal{G} = \Phi - \phi$ is a general convex function.*

Proof. Let ϕ be HOGAF. Then

$$\phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) = (1-\lambda)\phi(\xi(\mu)) + \lambda\phi(\nu) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \ \forall \mu, \nu \in \Omega_\xi. \tag{3.13}$$

From the HOG convexity of Φ, we have

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \leq (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu) - \mu_1\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \ \forall \mu, \nu \in \Omega_\xi. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) - \phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) \leq (1-\lambda)(\Phi(\xi(\mu)) - \phi(\xi(\mu))) + \lambda(\Phi(\nu) - \phi(\nu)).$$

Thus

$$\begin{aligned}
\mathcal{G}((1-\lambda)\xi(\mu) + \lambda\nu) &= \Phi((1-\lambda)\xi(\mu) + \lambda\nu) - \phi((1-\lambda)\xi(\mu) + \lambda\nu) \leq (1-\lambda)\Phi(\xi(\mu)) + \lambda\Phi(\nu) - (1-\lambda)\phi(\mu) - \lambda(\phi)(\nu) \\
&= (1-\lambda)(\Phi(\xi(\mu)) - \phi(\xi(\mu))) + \lambda(\Phi(\nu) - \phi(\nu)),
\end{aligned}$$

which shows that $\mathcal{G} = \Phi - \phi$ is a general convex function. □

4. General variational inequalities

In this section, we consider the higher order general variational inequalities (HOGVI). In this direction, we have the following result.

Theorem 4.1. *Let Φ be a differentiable HOGCF with modulus $\mu_1 > 0$. If $\mu \in \Omega_\xi$ is the minimum of the function Φ, then*

$$\Phi'(\nu) - \Phi(\xi(\mu)) \geq \mu_1\|\nu - \xi(\mu)\|^p, \quad \forall \mu, \nu \in \Omega_\xi. \tag{4.1}$$

Proof. Let $\mu \in \Omega_h$ be a minimum of the function Φ. Then

$$\Phi(\xi(\mu)) \leq \Phi(\nu), \quad \forall \mu, \nu \in \Omega_\xi. \tag{4.2}$$

Since Ω_ξ is a general convex set, so $\forall \mu, \nu \in \Omega_\xi, \quad \lambda \in [0, 1], \quad \nu_\lambda = (1-\lambda)h(\mu) + \lambda\nu \in \Omega_\xi$.
Setting $\nu = \nu_\lambda$ in (4.2), we have

$$0 \leq \lim_{\lambda \to 0}\left\{\frac{\Phi(\xi(\mu) + \lambda(\nu - \xi(\mu))) - \Phi(\xi(\mu))}{\lambda}\right\} = \langle \Phi'(\xi(\mu)), \nu - \xi(\mu) \rangle. \tag{4.3}$$

that is,

$$\Phi(\xi(\mu)+\lambda(\nu-\xi(\mu)))\leq\Phi(\xi(\mu))+\lambda(\Phi(\nu)-\Phi(\xi(\mu)))-\mu_1\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\nu-\xi(\mu)\|^p,\ \forall\mu,\nu\in\Omega_\xi.$$

Using (4.3), we have

$$\begin{aligned}\Phi(\nu)-\Phi(\xi(\mu)) &\geq \lim_{\lambda\to 0}\left\{\frac{\Phi(\xi(\mu)+\lambda(\nu-\xi(\mu)))-\Phi(\xi(\mu))}{\lambda}\right\}+\mu_1\|\nu-\xi(\mu)\|^p\\ &= \langle\Phi'(\xi(\mu)),\nu-\xi(\mu)\rangle+\mu_1\|\nu-\xi(\mu)\|^p.\end{aligned}$$

□

Remark 4.2. If $\mu\in\Omega_h$ satisfies the inequality

$$\langle\Phi'(\xi(\mu)),\nu-\xi(\mu)\rangle+\mu_1\|\nu-\xi(\mu)\|^p\geq 0,\quad\forall\mu,\nu\in\Omega_\xi,\tag{4.4}$$

then $\mu\in\Omega_\xi$ is the minimum of the HOGCF Φ. The inequality of the type (4.4) is called the higher order general variational inequality (HOGVI).

In this section, we consider another variational inequality of which (4.4) is a special case.
For given two operators $\mathcal{T},\xi$, we consider the problem of finding $\mu\in\Omega$ for a constant η such that

$$\langle\mathcal{T}\mu,\nu-\xi(\mu)\rangle+\eta\|\nu-\xi(\mu)\|^p\geq 0,\quad\forall\mu,\nu\in\Omega,\ p\geq 1,\tag{4.5}$$

which is called the HOGVI.

Special cases

(I). For $\mathcal{T}\mu=\Phi'(\xi(\mu))$, problem (4.5) is exactly GVI (4.4).

(II). For $\eta=0$, then problem (4.5) is equivalent to finding $\mu\in\Omega$, such that

$$\langle\mathcal{T}\mu,\nu-\xi(\mu)\rangle\geq 0,\quad\forall\nu\in\Omega,\tag{4.6}$$

which is known as the general variational inequality, see Noor [19].
(III). If $\Omega^\star=\{\mu\in\mathcal{H}:\langle,\forall\mu,\nu\in\Omega\}$ is a polar (dual) cone, then the problem (4.6) is equivalent to finding μ such that

$$\xi(\mu)\in\Omega,\quad\mathcal{T}\mu\in\Omega^\star,\quad\langle\mathcal{T}\mu,\xi(\mu)\rangle=0,\tag{4.7}$$

which is called the general complementarity problem. Obviously general complementarity problems include the complementarity problems. For more details, see [9, 11, 18, 19, 21, 38, 39] and the references therein

(IV). For $p=1$, then the problem (4.5) reduces to the problem of finding $\mu\in\Omega$ such that

$$\langle\mathcal{T}\mu,\nu-h(\mu)\rangle+\eta\|\nu-\xi(\mu)\|\geq 0,\quad\forall\nu\in\Omega,$$

which is called the approximate general variational inequality.

(V). If $\eta=0,\quad\xi=I$, then problem (4.5) reduces to finding $\mu\in\Omega$ such that

$$\langle\mathcal{T}\mu,\nu-\mu\rangle\geq 0,\quad\forall\nu\in\Omega,\tag{4.8}$$

is the classical variational inequality. See [1]-[3], [11]-[21], [23]-[29], [31]-[34] [36]-[41], [43]-[48] and the references therein.

For suitable and appropriate choice of the parameter μ and p, one can obtain several new and known classes of variational inequalities, see [13, 15], [20]-[22].

The auxiliary principle technique of Glowinski et al. [10], Lions et al. [12] as developed by Noor [17, 21, 22] is applied to suggest and analyze some iterative methods for solving (4.5).

For given $\mu \in \Omega$ satisfying (4.5), find $\omega \in \Omega$, such that

$$\langle \rho \mathcal{T} \omega, \nu - \xi(\omega) \rangle + \langle \omega - \mu, \nu - \omega \rangle + \eta \|\nu - \xi(\omega)\|^p \geq 0, \quad \forall \nu \in \Omega, \ p \geq 1, \tag{4.9}$$

where $\rho > 0$ is a parameter.
It is clear that the relation (4.8) defines a mapping connecting the problems (4.5) and (4.8). We not that, if $\omega(\mu) = \mu$, then w is a solution of problem (4.5). This simple fact enables to suggest an iterative method for solving HOGVI (4.5).

Algorithm 4.1. For given $\mu_0 \in \Omega$, find the approximate solution μ_{n+1} by the scheme

$$\langle \rho \mathcal{T} \mu_{n+1}, \nu - \xi(\mu_{n+1}) \rangle + \langle \mu_{n+1} - \mu_n, \nu - \mu_{n+1} \rangle + \eta \|\nu - \xi(\mu_{n+1})\|^p \geq 0, \quad \forall \nu \in \Omega, \ p \geq 1. \tag{4.10}$$

Algorithm 4.1 is known as the implicit method. Such type of methods have been studied extensively. See [13, 14] and the reference therein.

If $\eta = 0$, then Algorithm 4.1 reduces to:

Algorithm 4.2. For given $\mu_0 \in \Omega$, find the approximate solution μ_{n+1} by the scheme

$$\langle \rho \mathcal{T} \mu_{n+1}, \nu - \xi(\mu_{n+1}) \rangle + \langle \mu_{n+1} = \mu_n, \nu - \mu_{n+1} \rangle \geq 0, \quad \forall \nu \in \Omega.$$

For the convergence analysis of Algorithm 4.1, we need the following concept.

Definition 4.3. The operator $\mathcal{T}$ is said to be pseudo ξ-monotone with respect to $\eta\|\nu - h(\mu)\|^p$, $p \geq 1$, if

$$\begin{aligned} &\langle \rho \mathcal{T} \mu, \nu - \xi(\mu) \rangle + \eta \|\nu - \xi(\mu)\|^p \geq 0, \quad \forall \nu \in \Omega, \ p \geq 1, \\ &\Longrightarrow \\ &\langle \rho \mathcal{T} \nu, \xi(\nu) - \mu \rangle - \eta \|\nu - \xi(\mu)\|^p, \geq 0, \quad \forall \nu \in \Omega, \ p \geq 1 \end{aligned}$$

We now study the convergence analysis of Algorithm 4.1.

Theorem 4.4. *Let* $\mu \in \Omega$, *be a solution of* (4.5) *and* μ_{n+1} *be the approximate solution obtained from* Algorithm (4.1). *If* $\mathcal{T}$ *is a pseudo* ξ*-monotone with respect to* $\eta\|\nu - h(\mu)\|^p$, *then*

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu_n - \mu\|^2 - \|\mu_{n+1} - \mu_n\|^2. \tag{4.11}$$

Proof. Let $\mu \in \Omega$ be a solution of (4.5). Then

$$\langle \rho \mathcal{T} \mu, \nu - \xi(\mu) \rangle + \eta \|\nu - \xi(\mu)\|^p, \quad \forall \nu \in \Omega,$$

implies that

$$\langle \rho \mathcal{T} \nu, \xi(\nu) - \mu \rangle - \eta \|\nu - \xi(\mu)\|^p, \quad \forall \nu \in \Omega. \tag{4.12}$$

Taking $\nu = \mu_{n+1}$ in (4.12), we have

$$\langle \rho \mathcal{T} \mu_{n+1}, \xi(\mu_{n+1}) - \mu \rangle - \eta \|\xi(\mu_{n+1}) - \mu\|^p \geq 0. \tag{4.13}$$

Setting $\nu = \mu$ in (4.10), we have

$$\langle \rho \mathcal{T} \mu_{n+1}, \mu - \xi(\mu_{n+1}) \rangle + \langle \mu_{n+1} - \mu_n, \mu - \mu_{n+1} \rangle + \eta \|\mu - \xi(\mu_{n+1})\|^p \geq 0. \tag{4.14}$$

Combining (4.13) and (4.14), we have

$$\langle \mu_{n+1} - \mu_n, \mu_{n+1} - \mu \rangle \geq 0.$$

Using the inequality

$$2\langle \eta_1, \eta_2 \rangle = \|\eta_1 + \eta_2\|^2 - \|\eta_1\|^2 - \|\eta_2\|^2, \quad \forall \eta_1, \eta_2 \in \mathcal{H},$$

we obtain

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu_n - \mu\|^2 - \|\mu_{n+1} - \mu_n\|^2,$$

the required result (4.11). □

Theorem 4.5. *Let $\mathcal{T}$ be a pseudo ξ-monotone operator with respect to $\eta\|\nu - \xi(\mu)\|^p$. If μ_{n+1} is the approximate solution obtained from* Algorithm (4.1) *and $\mu \in \Omega$ is the exact solution of* (4.5)*, then*

$$\lim_{n\to\infty} \mu_n = \mu.$$

Proof. Let $\mu \in \Omega$ be a solution of (4.5). Then, from (4.11), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{\mu_n\}$ is bounded. From (4.11), we have

$$\sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\|^2 \leq \|\mu_0 - \mu\|^2.$$

This implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \tag{4.15}$$

Let $\hat{\mu}$ be a cluster point of $\{\mu_n\}$ and the subsequence $\{\mu_{n_j}\}$ of the sequence $\{\mu_n\}$ converge to $\hat{\mu} \in \mathcal{H}$. Replacing μ_n by μ_{n_j} in (4.15), taking the limit $n_j \to 0$ and from (4.15), we have

$$\langle \mathcal{T}\hat{\mu}, \nu - \hat{\mu} \rangle + \eta\|\nu - \hat{\mu}\|^p, \quad \forall \nu \in \Omega, \ p \geq 1.$$

This shows that $\hat{\mu} \in \Omega$ satisfies

$$\|\mu_{n+1} - \mu_n\|^2 \leq \|\mu_n - \hat{\mu}\|^2.$$

From the above inequality, we conclude that the sequence μ_n has exactly one cluster point $\hat{\mu}$ and

$$\lim_{n\to\infty} \mu_n = \hat{\mu}.$$

□

Algorithm 4.1 is an implicit method, which is equivalent to the following method.

Algorithm 4.3. For a given $\mu_0 \in \Omega$, find μ_{n+1} by the iterative schemes

$$\begin{aligned} \langle \rho\mathcal{T}\mu_n, \nu - \xi(y_n) \rangle &+ \langle y_n - \mu_n, \nu - y_n \rangle + \eta\|\nu - \xi(y_n)\|^p \geq 0, \quad \forall \nu \in \Omega, \\ \langle \rho\mathcal{T} y_n, \nu - \xi(y_n) \rangle &+ \langle \mu_n - y_n, \nu - y_n \rangle + \mu\|\nu - \xi(\mu_n)\|^p \geq 0, \quad \forall \nu \in \Omega. \end{aligned}$$

We now use the auxiliary principle technique involving the Bregman distance function.

For a given $\mu \in \Omega$ satisfying (4.5), find $\omega \in \Omega$ such that

$$\langle \rho\mathcal{T}\omega, \nu - \xi(\omega)) + \langle \varphi'(\xi(\omega)) - \varphi'(\mu), \nu - \xi(\omega) \rangle + \eta\|\nu - \xi(\omega)\|^p \geq 0, \quad \forall \nu \in \Omega, \ p \geq 1, \tag{4.16}$$

where $\varphi'(\mu)$ is the differential of a general convex function $\varphi(\mu)$ at $\mu \in \Omega$.

Remark 4.6. The function $\mathcal{B}(\omega, \mu) = \varphi(\omega) - \varphi(\mu) - \langle \varphi'(\mu), \omega - \mu \rangle$ associated with the strongly convex function $\varphi(\mu)$ is called the Bregman function. By the strongly convexity of the function $\varphi(\mu)$, the Bregman function $\mathcal{B}(.,.)$ is nonnegative and $\mathcal{B}(\omega, \mu) = 0$, if and only if $\mu = \omega, \forall \mu, \omega \in \Omega$.

We note that, if $\omega = \mu$, then clearly ω is solution of the problem (4.5). This fact is used to consider the following iterative method for solving HOGVI (4.5).

Algorithm 4.4. For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho \mathcal{T} \mu_{n+1}, \nu - \xi(\mu_{n+1}) \rangle + \langle \varphi'(\xi(\mu_{n+1})) - \varphi'(\mu_n), \nu - \xi(\mu_{n+1}) \rangle + \eta \|\nu - \xi(\mu_{n+1})\|^p \geq 0, \quad \forall \nu \in \Omega, \ p \geq 1,$$

where $\rho > 0$ is a constant. Algorithm 4.4 is called the proximal method for solving the problem HOGVI (4.5).

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this difficulty, we now consider another method for solving the problem (4.5).

For a given $\mu \in \Omega$ satisfying (4.5), find $\omega \in \Omega$ such that

$$\langle \rho \mathcal{T} \mu, \nu - \xi(\omega) \rangle + \langle \varphi'(\xi(\omega)) - \varphi', \nu - \xi(\omega) \rangle + \eta \|\nu - \xi(\omega)\|^p \geq 0, \quad \forall \nu \in \Omega, \tag{4.17}$$

where $\varphi'(\mu)$ is the differential of a strongly general function $\varphi(\mu)$ at $\mu \in \Omega$. Note that problems (4.17) and (4.16) are quite different problems.
For $\omega = \mu$, ω is a solution of (4.5). This fact allows us to suggest another method for solving the problem (4.5).

Algorithm 4.5. For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho \mathcal{T} \mu_n, \nu - \xi(\mu_{n+1}) \rangle + \langle \varphi'(\xi(\mu_{n+1}) - \varphi'(\mu_n), \nu - \xi(\mu_{n+1}) \rangle + \eta \|\nu - \xi(\mu_{n+1})\|^p \geq 0, \quad \forall \nu \in \Omega. \tag{4.18}$$

Remark 4.7. For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving the problem (4.5) and related optimization problems.

5. Applications

In this section, we derive some mew parallelogram laws of uniformly Banach spaces, which can be viewed as novel application of HOGCF.
Setting $\Phi(\mu) = \|\mu\|^p$ in Definition 2.11, we have

$$\|\xi(\mu) + t(\nu - \xi(\mu))\|^p = (1-\lambda)\|\xi(\mu)\|^p + \lambda\|\nu\|^p - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\nu - \xi(\mu)\|^p, \ \forall \mu, \nu \in \Omega, \ \lambda \in [0,1]. \tag{5.1}$$

Taking $\lambda = \frac{1}{2}$ in (5.1), we obtain the parallelogram law:

$$\|\xi(\mu) + \nu\|^p + \eta\|\nu - \xi(\mu)\|^p = 2^{p-1}\{\|\xi(\mu)\|^p + \|\nu\|^p\}, \quad \forall \mu, \nu \in \Omega. \tag{5.2}$$

Some special cases

(I). For $p = 2$, we obtain

$$\|\xi(\mu) + \nu\|^2 + \eta\|\nu - \xi(\mu)\|^2 = 2\{\|\xi(\mu)\|^2 + \|\nu\|^2\}, \quad \forall \mu, \nu \in \Omega$$

which is a parallelogram law involving an arbitrary function.

(II). For $\xi(\mu) = m\mu, \ m \in (0,1)$, we have a new m-parallelogram law as:

$$\|m\mu + \nu\|^p + \eta\|\nu - m\mu\|^p = 2^{p-1}\{\|m\mu\|^p + \|\nu\|^p\}, \quad \forall m\mu, \nu \in \Omega_m, \ m \in (0,1). \tag{5.3}$$

(III). For $\xi = I$, parallelogram law (5.2) reduces to

$$||\mu + \nu||^p + \eta||\nu - \mu||^p = 2^{p-1}\{||\mu||^p + ||\nu||^p\}, \quad \forall \nu, \mu \in \Omega, \; p \geq 1, \tag{5.4}$$

which is well known parallelogram law. See Xu [46], Bynum [5] and Chen et al. [6, 7] for the properties and applications of the parallelogram laws in prediction theory and applied sciences.

(IV). For $p = 2$, the parallelogram law (5.4) collapses to

$$||\mu + \nu||^2 + \eta||\nu - \mu||^2 = 2||\mu||^2 + ||\nu||^2, \quad \forall \mu, \nu \in \Omega,$$

which is the well known parallelogram law [3, 16] and characterize the inner product spaces.

Form these observations and comments, it is clear that the parallelogram law (5.2) is flexible and unifying ones. In spite of the recent activities, further research efforts are need to explore the applications of these parallelogram laws in various fields of pure and applied sciences.

6. Open problems

Let $\mathcal{T}, \xi, \phi : \mathcal{H} \to \mathcal{H}$ be nonlinear operators. We consider the problem of finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}\mu, \xi(\nu) - \phi(\mu)\rangle + \zeta||\xi(\nu) - \phi(\mu)||^p \geq 0, \quad \forall \nu \in \Omega, \; p \geq 0, \tag{6.1}$$

which is called the higher order extended general variational inequalities.

(i). If $\xi = I$ is the identity operator, then problem (6.1) reduces to finding $u \in \Omega_h$, such that

$$\langle \mathcal{T}\mu, \nu - \phi(\mu)\rangle + \zeta||\nu - \phi(\mu)||^p \geq 0, \quad \forall \nu \in \Omega, \; p \geq 0, \tag{6.2}$$

which is called the higher order variational inequality (4.5).

(ii). For $\phi = I$ the problem (6.1) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \xi(\nu) - \mu\rangle + \zeta||\xi(\nu) - \mu||^p \geq 0, \quad \forall \nu \in \Omega, \; p \geq 0, \tag{6.3}$$

which is known as the higher order variational inequality, introduced and studied by Noor et al. [28, 30], [35]-[37].

It is an interesting open problem to explore the applications, existence results, sensitivity analysis, merit functions, well-posedness and other aspects of the higher order extended general variational inequalities (6.1).

Conclusion

In this paper, the concepts of higher order strongly general convex functions are introduced and studied. Some interesting and important classes of known and new classes of convex functions are deduced as special cases. Relationship with other convex functions are highlighted. Characterizations of the general convex functions are discussed. It is shown that the optimality criteria of differentiable general convex functions is represented by the variational inequalities. Convergence criteria of the proposed methods is investigated. Our method of proof is very simple as compared with other approaches. Parallelogram laws are derived as novel applications. One may consider the applications of the main results.

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