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# Nadler's Fixed Point Theorem for Set-Valued Mappings in b-Metric Spaces

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**Abstract**

In the paper we present the fixed point theorem for set-valued contraction mappings in generalized b-metric spaces, which generalizes the famous Nadler's fixed point theorem for such mappings in metric spaces. Also some local fixed point theorems for such multi-valued mappings are presented.

*Keywords:* Metric space, b-metric space, Generalized b-metric space, Fixed point

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## 1. Introduction

In the paper we present some results on the existence of fixed points of a set-valued contraction mappings in a generalized b-metric (ball metric) spaces. A b-metric space were introduced by the author first for $s = 2$ in [9] and then for an arbitrary s in [10]. The results stated in this paper, generalize the famous Nadler's theorems [14] (theorem 5), [15] (theorem 1) and [3] (theorem 1). Note that Nadler's results were the first one for multi-valued mappings generalizing the well known contraction principle of Banach [1]. In the paper [3] there are also results generalizing the method of Diaz and Margolis [11] (see also [4]-[7]).

In this part we present some necessary ideas and definitions which we shall use in the paper.

**Definition 1.1** (*cf.* [2])**.** A b-metric on a set $X$ (nonempty) is a function $d : X \times X \to [0, \infty)$, satisfying the following conditions:

**(i)** $d(x, y) = 0$ iff $x = y$,

**(ii)** $d(x, y) = d(y, x)$,

**(iii)** $d(x, y) \leqslant s[d(x, z) + d(z, y)]$,

for all $x, y, z \in X$, and for some fixed $s \geqslant 1$. The pair $(X, d)$ is called a b-metric space. If $d$ is such that $d : X \times X \to [0, \infty]$ then $d$ is called a generalized b-metric (see [13], abbreviated gbms).

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**Definition 1.2.** A set $U \subset (X, d)$, where $d$ is a generalized b-metric in $X$, is said to be closed, iff for every sequence of points $x_n \in U$, $n \in \mathbb{N}$, $x_n \to x \in X$ (with respect to $d$) implies $x \in U$.
By $CLX$ we denote the family of all nonempty closed subsets of $X$ (see [3]).

**Definition 1.3.** Let

$$H(A,B) := \begin{cases} \max\left\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\right\}, \text{ if the max exists,} \\ \infty, \text{ otherwise,} \end{cases} \tag{1.1}$$

for $A, B \in CLX$.
Then $H$ is called the generalized Hausdorff distance induced by $d$ (see also e.g. [3]).

One has

**Lemma 1.4.** *If $a \in X$ and $B \in CL(X,d)$, $(X,d)$-gbms, then*

$$D(a,B) = 0 \Rightarrow a \in B.$$

*Proof.* If $D(a,B) = 0$, that is $\inf_{\xi\in B} d(a,\xi) = 0$, then for $\epsilon_n > 0$, there exists $\xi_n \in B$ such that $d(a,\xi_n) < \epsilon_n$. Let $\epsilon_n \to 0$, so $d(a,\xi_n) \to 0$. Clearly $\{\xi_n\}$ is a Cauchy sequence. Indeed, $d(\xi_n,\xi_{n+m}) \leqslant s[d(\xi_n,a) + d(a,\xi_{n+m})] \to s[0+0]$. So $\xi_n \to \xi \in B$, since $B \in CL(X,d)$.
But

$$d(a,\xi) \leqslant s[d(a,\xi_n) + d(\xi_n,\xi)] \to s[0+0].$$

Therefore,

$$[d(a,\xi) \to 0] \Rightarrow a = \xi \in B,$$

which means that $a \in B$.
Clearly, $a \in B \Rightarrow D(a,B) = 0$. □

**Lemma 1.5.** *If $(X,d)$ is a generalized b-metric space (gbms), then $H$ is a generalized b-metric (gbm) for $A, B \in CL(X,d)$ :*

$$H(A,B) = 0 \Leftrightarrow A = B, \tag{1.2}$$

$$H(A,B) = H(B,A), \tag{1.3}$$

$$H(A,B) \leqslant s[H(A,U) + H(U,B)], \tag{1.4}$$

*for all $A, B, U \in CL(X,d)$. Except that $H$ may have "infinite values" (see [3]).*

*Proof.* Let for $A, B \in CL(X,d)$

a) $H(A,B) < \infty$, $H(A,B) = \sup_{a\in A} D(a,B)$.

Then one has

$$\begin{aligned} H(A,B) &= \sup_{a\in A} D(A,B) \leqslant \sup_{a\in A}\left\{\inf_{b\in B} d(a,b)\right\} \\ &\leqslant \sup_{a\in A}\inf_{b\in B}\left\{s[d(a,\xi) + d(\xi,b)\right\}, \left(\xi \in U \in CL(X,d)\right) \\ &\leqslant s\sup_{a\in A}\left\{d(a,\xi) + d(\xi,b)\right\} \\ &\leqslant s\sup_{a\in A}\left\{d(a,\xi) + D(U,b)\right\} \leqslant s\sup_{a\in A}\left\{D(a,U) + D(U,b)\right\} \\ &\leqslant s[\sup_{a\in A} D(a,U) + D(U,b)] \leqslant s[H(A,U) + H(U,B)], \end{aligned}$$

i.e.

$$H(A,B) \leqslant s[H(A,U) + H(U,B)], \quad A,B,U \in CL(X,d).$$

If

b) $H(A,B) = \sup_{b \in B} D(b,A)$,

the proof is the same.

For the equation (1.2): by Lemma 1.4 and Definition 1.3,

$$H(A,B) = 0 \Rightarrow \sup_{a \in A} D(a,B) = 0 \Rightarrow D(a,B) = 0 \Rightarrow a \in B \Rightarrow A \subset B.$$

Similarly, $B \subset A$, i.e. $A = B$.

Conversely, if $A = B \Rightarrow H(A,B) = 0$ by the definition. The equation (1.3) is obvious. □

Let's note the following remark (see also [3]). Generalized b-metric can be reduce to b-metric by taking the minimum of the generalized b-metric and the real number 1. Such new b-metric preserves the topology but changes the Lipschitz structure of the of the generalized b-metric. Because we work with contraction mappings, we can not do such operation.

Let's also make the following Lemma.

**Lemma 1.6.** *Let $(X,d)$ be a generalized b-metric on X.*
*Let*

**(iv)** $\xi(x,y) := \min[d(x,y),1] = \begin{cases} d(x,y), & d(x,y) < 1, \\ 1, & d(x,y) \geqslant 1. \end{cases}$

*The function (iv) is a b-metric on X.*

*Proof.* First of all, we present the proof in the case

Case I) $d(x,y) > 1$.

For $d(x,y) > 1$, one has $\xi(x,y) = 1$. Also

$$d(x,y) \leqslant s[d(x,z) + d(z,y)], \quad x,y,z \in X, \quad s \geqslant 1.$$

Assume that $d(x,z) < 1$, $d(z,y) < 1$, then $\xi(x,z) = d(x,z)$, $\xi(z,y) = d(z,y)$ and one has

$$\begin{aligned} \xi(x,y) = 1 < d(x,y) &\leqslant s[d(x,z) + d(z,y)] \\ &\leqslant s[\xi(x,z) + \xi(z,y)]. \end{aligned}$$

If $d(x,z) \geqslant 1$, then $\xi(x,z) = 1$, and

$$\xi(x,y) \leqslant 1 \leqslant s \leqslant s[1 + \xi(z,y)] \leqslant s[\xi(x,z) + \xi(z,y)].$$

Case II) $d(x,y) < 1$.

Then $\xi(x,y) = d(x,y)$. One has

$$\xi(x,y) = d(x,y) \leqslant s[d(x,z) + d(z,y)].$$

Moreover, if $d(z,y) < 1$, then $\xi(z,y) = d(z,y)$, and

$$\xi(x,y) \leqslant s[d(x,z) + d(z,y)] \leqslant s[\xi(x,z) + \xi(z,y)].$$

If $d(z,y) \geqslant 1$, then $\xi(z,y) = d(z,y) = 1$ and hence

$$\xi(x,y) \leqslant s[d(x,z) + d(z,y)] \leqslant s[\xi(x,z) + \xi(z,y)],$$

so the inequality

$$\xi(x,y) \leqslant s[\xi(x,z) + \xi(z,y)],$$

holds true.

□

**Lemma 1.7.** *For every $a \in A$, $A \in CL(X,d)$ and every $\epsilon > 0$, there exists $b \in B$; $B \in CL(X,d)$ such that*

$$d(a,b) \leqslant H(A,B) + \epsilon. \tag{1.5}$$

*Proof.* Let

$H(A,B) < \infty$, $H(A,B) = \sup_{a\in A} D(a,B)$.

Then

$$\sup_{a\in A} D(a,B) \leqslant H(A,B) \leqslant H(A,B) + \epsilon.$$

Therefore there exists $b \in B$ such that

$$d(a,b) \leqslant H(A,B) + \epsilon.$$

If not, so for every $b \in B$, $d(a,b) > H(A,B) + \epsilon$, $\epsilon$-fixed, and hence

$$\inf_{b\in B} d(a,b) = D(a,B) \geqslant H(A,B) + \epsilon.$$

Hence

$$\sup_{a\in A} D(a,B) \geqslant H(A,B) + \epsilon,$$

and consequently,

$$H(A,B) = \sup_{a\in A} D(a,b) \geqslant H(A,B) + \epsilon,$$

what is impossible. □

**Definition 1.8** (*cf.* [3]). A function $T : X \to CL(X)$ is said to be a multi-valued contraction mapping (mvcm) iff there exists a real number $0 \leqslant \lambda < 1$ such that

$$H[T(x), T(y)] \leqslant \lambda d(x,y), \quad x, y \in X, d(x,y) < \infty.$$

Similarly, a mapping $T : X \to CL(X)$ is said to be an $(\lambda, \epsilon)$-uniformly locally contractive multi-valued mapping, where $\epsilon > 0$ and $0 \leqslant \lambda < 1$, iff

$$H[T(x), T(y)] \leqslant \lambda d(x,y), x, y \in X, d(x,y) < \epsilon.$$

**Definition 1.9** (*cf.* [3]). For a generalized b-metric space $(X,d)$, $x_0 \in X$, and $T : X \to CL(X)$ a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in X$, $n \in \mathbb{N}$, is said to be an iterative sequence of $T$ at $x_0$, iff $x_{n+1} \in T(x_n)$ for $n \in \mathbb{N}_0$. A point $x \in X$ is called a fixed point of $T$ iff $x \in T(x)$.

**Definition 1.10** (*cf.* [3, 8]). A generalized b-metric space $(X,d)$ is said to be $\epsilon$-chainable, $\epsilon > 0$, iff for every $x, y \in X$, $d(x,y) < \infty$, there exists an $\epsilon$-chain from $x$ to $y$, i.e. a finite set of points $x_0 = x, x_1 ..., x_n = y$ such that $d(x_{k-1}, x_k) < \epsilon$ for $k = 1, 2, ..., n$.
The following interesting results have been proved by Covitz and Nadler, Jr.

**Proposition 1.11** (*cf.* [3]). *Let $(X,d)$ be a generalized complete metric space, and let $x_0 \in X$. If $F : X \to CL(X)$ is an $(\lambda, \epsilon)$-uniformly locally contrative multi-valued mapping, then the following alternative holds: either*
*(A) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of $F$ at $x_0$, $d(x_{i-1}, x_i) \geqslant \epsilon$ for each $i \in \mathbb{N}$, or*
*(B) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of $F$ at $x_0$ such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of $F$.*

**Proposition 1.12** (*cf.* [14]). *Let $(X,d)$ be a complete metric space and let $x_0 \in X$. If $F : X \to CL(X)$ is a mvcm, then there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of $F$ at $x_0$ such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of $F$.*

As usual, by $\mathbb{N}$, $\mathbb{N}_0$ we denote the set of all natural numbers or the set of all natural numbers with zero, respectively. By " $\sim$ " we denote an equivalence relation in $X$.

## 2. Main results

Now we prove the following

**Theorem 2.1.** *Let* $(X, d)$ *be a generalized complete b-metric space and let* $x_0 \in X$*. If* $T : X \to CL(X, d)$ *is a* $(\lambda, \epsilon)$*-uniformly locally contractive multi-valued mapping, i.e.*

$$H[T(x), T(y)] \leqslant \lambda d(x, y), x, y \in X, d(x, y) < \epsilon, \tag{2.1}$$

*and*

$$0 \leqslant \lambda < 1, \tag{2.2}$$

*then the following alterative holds: either*
(I) *for each iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *of* $T$ *at* $x_0$, $d(x_{n-1}, x_n) \geqslant \epsilon$ *for* $n \in \mathbb{N}$*, or*
(II) *there exists an iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *of* $T$ *at* $x_0$ *such that* $x_n \to u \in X$*, and* $u \in T(u)$*.*

*Proof.* Assume that (I) does not hold. Then there exists $x_m \in T(x_{m-1})$, $x_{m-1}, x_m \in X$, and $d(x_{m-1}, x_m) < \epsilon$ for some $m \in \mathbb{N}$. Hence

$$H[T(x_{m-1}), T(x_m)] \leqslant \lambda d(x_{m-1}, x_m) < \lambda\epsilon < \epsilon.$$

Since $x_m \in T(x_{m-1})$, there exists an $x_{m+1} \in \epsilon T(x_m)$ such that, by Lemma 1.7,

$$d(x_m, x_{m+1}) \leqslant H[T(x_{m-1}), T(x_m)] + \lambda\overline{\epsilon},$$

where $\overline{\epsilon} > 0$. Let $\overline{\epsilon} < \min[\epsilon - \lambda\epsilon, 1]$.

Therefore

$$d(x_m, x_{m+1}) < \lambda\epsilon + \lambda\overline{\epsilon} < \lambda(\epsilon + 1).$$

We show that there exists a sequence (for T) such that:

(1)
$$x_{m+k+1} \in T(x_{m+k}),$$

(2)
$$d(x_{m+k}, x_{m+k+1}) < \epsilon, \tag{2.3}$$

(3)
$$d(x_{m+k}, x_{m+k+1}) < \lambda^k + k\lambda^{k-1}\overline{\epsilon},$$

for $k \in \mathbb{N}$. Assume that the equation (2.3) is true for $k \in \mathbb{N}$. By induction, for $k + 1$ one has: there exists an $x_{m+k+2} \in T(x_{m+k+1})$ such that (see Lemma 1.6)

$$\begin{aligned} d(x_{m+k+1}, x_{m+k+2}) &\leqslant H[T(x_{m+k}), T(x_{m+k+1})] + \lambda^k\overline{\epsilon} \\ &\leqslant \lambda d(x_{m+k}, x_{m+k+1}) + \lambda^k\overline{\epsilon} \\ &\leqslant \lambda[\lambda^k + k\lambda^{k-1}\overline{\epsilon}] + \lambda^k\overline{\epsilon} \\ &\leqslant \lambda^{k+1} + (k+1)\overline{\epsilon}\lambda^k, \end{aligned}$$

i.e.

$$d(x_{m+k+1}, x_{m+k+2}) \leqslant \lambda^{k+1} + (k+1)\overline{\epsilon}\lambda^k,$$

that is the equation (2.3) for $k + 1$. Clearly,

$$d(x_{m+k+1}, x_{m+k+2}) \leqslant \lambda\epsilon + \overline{\epsilon} < \epsilon.$$

This means that (1), (2), (3) are true for all $k \in \mathbb{N}$.

Now we verify that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. From Paluszyński, Stempak [16], for $x, y \in X$, $d(x, y) < \infty$, let

$$d_\epsilon(x, y) := \inf \begin{cases} \sum_{i=1}^{n} d^p(x_{i-1}, x_i) : x = x_0, x_1, ..., x_n = y, \\ \infty, \text{if} d(x, y) = \infty, \end{cases} \tag{2.4}$$

where $0 < p \leqslant 1$, is such that

$$(2s)^p = 2.$$

Evidently, $d_\epsilon$ is symmetric, satisfies the triangle inequality and $d_\epsilon \leqslant d^p$. They also proved that $d_\epsilon$ is a metric and $d_\epsilon \sim d^p$. Note that $p$ is such that $(2s)^p = 2$. For $n, r \in \mathbb{N}$, by the equation (2.4) one has

$$\begin{aligned} d_\epsilon(x_n, x_{n+r}) &\leqslant d_\epsilon(x_n, x_{n+r}) + \ldots + d_\epsilon(x_{n(r-1)} x_{n+r}) \\ &\leqslant d^p(x_n, x_{n+1}) + ... + d^p(x_{n(r-1)}, x_{n+r}) \\ &\leqslant \sum_{i=n}^{\infty} d^p(x_i, x_{i+1}) \\ &\leqslant \sum_{i=n}^{\infty} \lambda^{p(i+1)}(\epsilon + i + 1)^p. \end{aligned}$$

Put $\lambda^p = \xi < 1$, so

$$\begin{aligned} d_\epsilon(x_n, x_{n+r}) &\leqslant \sum_{i=n}^{\infty} \xi^{i+1}(\epsilon + i + 1) \\ &\leqslant (1 - \xi)^{-1}\xi^{n+1} + \sum_{i=n}^{\infty}(i + 1)\xi^{i+1} \\ &\leqslant (1 - \xi)^{-2}(n + 4)\xi^{n+1}. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, d_\epsilon)$ so $\{x_n\}$ is a Cauchy sequence in $(X, d)$ since $d_\epsilon \sim d^p$ (for details see Jung [12] and [2]). Consequently, $x_n \to u$ as $n \to \infty$ in $(X, d)$, but $(X, d)$ is complete, so $u \in X$.

Finally we prove that $u \in T(a)$. Indeed, for $n$ sufficiently large, we have

$$\begin{aligned} D(uT(u)) &= \inf_{\xi \in T(u)} d(u, \xi) \leqslant d(u, \xi) \\ &\leqslant s[d(u, x_{n+1}) + d(x_{n+1}, \xi)] \\ &\leqslant s[d(u, x_{n+1}) + D(x_{n+1}, T(u))] \\ &\leqslant s[d(u, x_{n+1}) + H(T(x_n), T(u))] \\ &\leqslant s[d(u, x_{n+1}) + \lambda d(x_n, u)] \to s[0 + 0] = 0, n \to \infty. \end{aligned}$$

Hence $D(u, T(u)) = 0$, but since $T(u)$ is closed, so $u \in T(u)$. □

Next result is the following

**Theorem 2.2.** *Let $(X, d)$ be a complete $\epsilon$-chainable generalized b-metric space and let $x_0 \in X$. If $T : X \to CL(X, d)$ is a $(\lambda, \epsilon)$-uniformly locally contractive multi-valued mapping, then the following alternative holds: either*
(III) *for each iterative sequence $\{x_n\}_{n=1}^{\infty}$ of $T$ at $x_0$, $d(x_{n-1}, x_n) = \infty$ for $n \in \mathbb{N}$;*
*or*
(IV) *there exists an iterative sequence $\{x_n\}_{n=1}^{\infty}$ of $T$ at $x_0$ such that $x_n \to x$ as $n \to \infty$, $x \in X$ and $x \in T(x)$.*

*Proof.* Suppose (III) does not hold. By Paluszyński and Stempak [16], define for $x, y \in X$,

$$d_\epsilon(x, y) := \begin{cases} \inf \sum_{i=1}^{n} d^p(x_{i-1}, x_i), x_0 = x, x_1, \ldots, x_n = y, \\ d(x_{i-1}, x_i) < \epsilon \text{ for } i = 1, \ldots, n, \\ \infty, \text{ if } d(x, y) = \infty, \end{cases}$$

where $(2s)^p = 2$.
Then $(X, d_\epsilon)$ is a generalized complete metric space, $d_\epsilon \sim d^p$, $0 < p \leqslant 1$ ( one can repeat the proof presented by Paluszyński and Stempak for $\epsilon$-chainable b-metric space). Let $H_\epsilon$ be the generalized Hausdorff metric on $CL(X, d_\epsilon)$ induced by $d_\epsilon$.
We can verify that $CL(X, d) = CL(X, d_\epsilon)$. Indeed, if $U \in CL(X, d)$, then by the Definition 1.2, one has $[x_n \in U$ and $x_n \overset{d}{\to} x \in X] \Rightarrow x \in U$, so as well $[x_n \in U, x_n \overset{d_\epsilon}{\to} x] \Rightarrow x \in U$ and consequently $CL(X, d) \subset CL(X, d_\epsilon)$. Conversely is the same, since $d_\epsilon \sim d^p$.
Now we want to prove that

$$H_\epsilon(A, B) \leqslant H^P(A, B), \quad A, B \in CL(X).$$

Let $H_\epsilon(A, B) = \sup_{a\in A} D_\epsilon(a, B)$. Then one has for $H_\epsilon(A, B) < \infty$,

$$\begin{aligned} H_\epsilon(A, B) &= \sup_{a\in A} D_\epsilon(a, b) \leqslant \sup_{a\in A}\{\inf_{b\in B} d_\epsilon(a, b)\} \\ &\leqslant \sup_{a\in A} \inf_{b\in B} d^p(a, b) \leqslant \sup_{a\in A}\{\inf_{b\in B} d(a, b)\}^p \\ &\leqslant \sup_{a\in A} D^p(a, B) \leqslant [\sup_{a\in A} D(a, B)]^p \leqslant H^p(A, B), \end{aligned}$$

i.e.

$$H_\epsilon(A, B) \leqslant H^p(A, B), \quad A, B \in CL(X, d) = CL(X, d_\epsilon).$$

□

Now let $x, z \in X$ and $d(x, z) < \infty$. If $x_0 = x, x_1, \ldots, x_n = z$, then

$$\begin{aligned} H_\epsilon[T(x), T(z)] &\leqslant \sum_{i=1}^{n} H_\epsilon[T(x_{i-1}), T(x_i)] \leqslant \sum_{i=1}^{n} H^p[T(x_{i-1}), T(x_i)] \\ &\leqslant \sum_{i=1}^{n} [\lambda d(x_{i-1}, x_i)]^P \leqslant \lambda^p \sum_{i=1}^{n} d^p(x_{i-1}, x_i). \end{aligned}$$

Since the inequality between the first and last terms of the above inequalities holds for all $\epsilon$-chains $x = x_0, x_1, \ldots, x_n = z$, $n \in \mathbb{N}$, connecting $x$ and $z$, it follows

$$H_\epsilon[T(x), T(z)] \leqslant \lambda^p d_\epsilon(x, z) \tag{2.5}$$

for all $x, z \in X$, $d(x, z) < \infty$, where $0 \leqslant \lambda^p < 1$.
Clearly $T$ is a $(\lambda^p, \epsilon)$-uniformly locally contractive multi-valued mapping with $d_\epsilon$, $H_\epsilon$, and $\epsilon > 0$ such that $d(x_{n-1}, x_n) < \epsilon$, which follows by the assumption that (III) does not hold. Therefore the sequence $\{x_n\}$ starting from $x_{n-1}$, does not satisfy the condition (I) (see the proof of Theorem 2.1). So our statement (IV) follows directly from (II) of Theorem 2.1.

*Remark* 2.3. In the proofs of Theorem 2.1 and Theorem 2.2 we utilize some ideas contained in [3].

*Remark* 2.4. If $(X, d)$ is a metric space, then from Theorem 2.1 we get the famous Nadler's fixed point theorem for multi-valued contraction mappings (see [2, 3, 8]).

*Remark* 2.5. If $(X, d)$ is an $\epsilon$-chainable metric space, we get from Theorem 2.2 Corollary 2 of [3].

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