Montes Taurus J. Pure Appl. Math. 4 (2), 1–10, 2022

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

p-integrable solution of boundary fractional differential and integro-differential equations with Riemann derivatives of order $(n - 1 < \delta \leq n)$

Dilsher N. Abdulqader[a], Shayma Adil Murad [b]

[a]*Department of Mathematics, College of Science, University of Duhok, Duhok 42001, IRAQ*
[b]*Department of Mathematics, College of Science, University of Duhok, Duhok 42001, IRAQ*

Abstract

This paper considers the existence of L_p-solutions of certain fractional differential and integro-differential equations involving the Riemann derivatives of order $(n - 1 < \delta \leq n)$, with boundary conditions. The results are established by means of the Hölder's inequality in a Banach space. Some special cases and examples are given to explain the main results.

Keywords: Fracional differential equations, integro-differential equations, Riemann fractional derivatives, existence and uniqueness, Hölder's inequality, L_p space

2010 MSC: 34A08, 26A33

1. Introduction

Recently, fractional differential equations pay the attention for many researchers in fields of science and engineering. This is, mainly, due to the importance of noninteger order derivatives in modeling certain physical phenomena see ([12, 17, 18]). The researchers get interested in looking at the theory of existence and uniqueness of solutions for differential equations involving the fractional derivatives for references see ([1, 3, 10], [13]-[16]). In addition, there have been some good results concerning the existence, uniqueness, of L_p solutions to some nonlinear fractional differential equations. As for some bibliographies, we refer readers to see ([2, 5, 7, 8, 11]) and the reference therein. Aghajani et al. [4] proved the solvability of a large class of nonlinear fractional integro-differential equations by establishing some fractional integral inequalities under some suitable conditions. Ahmed et al. [6] studied the existence and uniqueness of solutions for a nonlinear boundary value problem of fractional differential equations with higher order $(n - 1 < \delta \leq n)$ involving Riemann-Liouville fractional derivative using Banach's fixed point theorem. The L_p-solutions of fractional differential equations are investigated in [9].

In this paper, we consider the following fractional boundary value problems of the form

1. First case

$$D^{\delta} w(s) = h(s, w), \qquad I = [\Omega, B], \tag{1.1}$$

$$\Upsilon_r D^{(\delta - r)} w(\Omega) + \eta_r D^{(\delta - r)} w(B) = \Delta_r, \quad r = 1, 2, 3, ..., n. \tag{1.2}$$

---

†*Article ID:* MTJPAM-D-21-00050

Email addresses: dilsher.abdulqader@uod.ac (Dilsher N. Abdulqader), shaymaadil@uod.ac (Shayma Adil Murad)

Received:25 July 2021, Accepted:9 March 2022, Published:12 April 2022

⋆*Corresponding Author:* Shayma Adil Murad

2. Second case

$$D^{\delta}w(s) = \frac{1}{\Gamma(\gamma)}\int_{\Omega}^{s}(s-\tau)^{\gamma-1}z(s,\tau,w(\tau))d\tau + h(s,w),$$
$$\Upsilon_r D^{(\delta-r)}w(\Omega) + \eta_r D^{(\delta-r)}w(B) = \Delta_r, \quad r = 1,2,3,...,n,$$

where D^{δ} is the Riemann fractional derivative, $(k-1<\gamma\le k,\ n-1<\delta\le n)$, $\Upsilon_r, \eta_r, \Delta_r$ are constants, and $L_p[\Omega, B]$ $(1 \le p < \infty)$ is the space of all measurable function such that $|h|^p$ is Lebsegue integrable on $[\Omega, B]$ and for any $h \in L_p[\Omega, B]$, we define the norm as:

$$\|h\|_p = \left(\int_{\Omega}^{B}|h(s)|^p\,ds\right)^{\frac{1}{p}},$$

under this norm, it is known that the space $L_p[\Omega, B]$ is Banach space. we investigate the existence and uniqueness of solutions for the above boundary value problems in L_p-space. Moreover, we provide an illustrative examples.

2. Preliminaries

In this section, we give some basic definitions and lemmas of fractional calculus which will be used in this paper. For references see ([12, 17, 18]).

Definition 2.1. Let f be a function which is defined almost everywhere ($a.e.$) on $[\Omega, B]$, If $\delta > 0$, then:

$$_{\Omega}^{B}I^{\delta}h = \int_{\Omega}^{B}h(s)\frac{(B-s)^{\delta-1}}{\Gamma(\delta)}ds,$$

provided that this integral (Lebesgue) exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\delta > 0$ for a function function $f : [0,\infty) \to \mathbb{R}$, is defined as

$$^{RL}D^{\delta}h(s) = \frac{1}{\Gamma(n-\delta)}\frac{d^n}{dt^n}\int_0^s(s-\tau)^{n-\delta-1}h(\tau)d\tau, \quad n-1<\delta\le n,$$

where $n = [\delta]+1, [\delta]$ denotes the integer part of the real number δ.

Lemma 2.3. *Let $\delta > 0$, n be the smallest integer $n > \delta$ and let $h(t) \in L(\Omega, B)$. If $_{a}^{t}D^{\delta-1}h$ exists and is absolutely continuous on $[\Omega, B]$, then $_{\Omega}^{\Omega^+}I^{i-\delta}h = k_i$ exists for $i = 1,2,...,n$; $_{\Omega}^{s}D^{\delta}h$ exists a.e. on $[\Omega, B]$, is in $L(\Omega, B)$ and*

$$_{\Omega}^{s}I^{\delta}\ _{\Omega}^{\tau}D^{\delta}h(\tau) = -\sum_{i=1}^{n}\frac{\kappa_i(s-\Omega)^{\delta-i}}{\Gamma(\delta-i+1)} + h(s) \quad \text{a.e.} \quad \text{on} \quad \Omega\le s\le B.$$

Furthermore, the inequality holds everywhere on $(\Omega, B]$, if in addition, $h(t)$ is continuous on $(\Omega, B]$.

Lemma 2.4. If $h(s) \in L(\Omega, B)$ and $\delta, \gamma > 0$, then on $\Omega \le s \le B$, we have

(1) $\int_{\Omega}^{s}\ _{\Omega}^{\tau}D^{-\delta}h\,ds = \ _{\Omega}^{s}D^{-(\delta+1)}h.$

(2) If $\delta \ge 1$ then $_{\Omega}^{s}D^{-\delta}h$ is absolutely continuous at $s \in [\Omega, B]$.

(3) $\frac{d}{ds}\ _{\Omega}^{s}D^{-(\delta+1)}h = \ _{\Omega}^{s}D^{-\delta}h$, everywhere if $\delta \ge 1$ and a.e. if $\delta < 1$.

(4) $_{\Omega}^{s}D^{-(\delta+\gamma)}h = \ _{\Omega}^{s}D^{-\delta}\ _{\Omega}^{\tau}D^{-\gamma}h$ a.e. if $\delta+\gamma \le 1$.

Lemma 2.5. *If $h(s) \in L(\Omega, B)$ and $\delta > 0$, then $_{\Omega}^{s}I^{\delta}h$ exists for all $s \in [\Omega, B]$, if $\delta \ge 1$ and a.e. if $\delta < 1$.*

3. Main results (Existence result for fractional boundary value problem)

Lemma 3.1 (*cf.* [3]). *For any $w \in C(I)$, and $n-1 < \delta \leq n$, then the boundary value problem* (1.1)-(1.2) *has a solution*

$$\begin{aligned} w(s) &= \sum_{i=1}^{n} \frac{\kappa_i (s-\Omega)^{\delta-i}}{\Gamma(\delta-i+1)} + {}_{\Omega}^{s}I^{\delta} h(s,w) \\ \kappa_i &= \frac{1}{\eta_i + \Upsilon_i}\Big[-\Upsilon_i \sum_{m=1}^{i-1} \frac{\kappa_m (B-\Omega)^{i-m}}{\Gamma(i-m+1)} - \Upsilon_i \, {}_{\Omega}^{B}I^{i} h(s,w) + \Delta_i \Big]. \end{aligned} \tag{3.1}$$

For the forthcoming analysis, the following conditions need to be fulfilled:

(H_1) The function $h : I \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists a constant $\Psi > 0$ such that:

$$|h(s,w_1) - h(s,w_2)| \leq \Psi |w_1 - w_2|,$$

for all $w_1, w_2 \in \mathbb{R}$ and $s \in I$.

Let us set the following notation for convenience:

$$\Lambda_1 = \Big(\sum_{i=1}^{n-1} \frac{2^{(i+1)p} (B-\Omega)^{p(\delta-i)+1} \, \theta_i}{(p(\delta-i)+1) \, (\Gamma(\delta-i+1))^p} + \frac{2^{np} (B-\Omega)^{p(\delta-n)+1} \, \theta_n}{(p(\delta-n)+1) \, (\Gamma(\delta-n+1))^p} + \frac{2^p (B-\Omega)^{(p\delta)} \, \Psi^p}{(\Gamma(\delta))^p \, (\frac{p\delta-1}{p-1})^{p-1} \, p\delta} \Big)^{\frac{1}{p}},$$

$$\begin{aligned} \Lambda_2 = \Big[& \frac{2^{3p} \, \Psi^p \, B^{p\delta} \, \Upsilon_2^p}{(\eta_2+\Upsilon_2)^p \, (p(\delta-2)+1) \, (\frac{2p-1}{p-1})^{p-1} \, (\Gamma(\delta-1))^p} + \frac{2^p \, B^{p\delta} \, \Psi^p}{(\Gamma(\delta))^p (\frac{p\delta-1}{p-1})^{p-1} \, (p\delta)} \\ & + \frac{2^{2p} \, (\Upsilon_1 \Upsilon_2)^p \, \Psi^p \, B^{p\delta}}{(\eta_1+\Upsilon_1)^p (\eta_2+\Upsilon_2)^p (\Gamma(\delta-1))^p (p(\delta-2)+1)} + \frac{2^{3p} \, \Psi^p \, B^{p(\delta-1)+p} \, \Upsilon_1^p}{(\eta_1+\Upsilon_1)^p \, (p(\delta-1)+1) \, (\Gamma(\delta))^p} \Big]^{\frac{1}{p}}, \end{aligned}$$

where $(p(\delta-2) \neq -1)$, $(p(\delta-1) \neq -1)$.

Theorem 3.2. *Assume that* (H_1)-(H_2) *hold. If* $\Lambda_1 < 1$. *Then the boundary value problem* (1.1)-(1.2) *has a unique solution.*

Proof. Define the operator $F \in L_p[\Omega, B]$ by

$$Fw(s) = \sum_{i=1}^{n} \frac{\kappa_i (s-\Omega)^{\delta-i}}{\Gamma(\delta-i+1)} + {}_{\Omega}^{s}I^{\delta} h(s,w),$$

where

$$\kappa_i = \frac{1}{\eta_i + \Upsilon_i}\Big[-\Upsilon_i \sum_{m=1}^{i-1} \frac{\kappa_m (B-\Omega)^{i-m}}{\Gamma(i-m+1)} - \Upsilon_i \, {}_{\Omega}^{B}I^{i} h(s,w) + \Delta_i \Big].$$

In view of the continuity of the first term and h for all $t \in [\Omega, B]$. Now, by Lemma 2.4-(2), we obtain ${}_{\Omega}^{s}I^{\delta} h(s,w)$ is continuous. Then, the operator Fw defined above is continuous and it is measurable.
Firstly, we show that Fw is Lebesgue integrable. Indeed, since

$$|Fw(s)|^p \leq 2^p \Big| \sum_{i=1}^{n} \frac{\kappa_i (s-\Omega)^{\delta-i}}{\Gamma(\delta-i+1)} \Big|^p + \frac{2^p}{\Gamma(\delta)} \Big| \int_{\Omega}^{s} (s-\tau)^{\delta-1} h(\tau, w(\tau)) d\tau \Big|^p,$$

using Holder's inequality, we have

$$\Big(\int_{\Omega}^{s} (s-\tau)^{\delta-1} h(\tau, w(\tau)) d\tau \Big)^p \leq \frac{(s-\Omega)^{p\delta-1}}{\Big[\frac{p\delta-1}{p-1}\Big]^{p-1}} \int_{\Omega}^{s} (h(\tau, y))^p d\tau,$$

by Definition 2.1 and Lemma 2.4-(1) the following is obtained:

$$ {}_{\Omega}^{s}I\, {}_{\Omega}^{\tau}I(h(\tau,y))^p = {}_{\Omega}^{s}D^{-2}(h(\tau,y))^p = {}_{\Omega}^{s}I^2(h(\tau,y))^p. $$

It is clear that $\int_\Omega^s h^p$ is Lebesgue integrable for all $s \in [\Omega, B]$ by Lemma 2.5. So that, $|Fw(s)|^p$ is Lebesgue integrable. Then, $Fw : L_p[\Omega, B] \to L_p[\Omega, B]$. In the next step, we need to verify that F is a contraction. Let $w_1, w_2 \in L_p[\Omega, B]$. Consider

$$ \|Fw_1(s) - Fw_2(s)\|_p^p \le \int_\Omega^B \Big[\sum_{i=1}^{n} \frac{(s-\Omega)^{\delta-i}\,|\kappa_i - \zeta_i|}{\Gamma(\delta - i + 1)} + {}_{\Omega}^{s}I^{\delta}\Big|h(s, w_1(s)) - h(s, w_2(s))\Big|\Big]^p ds, $$

it follows that

$$ \|Fw_1(s) - Fw_2(s)\|_p^p \le 2^p \int_\Omega^B \Big(\Big[\sum_{i=1}^{n} \frac{(s-\Omega)^{\delta-i}\,|(\kappa_i - \zeta_i)|}{\Gamma(\delta - i + 1)}\Big]^p + \Big[{}_{\Omega}^{s}I^{\delta}\Big|h(s, w_1(s)) - h(s, w_2(s))\Big|\Big]^p\Big) ds. $$

By using the Lipshitz condition (H_2), the result is:

$$ \|Fw_1(s) - Fw_2(s)\|_p^p \le 2^p \Psi^p \int_\Omega^B \Big[{}_{\Omega}^{s}I^{\delta}\Big|w_1 - w_2\Big|\Big]^p ds $$

$$ +2^p \int_\Omega^B \Big(\frac{2^{(n-1)p}\,(s-\Omega)^{p(\delta-n)}\,|\kappa_n - \zeta_n|^p}{(\Gamma(\delta - n + 1))^p} + \sum_{i=1}^{n-1} \frac{2^{ip}|\kappa_i - \zeta_i|^p (s-\Omega)^{p(\delta-i)}}{(\Gamma(\delta - i + 1))^p}\Big) ds. \tag{3.2} $$

Now, the value of $|\kappa_i - \zeta_i|^p$ needs to be found, as follows:

$$ |\kappa_i - \zeta_i|^p \le \frac{2^p \Upsilon_i^p}{(\eta_i + \Upsilon_i)^p} \sum_{m=1}^{i-2} \frac{2^{mp}\,|\kappa_m - \zeta_m|^p\,(B-\Omega)^{p(i-m)}}{(\Gamma(i - m + 1))^p} + \frac{2^p \Upsilon_i^p}{(\eta_i + \Upsilon_i)^p} 2^{(i-2)p}(B-\Omega)^p\,|\kappa_{i-1} - \zeta_{i-1}|^p $$

$$ + \frac{2^p \Upsilon_i^p}{(\eta_i + \Upsilon_i)^p} \Big[\frac{1}{\Gamma(i)} \int_\Omega^B (B-\tau)^{i-1} |h(\tau, w_1(\tau)) - h(\tau, w_2(\tau))| d\tau \Big]^p, \tag{3.3} $$

when $i = n$, the equation (3.3) becomes:

$$ |\kappa_n - \zeta_n|^p \le \theta_n\Big(\int_\Omega^B |w_1(\tau) - w_2(\tau)|^p d\tau\Big), $$

where

$$ \theta_n = \frac{2^p\ \Upsilon_n^p}{(\eta_n + \Upsilon_n)^p} \sum_{m=1}^{n-2} \frac{2^{mp}\theta_m (B-\Omega)^{p(n-m)}}{(\Gamma(n - m + 1))^p} + \frac{2^p\ \Upsilon_n^p}{(\eta_n + \Upsilon_n)^p} \frac{\Psi^p (B-\Omega)^{(np-1)}}{(\frac{np-1}{p-1})^{p-1}\,(\Gamma(n))^p} + \frac{2^p\ \Upsilon_n^p}{(\eta_n + \Upsilon_n)^p} 2^{(n-2)p}\theta_{n-1}(B-\Omega)^p. $$

From the equation (3.2), below is found:

$$ \|Fw_1(s) - Fw_2(s)\|_p^p \le 2^p \int_\Omega^B \Big[\sum_{i=1}^{n-1} \frac{2^{ip}(s-\Omega)^{p(\delta-i)}\theta_i}{(\Gamma(\delta - i + 1))^p} + \frac{2^{(n-1)p}\,(s-\Omega)^{p(\delta-n)}\theta_n}{(\Gamma(\delta - n + 1))^p}\Big] \int_\Omega^B |w_1(\tau) - w_2(\tau)|^p d\tau ds $$

$$ +2^p \int_\Omega^B \Psi^p \Big[\frac{1}{\Gamma(\delta)} \int_\Omega^s (s-\tau)^{\delta-1} |w_1(\tau) - w_2(\tau)| d\tau\Big]^p ds, \tag{3.4} $$

by Hölder inequality and let $\sigma(s) = \int_\Omega^s |w_1(\tau) - w_2(\tau)|^p d\tau$. Then, the equation (3.4) becomes:

$$ \|Fw_1(s) - Fw_2(s)\|_p^p \le \Big(\frac{2^{np}\theta_n}{(\Gamma(\delta - n + 1))^p} \int_\Omega^B (s-\Omega)^{p(\delta-n)} ds + \sum_{i=1}^{n-1} \frac{2^{(i+1)p}\theta_i}{(\Gamma(\delta - i + 1))^p} \int_\Omega^B (s-\Omega)^{p(\delta-i)} ds\Big) $$

$$ \int_\Omega^B |w_1(\tau) - w_2(\tau)|^p d\tau + \frac{2^p \Psi^p}{(\frac{(p\delta-1)}{p-1})^{p-1} (\Gamma(\delta))^p} \int_\Omega^B (s-\Omega)^{(p\delta-1)} \sigma(s) ds, $$

so that

$$\|Fx(t) - Fw(s)\|_p^p \le \Big(\frac{2^{np}\theta_n}{(\Gamma(\delta-n+1))^p}\int_\Omega^B (s-\Omega)^{p(\delta-n)}ds + \sum_{i=1}^{n-1}\frac{2^{(i+1)p}\theta_i}{(\Gamma(\delta-i+1))^p}\int_\Omega^B (s-\Omega)^{p(\delta-i)}ds$$
$$+ \frac{2^p\Psi^p(B-\Omega)^{p\delta}}{(\frac{(p\delta-1)}{p-1})^{p-1}(\Gamma(\delta))^p\, p\delta}\Big)\|w_1 - w_2\|_p^p,$$
$$\|Fw_1(s) - Fw_2(s)\|_p \le \Lambda_1\, \|w_1(s) - w_2(s)\|_p,$$

By the given assumption: $\Lambda_1 < 1$, it follows that the operator F is a contraction mapping and has a fixed point. Thus, boundary value problem (1.1)-(1.2) has a unique solution. □

4. Special case when $(1 < \delta \le 2)$

In this section, we prove the existence and uniqueness for the boundary value problem which has the form

$$\begin{aligned} &D^{(\delta)}w(s) = h(s,w), \quad 1 < \delta \le 2\\ &\eta_1 D^{(\delta-1)}w(0) + \Upsilon_1 D^{(\delta-1)}w(B) = \Delta_1,\\ &\eta_2 D^{(\delta-2)}w(0) + \Upsilon_2 D^{(\delta-2)}w(B) = \Delta_2.\end{aligned} \tag{4.1}$$

The solution has the following form

$$w(s) = \frac{\Delta_1}{\Gamma(\delta)(\eta_1+\Upsilon_1)}s^{\delta-1} - \frac{\Upsilon_1\int_0^B h(\tau,w(\tau))d\tau}{\Gamma(\delta)(\eta_1+\Upsilon_1)}s^{\delta-1} + \frac{\Delta_2}{(\eta_2+\Upsilon_2)\Gamma(\delta-1)}s^{\delta-2} - \frac{s^{\delta-2}\Upsilon_2\Delta_1\, B}{(\eta_1+\Upsilon_1)(\eta_2+\Upsilon_2)\Gamma(\delta-1)}$$
$$+ \frac{B\, s^{\delta-2}\Upsilon_1\Upsilon_2\int_0^B h(\tau,w(\tau))d\tau}{(\eta_1+\Upsilon_1)(\eta_2+\Upsilon_2)\Gamma(\delta-1)} - \frac{s^{\delta-2}\Upsilon_2\int_0^B(B-\tau)h(\tau,w(\tau))d\tau}{(\eta_2+\Upsilon_2)\Gamma(\delta-1)} + {}_0^sI^\delta h(s,w(s)).$$

Theorem 4.1. *Assume that* (H_1)-(H_2) *hold. If* $\Lambda_2 < 1$. *Then the boundary value problem* (4.1) *has a unique solution.*

Proof. Define the operator F on $L_p[0,B]$ as:

$$Fw(s) = \frac{\Delta_1}{\Gamma(\delta)(\eta_1+\Upsilon_1)}s^{\delta-1} - \frac{\Upsilon_1\int_0^B h(\tau,w(\tau))d\tau}{\Gamma(\delta)(\eta_1+\Upsilon_1)}s^{\delta-1} + \frac{\Delta_2}{(\eta_2+\Upsilon_2)\Gamma(\delta-1)}s^{\delta-2} - \frac{s^{\delta-2}\Upsilon_2\Delta_1\, B}{(\eta_1+\Upsilon_1)(\eta_2+\Upsilon_2)\Gamma(\delta-1)}$$
$$+ \frac{B\, s^{\delta-2}\Upsilon_1\Upsilon_2\int_0^B h(\tau,w(\tau))d\tau}{(\eta_1+\Upsilon_1)(\eta_2+\Upsilon_2)\Gamma(\delta-1)} - \frac{s^{\delta-2}\Upsilon_2\int_0^B(B-\tau)h(\tau,w(\tau))d\tau}{(\eta_2+\Upsilon_2)\Gamma(\delta-1)} + {}_0^sI^\delta h(s,w(s)).$$

Let

$$|Fw(s)|^p \le 2^p\Big|{}_0^sI^\delta h(s,w)\Big|^p + \frac{2^{2p}s^{p(\delta-1)}\Delta_1^p}{(\Gamma(\delta)(\eta_1+\Upsilon_1))^p} + \frac{2^{3p}s^{p(\delta-1)}\Upsilon_1^p}{((\eta_1+\Upsilon_1)\Gamma(\delta))^p}\Big|\int_0^B h(\tau,w(\tau))d\tau\Big|^p$$
$$+\frac{2^{4p}\Delta_2^p}{(\Gamma(\delta-1)(\eta_2+\Upsilon_2))^p}s^{p(\delta-2)} + \frac{2^{5p}B^p\Upsilon_2^p\Delta_1^p}{(\Gamma(\delta-1)(\eta_1+\Upsilon_1)(\eta_2+\Upsilon_2))^p}s^{p(\delta-2)}$$
$$+\frac{2^{6p}s^{p(\delta-2)}(B\Upsilon_1\Upsilon_2)^p}{((\eta_2+\Upsilon_2))\Gamma(\delta-1)(\eta_1+\Upsilon_1))^p}\Big|\int_0^B h(\tau,w(\tau))d\tau\Big|^p + \frac{2^{6p}s^{p(\delta-2)}\Upsilon_2^p}{((\eta_2+\Upsilon_2)\Gamma(\delta-1))^p}\Big|\int_0^B(B-\tau)h(\tau,w(\tau))d\tau\Big|^p.$$

$|Fw(s)|^p$ is Lebesgue integrable, therefore, $F: L_p[0,B] \to L_p[0,B]$. Now we have to prove that F is a contraction mapping, let $w_1, w_2 \in L_p[0,B]$, consider

$$\|Fw_1(s) - Fw_2(s)\|_p^p \le \int_0^B\Big(2^p\Big[{}_0^sI^\delta\Big|h(s,w_1(s)) - h(s,w_2(s))\Big|\Big]^p$$
$$+\Big(\frac{2^{3p}\Upsilon_1^p\, s^{p(\delta-1)}}{((\eta_1+\Upsilon_1)\Gamma(\delta))^p} + \frac{2^{2p}(B\Upsilon_1\Upsilon_2)^p\, s^{p(\delta-2)}}{((\eta_1+\Upsilon_1)(\eta_2+\Upsilon_2)\Gamma(\delta-1))^p}\Big)\Big[\int_0^B\Big|h(\tau,w_1(\tau)) - h(\tau,w_2(\tau))\Big|d\tau\Big]^p$$
$$+\frac{2^{3p}s^{p(\delta-2)}\Upsilon_2^p}{((\eta_2+\Upsilon_2)(\Gamma(\delta-1))^p}\Big[\int_0^B(B-\tau)\Big|h(\tau,w_1(\tau)) - h(\tau,w_2(\tau))\Big|d\tau\Big]^p\Big)ds,$$

by using the Lipschitz condition (H_2), and Hölder inequality, the following is obtained:

$$\|Fw_1(s) - Fw_2(s)\|_p \leq \Lambda_2 \|w_1 - w_2\|_p.$$

We conclude that F is a contraction mapping and has a fixed point, If $\Lambda_2 < 1$, then the boundary value problem (4.1) has a unique solution. □

Example 4.2. Consider the following boundary value problem:

$$\begin{cases} D^{\delta} w(s) = \dfrac{\sin s + |w|}{\cos s + 5}, \quad 0 < \delta \leq 1 \\ 2w^{(\delta-1)}(0) + w^{(\delta-1)}(1) = 1. \end{cases} \tag{4.2}$$

Let $p = 2$ and $\delta = 0.6$, then the space is $L^2[0,1]$. Now by applying the condition (H_2), we have: $\Psi = \dfrac{1}{6}$, $\theta_1 = 0.0123456$ and $\Lambda_1 = 0.5657709 < 1$. Therefore, the boundary value problem (4.2) has a fixed point.

Example 4.3. To analyze the behavior of the operator Fw, for the following example:

$$\begin{cases} D^{0.6} w(s) = \dfrac{\sin s + |w|}{\cos s + 5}, \\ 8w^{(\delta-1)}(0) + w^{(\delta-1)}(1) = 1. \end{cases}$$

One can see that $|h(s, w(s))| \leq 0.332377347506373$, then see (Figure 1).

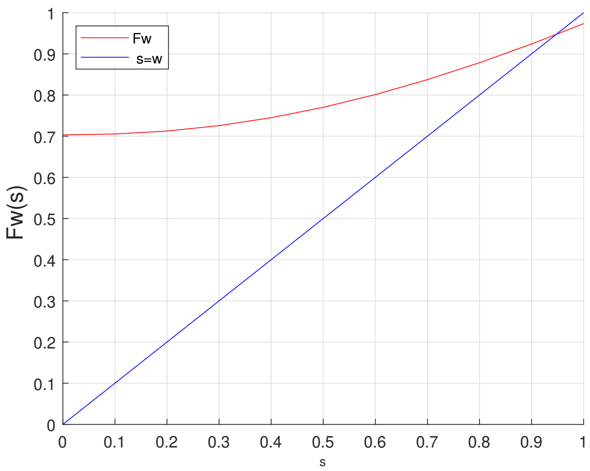

Figure 1. The behavior of the operator Fw for $s \in [0, 1]$.

Example 4.4. Consider the following boundary value problem:

$$\begin{cases} D^{\delta} w(s) = \dfrac{\cos s}{e^s + 4} + \dfrac{|w|}{e^s + 4}, \quad 1 < \delta \leq 2 \\ 2D^{(\delta-1)} w(0) + D^{(\delta-1)} w(1) = 1, \\ 4D^{(\delta-2)} w(0) + D^{(\delta-2)} w(1) = 1. \end{cases} \tag{4.3}$$

Let $\delta = \frac{5}{3}$, $p = 2$, by using (H_2), the outcome is: $\Psi = \frac{1}{5}$, $\theta_1 = 0.017777$ and

$$\theta_2 = \frac{2^p \Upsilon_2^p}{(\eta_2 + \Upsilon_2)^p}\Big[\frac{\Psi^p (B-\Omega)^{(2p-1)}}{(\frac{(2p-1)}{p-1})^{p-1}} + p_1 (B-\Omega)^p\Big] = 0.0049777.$$

Then $\Lambda_2 = 0.5523833 < 1$. Then the boundary value problem (4.3) has a fixed point. To analyze the behavior of the operator Fw, for the example (4.4), when $\delta = \frac{5}{2}$ and $|h(s, w(s))| \le 0.4$, then see (Figure 2).

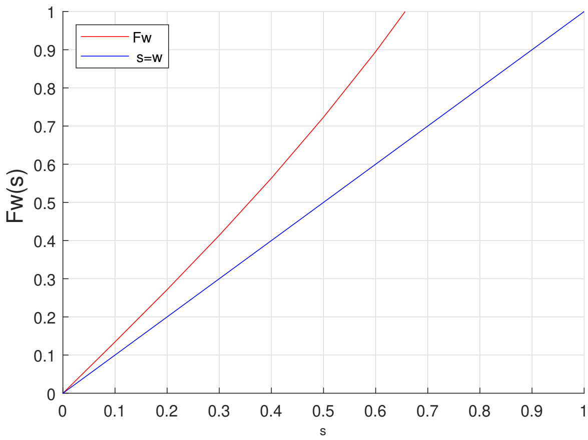

Figure 2. The behavior of the operator Fw for $s \in [0, 1]$.

5. Existence theorem for fractional integro-differential equation

The existence of solution is presented for the fractional integro-differential equation which has form

$$D^{\delta} w(s) = h(s, w) + \frac{1}{\Gamma(\gamma)} \int_{\Omega}^{s} (s-\tau)^{\gamma-1} z(s, \tau, w(\tau)) d\tau, \tag{5.1}$$

with boundary condition

$$\eta_r D^{(\delta-r)} w(\Omega) + \Upsilon_r D^{(\delta-r)} w(B) = \Delta_r, \quad (r = 1, 2, 3, ..., n). \tag{5.2}$$

The following assumptions is to be used throughout this section:

(H_3) $z : I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H_4) $z(s, \tau, w)$ satisfy Lipschitz condition then there exists a constant $\Phi > 0$ such that

$$|z(s, \tau, w_1(\tau)) - z(s, \tau, w_2(\tau))| \le \Phi |w_1 - w_2|$$

for each $w_1, w_2 \in \mathbb{R}$.

For convenience, we define the following notation:

$$\Lambda_3 = \Big[\Lambda_1 + \frac{2^{2p}\Phi^p}{p(\delta+\gamma)(\frac{p(\delta+\gamma)-1}{p-1})^{p-1}} \frac{(B-\Omega)^{p(\delta+\gamma)}}{(\Gamma(\delta+\gamma))^p}\Big]^{\frac{1}{p}}.$$

Lemma 5.1 (*cf.* [3])**.** *For any $w \in C(I)$, and $n-1 < \delta \leq n$, then the boundary value problem* (5.1)-(5.2) *has a solution*

$$w(s) = {}_{\Omega}^{s}I^{\delta}h(s,w) + {}_{a}^{s}I^{(\delta+\gamma)}z(s,\tau,w) + \sum_{i=1}^{n}\frac{\kappa_i}{\Gamma(\delta-i+1)}(s-\Omega)^{\delta-i},$$

where

$$\kappa_i = \frac{1}{(\eta_i+\Upsilon_i)}\Big(\Delta_i - \Upsilon_i\sum_{m=1}^{i-1}\frac{\kappa_m(B-\Omega)^{i-m}}{\Gamma(i-m+1)} - \Upsilon_i\,{}_{\Omega}^{B}I^{i}\,h(s,w) - \Upsilon_i\,{}_{\Omega}^{B}I^{(i+\gamma)}\,z(s,\tau,w)\Big).$$

Theorem 5.2. *Assume that* (H_1)*-*(H_4) *hold. If* $\Lambda_3 < 1$*. Then the boundary value problem* (5.1)-(5.2) *has a unique solution.*

Proof. Define the operator F on $L_p[\Omega, B]$ by:

$$Fw(s) = \sum_{i=1}^{n}\frac{\kappa_i}{\Gamma(\delta-i+1)}(s-\Omega)^{\delta-i} + {}_{\Omega}^{s}I^{\delta}f(s,y) + {}_{a}^{s}I^{(\delta+\gamma)}z(s,\tau,w).$$

$$\kappa_i = \frac{1}{(\eta_i+\Upsilon_i)}\Big(\Delta_i - \Upsilon_i\sum_{m=1}^{i-1}\frac{\kappa_m(B-\Omega)^{i-m}}{\Gamma(i-m+1)} - \Upsilon_i\,{}_{\Omega}^{B}I^{i}\,h(s,w) - \Upsilon_i\,{}_{\Omega}^{B}I^{(i+\gamma)}\,z(s,\tau,w)\Big).$$

Observe that continuity of Fw follows from the continuity of the first term and ${}_{\Omega}^{s}I^{\delta}h(s,w)$ with ${}_{a}^{s}I^{(\delta+\gamma)}z(s,\tau,w)$. Hence, the operator Fw is continuous and it is measurable.
Firstly, we prove that Fw is Lebesgue integrable, as following equation:

$$\Big|Fw(s)\Big|^p \leq 2^p\Big|\sum_{i=1}^{n}\frac{\kappa_i(s-\Omega)^{\delta-i}}{\Gamma(\delta-i+1)}\Big|^p + 2^{2p}\Big|\frac{1}{\Gamma(\delta)}\int_{\Omega}^{s}(s-\tau)^{\delta-1}h(\tau,w(\tau))d\tau\Big|^p$$
$$+2^{2p}\Big|\frac{1}{\Gamma(\delta+\gamma)}\int_{\Omega}^{s}(s-\tau)^{\delta+\gamma-1}z(s,\tau,w(\tau))d\tau\Big|^p. \tag{5.3}$$

The first and second terms of the equation (5.3) are Lebesgue integrable by Theorem 3.2, then it is a enough to show that the $\Big|\int_{\Omega}^{s}(s-\tau)^{\delta+\gamma-1}z(s,\tau,w(\tau))d\tau\Big|^p$ is Lebesgue integrable, using Hölder's inequality yields

$$\Big[\int_{\Omega}^{s}(s-\tau)^{\delta+\gamma-1}z(s,\tau,w(\tau))d\tau\Big]^p \leq \frac{(s-\Omega)^{p(\delta+\gamma)-1}}{[\frac{p(\delta+\gamma)-1}{p-1}]^{(p-1)}}\int_{\Omega}^{s}(z(s,\tau,w(\tau)))^p d\tau,$$

by Definition 2.1, and Lemma 2.4-(1) the following is obtained:

$${}_{\Omega}^{s}I\,{}_{\Omega}^{\tau}I(z(s,\tau,w(\tau)))^p = {}_{\Omega}^{s}I^{2}(z(s,\tau,w(\tau)))^p.$$

It is clear that $\int_{\Omega}^{s}z^p$ is Lebesgue integrable for all $s \in [\Omega, B]$ by Lemma 2.5. So that, $|Fw(s)|^p$ is Lebesgue integrable. Then, $Fw : L_p[\Omega, B] \to L_p[\Omega, B]$.
Next, we show that the operator F is a contraction. Let $w_1, w_2 \in L_p[\Omega, B]$. Then

$$||Fw_1(s) - Fw_2(s)||_p^p \leq 2^p\int_{\Omega}^{B}\Big(2^{(n-1)p}\frac{|\kappa_n-\zeta_n|^p(s-\Omega)^{p(\delta-n)}}{(\Gamma(\delta-n+1))^p} + \sum_{i=1}^{n-1}\frac{2^{ip}|\kappa_i-\zeta_i|^p(s-\Omega)^{p(\delta-i)}}{(\Gamma(\delta-i+1))^p}\Big)ds$$
$$+2^{2p}\int_{\Omega}^{B}\Big[\frac{1}{\Gamma(\delta)}\int_{\Omega}^{s}(s-\tau)^{\delta-1}\Big|h(\tau,w_1(\tau)) - h(\tau,w_2(\tau))\Big|d\tau\Big]^p ds$$
$$+2^{2p}\int_{\Omega}^{B}\Big(\int_{\Omega}^{s}\frac{(s-\tau)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)}\Big|z(s,\tau,w_1(\tau)) - z(s,\tau,w_2(\tau))\Big|d\tau\Big)^p ds, \tag{5.4}$$

where

$$|\kappa_i-\zeta_i|^p \le \Big(\frac{\Upsilon_i}{\eta_i+\Upsilon_i}\Big)^p \Big({}_{\Omega}^{B}I^i|h(s,w_1)-h(s,w_2)| + \sum_{m=1}^{i-1}\frac{|\kappa_m-\zeta_m|(B-\Omega)^{i-m}}{\Gamma(i-m+1)} + {}_{\Omega}^{B}I^{(i+\gamma)}\Big|z(s,\tau,w_1)-z(s,\tau,w_2)\Big|\Big)^p. \quad (5.5)$$

To find the value of $|\kappa_i-\zeta_i|^p$, we use the same steps of proof in Theorem 3.2 for $(i=1,2,...,n)$.
In general when $i=n$, the equation (5.5) becomes:

$$|\kappa_n-\zeta_n|^p \le \mu_n \int_{\Omega}^{B} |w_1(\tau)-w_2(\tau)|^p d\tau,$$

where

$$\mu_n = \theta_n + \frac{2^{2p}\Phi^p\Upsilon_n^p}{(\eta_n+\Upsilon_n)^p\,(\Gamma(n+\gamma))^p}\frac{(B-\Omega)^{p(n+\gamma)-1}}{(\frac{p(n+\gamma)-1}{p-1})^{p-1}}.$$

By using the Hölder inequality and the conditions (H_4), (H_2), the equation (5.4) becomes:

$$\|Fw_1(s)-Fw_2(s)\|_p \le \Big[\frac{2^{np}\mu_n}{(\Gamma(\delta-n+1))^p}\int_{\Omega}^{B}(s-\Omega)^{p(\delta-n)}ds + \sum_{i=1}^{n-1}\frac{2^{(i+1)p}\mu_i}{(\Gamma(\delta-i+1))^p}\int_{\Omega}^{B}(s-\Omega)^{p(\delta-i)}ds$$
$$+\frac{2^{2p}\Psi^p\,(B-\Omega)^{(p\delta)}}{(\frac{(p\delta-1)}{p-1})^{p-1}(\Gamma(\delta))^p\,p\delta} + \frac{2^{2p}\Phi^p\,(B-\Omega)^{p(\delta+\gamma)}}{(\frac{p(\delta+\gamma)-1)}{p-1})^{p-1}(\Gamma(\delta+\gamma))^p\,p(\delta+\gamma)}\Big]\|w_1-w_2\|_p^p,$$

and so

$$\|Fw_1(s)-Fw_2(s)\|_p \le \Lambda_3\|w_1-w_2\|_p,$$

therefore, F is a contraction , if $\Lambda_3<1$. Hence, by the Banach contraction principle, the boundary value problem (5.1)-(5.2) has a unique solution. □

6. Examples

In this section, we present some examples to check our results.

Example 6.1. Consider the following boundary value problem:

$$\begin{cases} D^{\delta}w(s) = \dfrac{e^{-2s}+1}{(s+2e)^4}w(s) + \int_{\Omega}^{s}\frac{8}{(s+5)^3}\frac{w(\tau)}{1+w(\tau)}d\tau, \quad 0<\delta\le 1 \\ \eta_1 w^{(\delta-1)}(\Omega) + \Upsilon_1 w^{(\delta-1)}(B) = 1. \end{cases} \quad (6.1)$$

Let $\Omega=0,\ B=2,\ p=2,\ \delta=\frac{1}{4}, \gamma=1\ \Upsilon_1=1, \eta_1=3.$
By using (H_2) and (H_4) , the result is: $\Psi=0.0022894,\ \ \Phi=0.064,\ \ \mu_1=0.01093314$. Since $p=2$ then the space is $L^2[0,2]$. Thus, $\Lambda_3=0.3399867<1$. Then, the boundary value problem (6.1) has a fixed point.

Example 6.2. Consider the following boundary value problem:

$$\begin{cases} D^{\delta}w(s) = \dfrac{e^{-2s}+1}{(s+2e)^4}w(s) + \int_{\Omega}^{s}\frac{5}{(s+1)^3}\frac{w(\tau)}{1+w(\tau)}d\tau, \quad 1<\delta\le 2 \\ \eta_1 w^{(\delta-1)}(\Omega) + \Upsilon_1 w^{(\delta-1)}(B) = 1, \qquad \eta_2 w^{(\delta-2)}(\Omega) + \Upsilon_2 w^{(\delta-2)}(B) = 1. \end{cases} \quad (6.2)$$

Let $\Omega=1,\ B=1.5,\ p=2,\ \delta=\frac{5}{4},\ \gamma=1,\ \ \Upsilon_1=\eta_1=1, \Upsilon_2=1, \eta_2=3$. By using (H_2) and (H_4), the result is: $\Psi=0.0006614665,\ \Phi=0.6250,\ \mu_1=0.065105041742528,\ \ \mu_2=0.004679417810830$ and $\Lambda_3=0.531735694339127<1$. Then, the boundary value problem (6.2) has a unique solution.

Acknowledgment

The authors are grateful to the reviewer for a careful reading of the manuscript.

Author Contributions: This paper is a part from M.Sc. Thesis, entitled "Some Existence Results for Certain Fractional Differential and Integro-Differential Equations with Analytical Approximate Solutions". University of Duhok, January 2020.

Conflict of interest: The authors declare no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] M. I. Abbas, *Existence and uniqueness of solution for a boundary value problem of fractional order involving two Caputo's fractional derivatives*, Adv. Differ. Equ. **2015 (252)**, 1–19, 2015.

[2] J. Gh. Abdulahad and Sh. A. Murad, *Local existence theorem of fractional differential equations in L^p space*, Raf. J. of Comp. & Math's. **9 (2)**, 71–78, 2012.

[3] D. N. Abdulqader and Sh. A. Murad, *Existence and uniqueness results for certain fractional boundary value problems*, Journal of Duhok University **22 (2)**, 76–88, 2019.

[4] A. Aghajani, Y. Jalilian and J. J. Trujillo, *On the existence of solutions of fractional integro-differential equations*, Fract. Calc. Appl. Anal. **15 (1)**, 44–69, 2012.

[5] R. P. Agrwal, A. Asma, V. Lupulescu and D. O'Regan, *L^p-solutions for a class of fractional integral equations*, J. Integral Equations Appl. **29 (2)**, 251–270, 2017.

[6] H. Ahmed, H. Boulares, A. Ardjouni and A. Chaoui, *On the study of fractional differential equations in a weighted sobolevs pace*, Bull. Int. Math. Virtual Inst. **9**, 333–343, 2019.

[7] S. Arshad, V. Lupulescu and D. O'Regan, *L^p-solutions for fractional integral equations*, Fract. Calc. Appl. Anal. **17 (1)**, 259–276, 2014.

[8] T. A. Barton and I. K. Purnaras, *L^p-solutions of singular integro-differential equations*, J. Math. Anal. Appl. **386**, 830–841, 2012 .

[9] T. A. Barton and B. Zhang, *L^p-solutions of fractional differential equations*, Nonlinear Stud. **19 (2)**, 161–177, 2012.

[10] L. Ibnelazyz, K. Guida, S. Melliani and K. Hilal, *On a nonlocal multi point and integral boundary value problem of nonlinear fractional integrodifferential equations*, J. Funct. Spaces. **2020**, 1–8, 2020.

[11] A. Karoui and A. Jawahdou. *Existence and approximate L^p and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types*, Appl. Math. Comput. **216 (7)**, 2077–2091, 2010.

[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V, Amsterdam, 2006.

[13] Sh. A. Murad and Z. A. Ameen, *Existence and Ulam stability for fractional differential equations of mixed Caputo-Riemann derivatives*, AIMS Math. **7 (4)**, 6404–6419, 2022.

[14] Sh. A. Murad and S. Hadid, *Existence and uniqueness theorem for fractional differential equation with integral boundary condition*, J. Frac. Calc. Appl. **3 (6)**, 1–9, 2012.

[15] Sh. A. Murad and A. S. Rafeeq, *Existence of solutions of integro-fractional differential equation when $\alpha \in (2,3]$ through fixed point theorem*, J. Math. Comput. Sci. **11 (5)**, 6392–6402, 2021.

[16] Sh. A. Murad, H. J. Zekri and S. Hadid, *Existence and uniqueness theorem of fractional mixed Volterra-Fredholm integrodifferential equation with integral boundary conditions*, Int. J. Differ. Equ. **2011**, 1–15, 2011.

[17] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York, London, 1974.

[18] I. Podlubny, *Fractional differential equation*, Mathematics in Science and Egineering, Academic Press, San Diego, 1999.