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Berezin radius and Cauchy-Schwarz inequality

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Abstract

In this manuscript, some refinements of the Cauchy-Schwarz inequality for contraction operators on the reproducing kernel Hilbert space are given in terms of the Berezin transform. We show several additional inequalities for the Berezin norm and Berezin radius of operators using these refinements.

Keywords: Cauchy-Schwarz inequality, Berezin transform, Berezin radius

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1. Introduction

In mathematical analysis, inequalities are used to analyze the characteristics of operators in the form of lower and upper limits. In practically all domains of science and engineering, mathematical inequalities are the greatest means to describe and suggest solutions to real-world issues. The boundedness attribute of many types of operators examined in analysis courses, including mathematical and functional analysis, is a critical aspect in creating theory and applications. Lower and upper limits, for example, are used to establish the operator norm, which is important in addressing related issues. Many researchers in mathematics and mathematical physics are interested in the Berezin symbol of an operator defined on the reproducing kernel Hilbert space. In this context, several mathematicians have conducted substantial research on the Berezin radius inequality given in (1.1) (see [8, 9]). In fact, it is of interest to academics to get refinements and extensions of this disparity [11, 12]. The purpose of this research is to investigate certain refinements of the Cauchy-Schwarz inequality using the Berezin transform for contraction operators on the reproducing kernel Hilbert space. Furthermore, we used the previously described refinements to show several additional inequalities for the Berezin norm and Berezin radius of operators. Related results are contained in [15]. We will now outline the preliminary concepts needed to proceed with the findings of this investigation.

Throughout this research, we will be focusing on reproducing kernel Hilbert space (RKHS). These are complete inner-product spaces made up of complex-valued functions defined on a set Ω with limited point evaluation. In a formal sense, let Ω be a subset of a topological space with a nonempty boundary $\partial\Omega$. Let $\mathcal{H}$ represent an infinite-dimensional Hilbert space of complex-valued functions defined on Ω. If the following two requirements are met, we say that $\mathcal{H}$ is an RKHS:

(a) the evaluation functionals $f \to f(\xi)$ are continuous on $\mathcal{H}$ for any $\xi \in \Omega$;

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(b) for each $\xi \in \Omega$, there exists $f_\xi \in \mathcal{H}$ such that $f_\xi(\xi) \neq 0$ (alternatively, there is no $\xi_0 \in \Omega$ such that $f(\xi_0) = 0$ for each $f \in \mathcal{H}$).

The assumption (a) implies, according to the basic Riesz representation theorem, that for any $\xi \in \Omega$ there exists a unique function $k_\xi \in \mathcal{H}$ such that

$$f(\xi) = \left\langle f, k_\xi \right\rangle_{\mathcal{H}}, \ f \in \mathcal{H}.$$

The reproducing kernel of $\mathcal{H}$ at point ξ is the function $k : \Omega \times \Omega \to \mathbb{C}$ defined by $k_\xi(z)$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, the Bergman space $L_a^2(\mathbb{D})$, the Dirichlet space $D^2(\mathbb{D})$ and the Fock space $F(\mathbb{C})$. Aronzajn [1], for example, provides a comprehensive treatment of the theory of RKHSs and reproducing kernels. Every reproducing kernel Hilbert space is separable, as is widely known. So, if $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$, it can be demonstrated that

$$k_\xi(z) = \sum_{n=0}^{\infty} \overline{e_n(\xi)} e_n(z).$$

We have $k_\xi \neq 0$ because of assumption (b), and we designate the normalized reproducing kernel by $\widehat{k_\xi}$, that is to say $\widehat{k_\xi} := \frac{k_\xi}{\|k_\xi\|_{\mathcal{H}}}$.

Linear operators induced by functions are frequently encountered in functional analysis; they include Hankel operators, composition operators, and Toeplitz operators. The inducing function is sometimes referred to as the symbol of the resultant operator. In many circumstances, a linear operator on a Hilbert space $\mathcal{H}$ also gives rise to a function on Ω. Hence, we frequently examine operators induced by functions, and we may similarly research functions induced by operators. The Berezin symbol is an outstanding exemplar of an operator-function link. More accurately, if $\mathcal{B}(\mathcal{H})$ is the Banach algebra of all linear bounded operator on $\mathcal{H}$, then the Berezin transform (symbol) $\widetilde{X}$ of any operator $X \in \mathcal{B}(\mathcal{H})$ is the complex-valued function defined on the Ω by the formula

$$\widetilde{X}(\xi) := \left\langle X\widehat{k_\xi}, \widehat{k_\xi} \right\rangle_{\mathcal{H}}, \ \xi \in \Omega.$$

F. Berezin proposed the Berezin transform in [5] and it has proven to be a fundamental tool in operator theory, since many essential features of significant operators are contained in their Berezin transforms. The Berezin range (set) of operator X is defined by

$$\mathrm{Ber}(X) = \left\{ \left\langle X\widehat{k_\xi}, \widehat{k_\xi} \right\rangle : \xi \in \Omega \right\} = \mathrm{Range}\left(\widetilde{X}\right),$$

and Berezin radius (number) $\mathrm{ber}(X)$ of operator X is the following number (see [13])

$$\mathrm{ber}(X) := \sup_{\xi \in \Omega} \left| \widetilde{X}(\xi) \right|.$$

The Berezin radius of an operator X meets the following qualities:

1) $\mathrm{ber}(\alpha X) = |\alpha| \, \mathrm{ber}(X)$, for all $\alpha \in \mathbb{C}$;

2) $\mathrm{ber}(X + Y) \leq \mathrm{ber}(X) + \mathrm{ber}(Y)$ for all $X, Y \in \mathcal{B}(\mathcal{H})$.

Since, $\left| \widetilde{X}(\xi) \right| \leq \|X\|$, Berezin transform is a bounded function on Ω. Also, it is trivial by Cauchy-Schwarz inequality that $\mathrm{ber}(X) \leq \|X\|$. The Berezin norm $\|X\|_{\mathrm{Ber}}$ of operator $X \in \mathcal{B}(\mathcal{H}(\Omega))$ is defined by

$$\|X\|_{\mathrm{Ber}} := \sup_{\xi \in \Omega} \left\| X\widehat{k_\xi} \right\|_{\mathcal{H}}.$$

It is simple to demonstrate that $\|X\|_{\mathrm{Ber}}$ determines a new norm in the algebra $\mathcal{B}(\mathcal{H}(\Omega))$, since the set of reproducing kernels span $\mathcal{H}$. It is also known that the Berezin norm $\|X\|_{\mathrm{Ber}}$ is not equivalent to the usual operator norm $\|X\|$, while the inequality $\|X\|_{\mathrm{Ber}} \leq \|X\|$ is trivial; also it is apparent that $\mathrm{ber}(X) \leq \|X\|_{\mathrm{Ber}}$.

On the other hand, the numerical range and numerical radius of X are connected to the the Berezin range and the Berezin radius of an operator X. Remember that

$$W(X) := \{\langle Xx, x \rangle : x \in \mathcal{H}, \ \|x\| = 1\},$$

$$w(X) := \sup\{|\langle Xx, x\rangle| : \langle Xx, x\rangle \in W(X)\},$$

and

$$\|X\| = \sup_{\|x\|=1} \|Xx\|$$

determine the numerical range, numerical radius, and operator norm, respectively. Clearly, $\mathrm{Ber}(X) \subset W(X)$ and $\mathrm{ber}(X) \le w(X)$. Furthermore, it is commonly understood that

$$\frac{1}{2}\|X\| \le w(X) \le \|X\|$$

and

$$\mathrm{ber}(X) \le w(X) \le \|X\| \tag{1.1}$$

for every $X \in \mathcal{B}(\mathcal{H}(\Omega))$. The numerical range of an operator has some interesting properties. For example, it is commonly understood that an operator's spectrum is confined inside the closure of its numerical range. We refer to [7, 14] for more recent studies on numerical radius inequalities and their applications.

2. Main results

In this part, we provide some new inequalities including refinements of the Berezin radius inequalities for operators working on the RKHS, utilizing the bounded function $\widetilde{X}$. Refinements of the well-known Cauchy-Schwarz inequality using contraction operators will lead to some applications such as Berezin radius and operator norm inequalities.

If $\langle Xx, x\rangle \ge 0$ for every $x \in \mathcal{H}$, an operator X is called to be positive in $\mathcal{B}(\mathcal{H})$ and is denoted as $X \ge 0$. A contraction is an operator $X \in \mathcal{B}(\mathcal{H})$ if X is positive and $X \le I$, where I is the identity operator on $\mathcal{H}$. For $X \in \mathcal{B}(\mathcal{H})$, we designate with $|X|$ the absolute value operator of X, that is, $|X| = (X^*X)^{\frac{1}{2}}$, where X^* is the adjoint operator of X. So, $|X^*| = (XX^*)^{\frac{1}{2}}$.

In [15], the following Cauchy-Schwarz type inequality for contractions is demonstrated.

Lemma 2.1. *Suppose that $X \in \mathcal{B}(\mathcal{H})$ is a contraction. Then we have*

$$|\langle Xx, y\rangle| \le \frac{\|x\|\|y\| + |\langle x, y\rangle|}{2}$$

for $x, y \in \mathcal{H}$.

The polar decomposition $X = U|X|$ demonstrates that the inequality

$$\begin{aligned} |\langle Xx, y\rangle| &\le \frac{\|X\|}{2}\left(|\langle Ux, y\rangle| + \|x\|\|U^*y\|\right) \\ &\le \frac{\|X\|}{2}\left(|\langle Ux, y\rangle| + \|x\|\|y\|\right) \end{aligned}$$

holds for $x, y \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$ with polar decomposition $X = U|X|$. We can see that U is a partial isometry, and so $\|U\| = \|U^*\| \le 1$.

In the aforementioned inequality, we have

$$\sup_{\xi\in\Omega}\left|\left(\widetilde{XY}\right)(\xi)\right| \le \frac{\|Y\|_{\mathrm{ber}}}{2}\left(\sup_{\xi\in\Omega}\left|\left\langle X\widehat{k}_{\xi}, \widehat{k}_{\xi}\right\rangle\right| + \sup_{\xi\in\Omega}\left\|X\widehat{k}_{\xi}\right\|\left\|\widehat{k}_{\xi}\right\|\right)$$

for $X, Y \in \mathcal{B}(\mathcal{H})$, and $\xi \in \Omega$ when Y is a positive operator and from the inequalities $\frac{1}{2}\|X\| \le \mathrm{ber}(X)$ and $\|X\|_{\mathrm{Ber}} \le \|X\|$,

$$\begin{aligned} \mathrm{ber}(XY) &\le \frac{\|Y\|}{2}\left(\mathrm{ber}(X) + \|X\|_{\mathrm{Ber}}\right) \\ &\le \frac{3}{2}\mathrm{ber}(X)\|Y\|. \end{aligned}$$

Specifically, if Y is a contraction, then we have

$$\operatorname{ber}(XY) \leq \frac{3}{2}\operatorname{ber}(X).$$

We may get the following conclusion from the aforementioned cases:

Corollary 2.2. *Assume $X, Y \in \mathcal{B}(\mathcal{H})$ and that Y is positive. Then we have*

$$\operatorname{ber}(XY) \leq \frac{3}{2}\|Y\|\operatorname{ber}(X).$$

By applying Lemma 2.1, we also prove the following theorem which provides two new inequalities involving the Berezin norm and Berezin radius of operators.

Theorem 2.3. *Assume $X, Y, Z \in \mathcal{B}(\mathcal{H})$ and that X is a contraction. Then we have*

$$\operatorname{ber}(YXZ) \leq \frac{1}{4}\left\||Z|^2 + |Y^*|^2\right\|_{\operatorname{ber}} + \frac{1}{2}\operatorname{ber}(YZ) \tag{2.1}$$

and

$$\operatorname{ber}(YXZ) \leq \frac{1}{2}\|Z\|_{\operatorname{Ber}}\|Y\|_{\operatorname{Ber}} + \frac{1}{2}\operatorname{ber}(YZ). \tag{2.2}$$

Proof. In Lemma 2.1, we have

$$\left|\left\langle YXZ\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right| \leq \frac{\left\|Z\widehat{k_\xi}\right\|\left\|Y^*\widehat{k_\xi}\right\| + \left|\left\langle Z\widehat{k_\xi}, Y^*\widehat{k_\xi}\right\rangle\right|}{2}$$

by replacing x with $Z\widehat{k_\xi}$ and y with $Y\widehat{k_\xi}$. Thus, by the AM-GM inequality, we obtain

$$\begin{aligned}
\left|\left\langle YXZ\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right| &\leq \frac{\left\|Z\widehat{k_\xi}\right\|\left\|Y^*\widehat{k_\xi}\right\| + \left|\left\langle YZ\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|}{2} \\
&= \frac{\sqrt{\left\langle |Z|^2\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\left\langle |Y^*|^2\widehat{k_\xi}, \widehat{k_\xi}\right\rangle} + \left|\left\langle YZ\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|}{2} \\
&\leq \frac{1}{2}\left(\frac{1}{2}\left(\left\langle |Z|^2\widehat{k_\xi}, \widehat{k_\xi}\right\rangle + \left\langle |Y^*|^2\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right) + \frac{1}{2}\left|\left\langle YZ\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|\right) \\
&= \frac{1}{4}\left\langle\left(|Z|^2 + |Y^*|^2\right)\widehat{k_\xi}, \widehat{k_\xi}\right\rangle + \frac{1}{2}\left|\left\langle YZ\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|.
\end{aligned}$$

Therefore, we have

$$\sup_{\xi\in\Omega}\left|\left(\widetilde{YXZ}\right)(\xi)\right| \leq \frac{1}{4}\sup_{\xi\in\Omega}\left\langle\left(|Z|^2 + |Y^*|^2\right)\widehat{k_\xi}, \widehat{k_\xi}\right\rangle + \frac{1}{2}\sup_{\xi\in\Omega}\left|\left(\widetilde{YZ}\right)(\xi)\right|$$

and

$$\operatorname{ber}(YXZ) \leq \frac{1}{4}\left\||Z|^2 + |Y^*|^2\right\|_{\operatorname{ber}} + \frac{1}{2}\operatorname{ber}(YZ).$$

So, we get the inequality (2.1).

Now, letting $x = Z\widehat{k_\xi}$ and $y = Y^*\widehat{k_\mu}$ in Lemma 2.1, then we get

$$\begin{aligned}
\left|\left\langle YXZ\widehat{k_\xi}, \widehat{k_\mu}\right\rangle\right| &\leq \frac{\left\|Z\widehat{k_\xi}\right\|\left\|Y^*\widehat{k_\mu}\right\| + \left|\left\langle Z\widehat{k_\xi}, Y^*\widehat{k_\mu}\right\rangle\right|}{2} \\
&= \frac{\left\|Z\widehat{k_\xi}\right\|\left\|Y^*\widehat{k_\mu}\right\| + \left|\left\langle YZ\widehat{k_\mu}, \widehat{k_\mu}\right\rangle\right|}{2}.
\end{aligned}$$

By taking supremum over $\xi \in \Omega$ with $\xi = \mu$, we deduce

$$\sup_{\xi\in\Omega}\left|\left\langle YXZ\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| \leq \frac{1}{2}\sup_{\xi\in\Omega}\left\|Z\widehat{k}_\xi\right\|\left\|Y^*\widehat{k}_\xi\right\| + \frac{1}{2}\sup_{\xi\in\Omega}\left|\left\langle YZ\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|$$

and

$$\mathrm{ber}(YXZ) \leq \frac{1}{2}\|Z\|_{\mathrm{Ber}}\|Y\|_{\mathrm{Ber}} + \frac{1}{2}\mathrm{ber}(YZ),$$

which proves inequality (2.2). This completes the proof. □

We are interested in power inequalities while dealing with Berezin radius inequalities. As an example of such disparities, we recommend the reader to [3, 8, 12]. We utilize Theorem 2.3 to get a power inequality for the Berezin radius in subsequent.

Corollary 2.4. *Assume $X, Y, Z \in \mathcal{B}(\mathcal{H})$ are such that X is a contraction. Then we have*

$$\mathrm{ber}^r(YXZ) \leq \frac{1}{4}\left\||Z|^{2r} + |Y^*|^{2r}\right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}^r(YZ), \tag{2.3}$$

for $r \geq 1$.

Proof. This is demonstrated by Theorem 2.3 and the facts that $t \to t^r$, $r \geq 1$ is a convex increasing function on $[0, \infty)$ and that

$$\left\|f\left(\frac{X+Y}{2}\right)\right\|_{\mathrm{ber}} \leq \frac{1}{2}\|f(X) + f(Y)\|_{\mathrm{ber}},$$

for every increasing convex function $f : [0, \infty) \to [0, \infty)$ and positive operators X, Y (see [6, Corollary 2.2]). □

Theorem 2.3 is then used to create a refinement of the inequality $\mathrm{ber}(Z) \leq \|Z\|$.

Corollary 2.5. *Assume $Z \in \mathcal{B}(\mathcal{H})$ is a given operator with polar decomposition $Z = U|Z|$. Then we have*

$$\begin{aligned}
\mathrm{ber}(Z) &\leq \frac{1}{2}\left(\|Z\| + \|Z\|^{\frac{1}{2}}\,\mathrm{ber}\left(U|Z|^{\frac{1}{2}}\right)\right) \\
&\leq \frac{1}{2}\left(\|Z\| + \|Z\|^{\frac{1}{2}}\left\|U|Z|^{\frac{1}{2}}\right\|\right) \\
&\leq \frac{1}{2}\left(\|Z\| + \|Z\|^{\frac{1}{2}}\|U\|\,\|Z\|^{\frac{1}{2}}\right) \\
&\leq \|Z\|.
\end{aligned}$$

Proof. We demonstrate the first inequality, which leads to the another inequalities. $Z = U|Z|$ denotes Z's polar decomposition. Then, for every $\xi, \eta \in \Omega$, we have

$$\left|\left\langle Z\widehat{k}_\xi, \widehat{k}_\eta\right\rangle\right| = \left|\left\langle U|Z|\widehat{k}_\xi, \widehat{k}_\eta\right\rangle\right| = \left|\left\langle |Z|^{\frac{1}{2}}\widehat{k}_\xi, |Z|^{\frac{1}{2}}U^*\widehat{k}_\eta\right\rangle\right|. \tag{2.4}$$

Let X be any positive operator. Theorem 2.3 implies

$$\left|\left\langle X\widehat{k}_\xi, \widehat{k}_\eta\right\rangle\right| \leq \frac{\|X\|}{2}\left(\left|\left\langle \widehat{k}_\xi, \widehat{k}_\eta\right\rangle\right| + \left\|\widehat{k}_\xi\right\|\left\|\widehat{k}_\eta\right\|\right).$$

For $\xi = \eta$, this together with (2.4) imply

$$\begin{aligned}
\left|\left\langle Z\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| &= \left|\left\langle |Z|^{\frac{1}{2}}\widehat{k}_\xi, |Z|^{\frac{1}{2}}U^*\widehat{k}_\xi\right\rangle\right| \\
&\leq \frac{\left\||Z|^{\frac{1}{2}}\right\|}{2}\left(\left|\left\langle \widehat{k}_\xi, |Z|^{\frac{1}{2}}U^*\widehat{k}_\xi\right\rangle\right| + \left\|\widehat{k}_\xi\right\|\left\||Z|^{\frac{1}{2}}U^*\widehat{k}_\xi\right\|\right) \\
&= \frac{\|Z\|^{\frac{1}{2}}}{2}\left(\left|\left\langle U|Z|^{\frac{1}{2}}\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| + \left\|\widehat{k}_\xi\right\|^2\|Z\|^{\frac{1}{2}}\|U^*\|\right).
\end{aligned}$$

So,

$$\left|\left\langle Z\widetilde{k}_{\xi},\widetilde{k}_{\xi}\right\rangle\right| \le \frac{\|Z\|^{\frac{1}{2}}}{2}\left(\left|\left\langle U\,|Z|^{\frac{1}{2}}\,\widetilde{k}_{\xi},\widetilde{k}_{\xi}\right\rangle\right| + \left\|\widetilde{k}_{\xi}\right\|^{2}\|Z\|^{\frac{1}{2}}\,\|U^{*}\|\right).$$

$\|U^*\| = \|U\| = 1$ and taking the supremum over $\xi \in \Omega$, we obtain

$$\mathrm{ber}\,(Z) = \frac{\|Z\|^{\frac{1}{2}}}{2}\left(\mathrm{ber}\left(U\,|Z|^{\frac{1}{2}}\right) + \|Z\|^{\frac{1}{2}}\right).$$

As expected, this completes the demonstration of the first inequality. □

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [2, 4, 10].

3. Conclusion

This study contributes to the operator theory and mathematical inequality research fields in the following way. We investigated certain refinements of the Cauchy-Schwarz inequality using the Berezin symbol for contraction operators on the RKHS. Furthermore, we used the previously described refinements to show several additional inequalities for the Berezin norm and Berezin number of operators. It is generally known on many RKHSs that certain features of particular operators may be derived by studying solely the operator's action on the set of normalized reproducing kernels. One may ask if such results exist between the Berezin and numerical ranges. Is there any property of $W(X)$ that can be derived from $\mathrm{Ber}(X)$ given an operator X on an RKHS $\mathcal{H}$? Our future work will focus on this kind of question.

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