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Some new fractional Hilbert type integral inequalities

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Abstract

This paper introduced the new notion of generalized fractional Hilbert type integral operators. The norm inequalities for these operators are established. The corresponding fractional Hilbert type integral inequalities with the best possible constant factor are also provided. The new work extends some previous work on Hilbert inequalities and opens a new direction for further study in this active domain of research.

Keywords: Hilbert type integral inequality, fractional integral operator, norm

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1. Introduction

Fractional integral operator is one of the important operators in harmonic analysis with background of partial differential equations. Many mathematician have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations. Hence, fractional integral inequalities are very important in the theory and applications of differential equations. Such inequalities are also of great importance in the mathematical modeling of the fractional boundary value problems. (see [1, 4, 6, 10, 11, 13], [17]-[19], [22, 23, 25], [27]-[30] and the references given therein). Some researchers have found that different fractional integral operators with different singular or non-singular kernels need to be identified by real-word problems in different fields of engineering and science. First, we recall the following definitions and some related results.

Definition 1.1 (*cf.* [13, 17, 28])**.** Let $f \in L[a,b]$, then Riemann-Liouville fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$T_1(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \;\; x > a, \tag{1.1}$$

and

$$T_2(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \;\; x < b, \tag{1.2}$$

respectively, where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{1.3}$$

is the Gamma function and when $\alpha = 0, T_1(f,x) = T_2(f,x) = f(x)$.

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Definition 1.2 (*cf.* [22])**.** Let $f \in L[a,b]$, then Riemann-Liouville k-fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$T_3(f,x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t)dt, \ \ x > a, \tag{1.4}$$

and

$$T_4(f,x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{(\alpha/k)-1} f(t)dt, \ \ x < b, \tag{1.5}$$

respectively, where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-(t^k/k)} dt, \ \ (\alpha > 0) \tag{1.6}$$

is the k-Gamma function.

The Mellin transform of f is defined by

$$M(f,\alpha) = \int_0^\infty t^{\alpha-1} f(t)dt, \ \ (\alpha > 0).$$

Therefore, the Mellin transform of the exponential function $e^{-(t^k/k)}$ is the k-Gamma function.

Definition 1.3 (*cf.* [23, 27])**.** Let $t^r f(t) \in L^1[a,b], a \geq 0$, then the generalized Riemann-Liouville fractional integral of f of order $\alpha \geq 0$ and $r \geq 0$, are defined by

$$T_5(f,x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t)dt, \ \ x > a, \tag{1.7}$$

$$T_6(f,x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{\alpha-1} t^r f(t)dt, \ \ x < b, \tag{1.8}$$

and there are also (k,r) fractional of Riemann type operators:

$$T_7(f,x) = \frac{(r+1)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{(\alpha/k)-1} t^r f(t)dt, \ \ x > a, \tag{1.9}$$

$$T_8(f,x) = \frac{(r+1)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{(\alpha/k)-1} t^r f(t)dt, \ \ x < b, \tag{1.10}$$

respectively, where $k, \alpha > 0, r \geq 0, x \in [a,b]$.

In particular, if $r = 0$, then Definition 1.3 reduces to Definition 1.1 and Definition 1.2.

Definition 1.4 (*cf.* [11, 25])**.** Let f be a conformable integrable function on $[a,b] \subset [0,\infty)$. The left-sided and right-sided generalized conformable fractional integrals T_9 and T_{10} of f of order $\alpha > 0$ are defined by

$$T_9(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{x^{r+s} - t^{r+s}}{r+s} \right)^{\alpha-1} t^{r+s-1} f(t)dt, \ \ x > a, \tag{1.11}$$

and

$$T_{10}(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\frac{t^{r+s} - x^{r+s}}{r+s} \right)^{\alpha-1} t^{r+s-1} f(t)dt \ \ x < b, \tag{1.12}$$

respectively, where $r \in \mathbb{R}, s \in (0,1], r+s \neq 0$.

In particular, if $s = 1$, then T_9, T_{10} reduce to T_5, T_6, respectively.

Definition 1.5 (*cf.* [11, 30])**.** Let $f \in L[a,b]$, $g : [a,b] \to (0,\infty)$ be an increasing function, and $g' \in C[a,b]$, $\alpha > 0$. Then g-Riemann-Liouville fractional integrals of f with respect to the function g on $[a,b]$ are defined by

$$T_{11}(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x g'(t)[g(x) - g(t)]^{\alpha-1} f(t)dt, \quad x > a, \tag{1.13}$$

and

$$T_{12}(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b g'(t)[g(t) - g(x)]^{\alpha-1} f(t)dt, \quad x < b, \tag{1.14}$$

respectively.

In 2020, Kuang ([18]) introduced the new notion of the generalized fractional integral operators and fractional area balance operators:

Definition 1.6 (*cf.* [18])**.** Let $f \in L[a,b]$, $g : [a,b] \to (0,\infty)$ be an increasing function, and $g \in AC[a,b]$, $k, c, \alpha > 0$, $a \geq 0$, where $AC[a,b]$ denotes the class of absolutely continuous functions on $[a,b]$. Then the generalized fractional integral operator T_{13} with respect to the function g on $[a,b]$ is defined by

$$T_{13}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_a^b g'(t)|g(x) - g(t)|^{(\alpha/k)-1} f(t)dt, \tag{1.15}$$

where $\Gamma_k(\alpha)$ is defined by (1.6).

Let

$$T_{14}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_a^x g'(t)[g(x) - g(t)]^{(\alpha/k)-1} f(t)dt, \ \ x > a, \tag{1.16}$$

and

$$T_{15}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_x^b g'(t)[g(t) - g(x)]^{(\alpha/k)-1} f(t)dt, \ \ x < b. \tag{1.17}$$

Then

$$T_{13}(f,x) = T_{14}(f,x) + T_{15}(f,x). \tag{1.18}$$

For suitable and appropriate choice of the parameters and function, one can obtain various new and old results. Hence, Definition 1.6 unified and generalized many known and new classes of fractional integral operators. Various generalizations of classical inequalities by means of fractional integral operators is considered as an interesting subject area. For instance, one of the best-known inequalities, the Hermite-Hadamard inequality, that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.19}$$

which is a generalized by means of several fractional integral operators, is given (see [3], [7]-[9], [12, 20, 21, 24, 26, 31, 33] and the references given therein). Another famous inequality is the Hilbert inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1.20}$$

where $||f||_p = (\int_0^\infty |f(x)|^p dx)^{1/p}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, the constant $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The corresponding Hilbert operator

$$T_0(f,x) = \int_0^\infty \frac{f(y)}{x+y} dy \tag{1.21}$$

satisfying

$$||T_0 f||_p \leq ||T_0|| \cdot ||f||_p, \tag{1.22}$$

where

$$||T_0|| = \frac{\pi}{\sin(\pi/p)}. \tag{1.23}$$

The Hilbert inequality is one of the most interesting inequalities in analysis and applications. The Hilbert type inequalities have been generalized by many researches. In 2008 and 2012, the author [15, 16], obtained some new inequalities related to the generalized Hilbert operator

$$T_{16}(f,x) = \int_0^\infty K(x,y)f(y)dy \tag{1.24}$$

with the general kernel $K(x,y)$ on the Herz spaces and the weighted Morrey-Herz spaces, respectively (see [2, 5], [14]-[16], [32] and the references given therein). The aim of this paper is to introduce the new notion of generalized fractional Hilbert type integral operators and establish some new fractional Hilbert type integral inequalities. In Section 2, we define generalized fractional Hilbert type integral operators. In Section 3, some new fractional Hilbert type integral inequalities with the best possible constant factor are derived. The corresponding norm inequalities are established in Section 4. The new work extends some previous work on Hilbert inequalities and opens a new direction for further study in the active domain of research.

2. Generalized fractional Hilbert type integral operators

Definition 2.1. Let $f \in L(0,\infty)$, $K(x,y)$ is a non-negative measurable function on $(0,\infty)\times(0,\infty)$, $c > 0$, then the generalized fractional Hilbert type integrals operator T_{17} is defined by

$$T_{17}(f,x) = \frac{c}{k\Gamma_k(\alpha)}\int_0^\infty (K(x,y))^{(\alpha/k)-1} f(y)dy, \tag{2.1}$$

where $\Gamma_k(\alpha)$ is defined by (1.6).

If K satisfies the following conditions:

$$K(tx,y) = t^{-\lambda_1}K\left(x, t^{-(\frac{\lambda_1}{\lambda_2})}y\right); K(x,ty) = t^{-\lambda_2}K\left(xt^{-(\frac{\lambda_2}{\lambda_1})}, y\right),$$

where $t, \lambda_1, \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, then we call K is a non-homogeneous kernel. In particular, if $K(x,y) = \frac{1}{x^{\lambda_1}+y^{\lambda_2}}$, then

$$K(x,y) = x^{-\lambda_1}K\left(1, x^{-(\frac{\lambda_1}{\lambda_2})}y\right) = y^{-\lambda_2}K\left(xy^{-(\frac{\lambda_2}{\lambda_1})}, 1\right);$$

if $\lambda_1 = \lambda_2 = \lambda$, then $K(tx,ty) = t^{-\lambda}K(x,y)$, that is, K is a homogeneous kernel of degree $-\lambda$. If $K(x,y) = \frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}$, where $\lambda, \lambda_1, \lambda_2 > 0$, then

$$T(f,x) = \frac{c}{k\Gamma_k(\alpha)}\int_0^\infty \left\{\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right\}^{(\alpha/k)-1} f(y)dy, \tag{2.2}$$

is said to be the fractional Hilbert type integral operator with the non-singular kernel and

$$T_{18}(f,x) = \frac{c}{k\Gamma_k(\alpha)}\int_0^\infty \left\{\frac{1}{|x^{\lambda_1}-y^{\lambda_2}|^\lambda}\right\}^{(\alpha/k)-1} f(y)dy \tag{2.3}$$

is said to be the fractional Hilbert type integral operator with the singular kernel. If $c = k = 1, \alpha = 2$, then (2.1) reduces to (1.24). Hence, the generalized fractional Hilbert type integral operators are extensions of the generalized Hilbert type integral operators.

3. Main results

Theorem 3.1. *Let* $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda, \alpha, \lambda_k > 0$, $k = 1,2,3,4$, *and*

$$\omega_1(x) = x^{\frac{\lambda_1}{\lambda_2}(\lambda_3+1)-\lambda_1\lambda(\frac{\alpha}{k}-1)-\lambda_4(p-1)}, \tag{3.1}$$

and

$$\omega_2(x) = x^{\frac{\lambda_2}{\lambda_1}(\lambda_4+1)-\lambda_2\lambda(\frac{\alpha}{k}-1)-\lambda_3(q-1)}. \tag{3.2}$$

Let f and g are non-negative measurable functions on $(0,\infty)$ and $f \in L^p_{\omega_1}(0,\infty)$, $g \in L^q_{\omega_2}(0,\infty)$. If

$$c_1 = \frac{1}{\lambda_2}\int_0^\infty \left(\frac{1}{(1+u)^\lambda}\right)^{\frac{\alpha}{k}-1} u^{\frac{\lambda_3+1}{\lambda_2}-1}du < \infty, \tag{3.3}$$

and

$$c_2 = \frac{1}{\lambda_1}\int_0^\infty \left(\frac{1}{(1+u)^\lambda}\right)^{\frac{\alpha}{k}-1} u^{\frac{\lambda_4+1}{\lambda_1}-1}du < \infty, \tag{3.4}$$

then

$$\int_0^\infty\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1} f(x)g(y)dxdy \le c_1^{1/p}c_2^{1/q}\|f\|_{p,\omega_1}\|g\|_{q,\omega_2}. \tag{3.5}$$

Proof. By Hölder inequality, we obtain

$$\begin{aligned}
\int_0^\infty\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1} f(x)g(y)dxdy &= \int_0^\infty\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})}\right)^{\frac{\alpha}{k}-1}\left(\frac{y^{\lambda_3/p}}{x^{\lambda_4/q}}f(x)\right)\left(\frac{x^{\lambda_4/q}}{y^{\lambda_3/p}}g(y)\right)dxdy \\
&\le \left\{\int_0^\infty\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1}\left(\frac{y^{\lambda_3}}{x^{\lambda_4(p-1)}}f^p(x)\right)dxdy\right\}^{1/p} \\
&\times \left\{\int_0^\infty\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1}\left(\frac{x^{\lambda_4}}{y^{\lambda_3(q-1)}}g^q(y)\right)dxdy\right\}^{1/q} \\
&= \left\{\int_0^\infty\left[\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1}\left(\frac{y^{\lambda_3}}{x^{\lambda_4}(p-1)}\right)dy\right]f^p(x)dx\right\}^{1/p} \\
&\times \left\{\int_0^\infty\left[\int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1}\left(\frac{x^{\lambda_4}}{y^{\lambda_3(q-1)}}\right)dx\right]g^q(y)dy\right\}^{1/q} \\
&= \left\{\int_0^\infty I_1 f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty I_2 g^q(y)dy\right\}^{1/q},
\end{aligned} \tag{3.6}$$

where

$$I_1 = \int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1}\frac{y^{\lambda_3}}{x^{\lambda_4(p-1)}}dy, \tag{3.7}$$

and

$$I_2 = \int_0^\infty \left(\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right)^{\frac{\alpha}{k}-1}\frac{x^{\lambda_4}}{y^{\lambda_3(q-1)}}dx. \tag{3.8}$$

The change of variable $u = y^{\lambda_2}x^{-\lambda_1}$ and using (3.1) and (3.3), gives

$$\begin{aligned}
I_1 &= \left\{\int_0^\infty \frac{1}{(1+y^{\lambda_2}x^{-\lambda_1})^{\lambda((\alpha/k)-1)}}y^{\lambda_3}dy\right\}x^{-\lambda_1\lambda((\alpha/k)-1)-\lambda_4(p-1)} \\
&= \left\{\frac{1}{\lambda_2}\int_0^\infty \frac{1}{(1+u)^{\lambda((\alpha/k)-1)}}u^{\frac{\lambda_3+1}{\lambda_2}-1}du\right\}x^{\frac{\lambda_1}{\lambda_2}(\lambda_3+1)-\lambda_1\lambda(\frac{\alpha}{k}-1)-\lambda_4(p-1)} \\
&= c_1\omega_1(x).
\end{aligned} \tag{3.9}$$

Similarly, the change of variable $u = y^{-\lambda_2}x^{\lambda_1}$ and using (3.2) and (3.4), gives

$$I_2 = c_2\omega_2(y). \tag{3.10}$$

By combining (3.6), (3.9) and (3.10), we get the desired result. The proof is thus completed. □

Remark 3.2. Inequality (3.5) is a very general basic result, because it contains six parameters $\lambda, \alpha, \lambda_k > 0, k = 1, 2, 3, 4$, that are independent of each other. For suitable and appropriate choice of these parameters, one can obtain various new and old results.

For example,

Corollary 3.3. *Under the assumptions of Theorem* 3.1*, if*

$$\max\left\{\frac{\lambda_3+1}{\lambda_2}, \frac{\lambda_4+1}{\lambda_1}\right\} < \lambda\left(\frac{\alpha}{k}-1\right), \tag{3.11}$$

then

$$\begin{aligned}\int_0^\infty\int_0^\infty \left\{\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right\}^{\frac{\alpha}{k}-1} f(x)g(y)dxdy &\leq \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}\left\{B\left(\frac{\lambda_3+1}{\lambda_2}, \lambda\left(\frac{\alpha}{k}-1\right)-\frac{\lambda_3+1}{\lambda_2}\right)\right\}^{1/p} \\ &\times \left\{B\left(\frac{\lambda_4+1}{\lambda_1}, \lambda\left(\frac{\alpha}{k}-1\right)-\frac{\lambda_4+1}{\lambda_1}\right)\right\}^{1/q} \|f\|_{p,\omega_1}\|g\|_{q,\omega_2}. \end{aligned} \tag{3.12}$$

If taking $\lambda_1 = \lambda_2, \lambda_3 = \frac{\lambda\lambda_1}{p}\left(\frac{\alpha}{k}-1\right)-1$, $\lambda_4 = \frac{\lambda\lambda_1}{q}\left(\frac{\alpha}{k}-1\right)-1$ in Corollary 3.3, we derive the following Corollary 3.4:

Corollary 3.4. *Let* $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda, \alpha, \lambda_1 > 0$, *and*

$$\omega_1(x) = x^{(\lambda\lambda_1(1-\frac{\alpha}{k})+1)(p-1)}, \tag{3.13}$$

and

$$\omega_2(x) = x^{(\lambda\lambda_1(1-\frac{\alpha}{k})+1)(q-1)}. \tag{3.14}$$

Let f and g are non-negative measurable functions on $(0,\infty)$ *and* $f \in L^p_{\omega_1}(0,\infty)$, $g \in L^q_{\omega_2}(0,\infty)$, *then*

$$\int_0^\infty\int_0^\infty \left\{\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right\}^{\frac{\alpha}{k}-1} f(x)g(y)dxdy \leq \frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right)\|f\|_{p,\omega_1}\|g\|_{q,\omega_2}, \tag{3.15}$$

where the constant factor $\frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right)$ *is the best possible .*

Proof. Taking $\lambda_1 = \lambda_2, \lambda_3 = \frac{\lambda\lambda_1}{p}\left(\frac{\alpha}{k}-1\right)-1$, $\lambda_4 = \frac{\lambda\lambda_1}{q}\left(\frac{\alpha}{k}-1\right)-1$ in Corollary 3.3, we have

$$\omega_1 = x^{\frac{\lambda\lambda_1}{p}(\frac{\alpha}{k}-1)-\lambda\lambda_1(\frac{\alpha}{k}-1)-[\frac{\lambda\lambda_1}{q}(\frac{\alpha}{k}-1)-1](p-1)} = x^{(\lambda\lambda_1(1-\frac{\alpha}{k})+1)(p-1)},$$

and

$$\omega_2(x) = x^{\frac{\lambda\lambda_1}{q}(\frac{\alpha}{k}-1)-\lambda\lambda_1(\frac{\alpha}{k}-1)-[\frac{\lambda\lambda_1}{p}(\frac{\alpha}{k}-1)-1](q-1)} = x^{(\lambda\lambda_1(1-\frac{\alpha}{k})+1)(q-1)},$$

$$\begin{aligned} c_1 &= \frac{1}{\lambda_1}B\left(\frac{\lambda_3+1}{\lambda_2}, \lambda\left(\frac{\alpha}{k}-1\right)-\frac{\lambda_3+1}{\lambda_2}\right) \\ &= \frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \lambda\left(\frac{\alpha}{k}-1\right)-\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right)\right) \\ &= \frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right). \end{aligned}$$

Similarly, we have

$$c_2 = \frac{1}{\lambda_1}B\left(\frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right)\right) = \frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right).$$

By Corollary 3.3, we get the desired resuly (3.15). To prove the constant factor $\frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right),\frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right)$ is the best possible, setting f_ε and g_ε as follows:

$$f_\varepsilon(t)=\begin{cases}0, & 0<t\le 1,\\ t^{\frac{\lambda\lambda_1}{q}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{p}-1}, & 1<t<\infty,\end{cases}$$

and

$$g_\varepsilon(t)=\begin{cases}0, & 0<t\le 1,\\ t^{\frac{\lambda\lambda_1}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q}-1}, & 1<t<\infty.\end{cases}$$

We get

$$\|f_\varepsilon\|_{p,\omega_1}^p=\int_1^\infty t^{\frac{p\lambda\lambda_1}{q}(\frac{\alpha}{k}-1)-\varepsilon-p}t^{(\lambda\lambda_1(1-\frac{\alpha}{k})+1)(p-1)}dt=\int_1^\infty t^{-\varepsilon-1}dt=\frac{1}{\varepsilon},$$

Similarly, $\|g_\varepsilon\|_{q,\omega_2^q}=\frac{1}{\varepsilon}$. If there exists a positive constant M, $M\le\frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right),\frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right)$, such that (3.15) is valid. It follows that

$$\begin{aligned}I&=\int_0^\infty\int_0^\infty\left\{\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^\lambda}\right\}^{\frac{\alpha}{k}-1}f_\varepsilon(x)g_\varepsilon(y)dxdy\\&\le M\|f_\varepsilon\|_{p,\omega_1}\|g_\varepsilon\|_{q,\omega_2}=M\left(\frac{1}{\varepsilon}\right)^{1/p}\left(\frac{1}{\varepsilon}\right)^{1/q}=M\frac{1}{\varepsilon}.\end{aligned}\tag{3.16}$$

The change of variable $u=y^{\lambda_1}x^{-\lambda_1}$ gives

$$\begin{aligned}I&=\int_0^\infty\int_0^\infty\left\{\frac{1}{(x^{\lambda_1}+y^{\lambda_1})^\lambda}\right\}^{\frac{\alpha}{k}-1}f_\varepsilon(x)g_\varepsilon(y)dxdy\\&=\int_1^\infty\left\{\int_1^\infty\frac{1}{(x^{\lambda_1}+y^{\lambda_1})^{\lambda((\alpha/k)-1)}}y^{\frac{\lambda\lambda_1}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q}-1}dy\right\}x^{\frac{\lambda\lambda_1}{q}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{p}-1}dx\\&=\frac{1}{\lambda_1}\int_1^\infty x^{-\varepsilon-1}\left\{\int_{x^{-\lambda_1}}^\infty\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k})-1)}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\}dx\\&=\frac{1}{\lambda_1}\int_1^\infty x^{-\varepsilon-1}\left\{\int_{x^{-\lambda_1}}^1\frac{1}{(1+u)^{\lambda((\alpha/k)-1)}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\}dx\\&+\frac{1}{\lambda_1}\int_1^\infty x^{-\varepsilon-1}\left\{\int_1^\infty\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k}-1))}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\}dx=I_1+I_2,\end{aligned}\tag{3.17}$$

where

$$\begin{aligned}I_2&=\frac{1}{\lambda_1}\int_1^\infty x^{-\varepsilon-1}\left\{\int_1^\infty\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k}-1))}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\}dx\\&=\frac{1}{\lambda_1\varepsilon}\left\{\int_1^\infty\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k}-1))}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\}.\end{aligned}\tag{3.18}$$

Using the Fubini theorem, we get

$$\begin{aligned}I_1&=\frac{1}{\lambda_1}\int_1^\infty x^{-\varepsilon-1}\left\{\int_{x^{-\lambda_1}}^1\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k}-1))}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\}dx\\&=\frac{1}{\lambda_1}\int_0^1\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k}-1))}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}\left\{\int_{u^{-\frac{1}{\lambda_1}}}^\infty x^{-\varepsilon-1}dx\right\}du\\&=\frac{1}{\lambda_1\varepsilon}\int_0^1\frac{1}{(1+u)^{\lambda((\frac{\alpha}{k})-1)}}u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)+\frac{\varepsilon}{p\lambda_1}-1}du.\end{aligned}\tag{3.19}$$

Thus, by (3.16), (3.17), (3.18), and (3.19), we conclude that

$$\frac{1}{\lambda_1}\int_0^1 \frac{1}{(1+u)^{\lambda((\frac{\alpha}{k})-1)}} u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)+\frac{\varepsilon}{p\lambda_1}-1}du + \frac{1}{\lambda_1}\left\{\int_1^\infty \frac{1}{(1+u)^{\lambda((\frac{\alpha}{k})-1)}} u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-\frac{\varepsilon}{q\lambda_1}-1}du\right\} \le M. \tag{3.20}$$

By letting $\varepsilon \to 0^+$ in (3.20), and using the Fatou Lemma, we get

$$\frac{1}{\lambda_1}\left\{\int_0^\infty \frac{1}{(1+u)^{\lambda((\frac{\alpha}{k})-1)}} u^{\frac{\lambda}{p}(\frac{\alpha}{k}-1)-1}du\right\} = \frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right) \le M. \tag{3.21}$$

Hence, $M = \frac{1}{\lambda_1}B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right)$ is the best possible constant factor of (3.15). The Corollary 3.4 is proved. □

4. Norm inequalities for fractional Hilbert type operators

Theorem 4.1. *Under the assumptions of Theorem* 3.1*, let*

$$\omega(y) = y^{-[\lambda\lambda_1(1-\frac{\alpha}{k})+1]}. \tag{4.1}$$

Then integral operator T is defined by (2.2)*: $T : L^p_{\omega_1}(0,\infty) \to L^p_{\omega}(0,\infty)$ exists as a bounded operator and*

$$\|Tf\|_{p,\omega} \le c(p)\|f\|_{p,\omega_1}, \tag{4.2}$$

where

$$c(p) = \frac{c}{k\Gamma_k(\alpha)} c_1^{1/p} c_2^{1/q}, \tag{4.3}$$

and c_1, c_2, ω_1 and ω_2 are defined by (3.3)*,* (3.4)*,* (3.1) *and* (3.2)*, respectively. This implies that*

$$\|T\| = \sup_{f\neq 0}\frac{\|Tf\|_{p,\omega}}{\|f\|_{p,\omega_1}} \le \frac{c}{k\Gamma_k(\alpha)} c_1^{1/p} c_2^{1/q}.$$

Proof. Let $g(y) = \operatorname{sgn}T(f,y)|T(f,y)|^{p-1}\omega(y)$, by using Theorem 3.1, we have

$$\begin{aligned} \|Tf\|^p_{p,\omega} &= \int_0^\infty T(f,y)\operatorname{sgn}T(f,y)|T(f,y)|^{p-1}\omega(y)dy \\ &= \int_0^\infty T(f,y)g(y)dy \\ &= \frac{c}{k\Gamma_k(\alpha)}\int_0^\infty\int_0^\infty \left\{\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^{\lambda}}\right\}^{(\frac{\alpha}{k}-1)} f(x)g(y)dxdy \\ &\le \frac{c}{k\Gamma_k(\alpha)} c_1^{1/p} c_2^{1/q}\|f\|_{p,\omega_1}\|g\|_{q,\omega_2}. \end{aligned} \tag{4.4}$$

Note that

$$\begin{aligned} \|g\|_{q,\omega_2} &= \left(\int_0^\infty |g(y)|^q\omega_2(y)dy\right)^{1/q} \\ &= \left\{\int_0^\infty \left(|T(f,y)|^{p-1}\omega(y)\right)^q\omega_2(y)dy\right\}^{1/q} \\ &= \left\{\left(\int_0^\infty |T(f,y)|^p\omega(y)dy\right)^{1/p}\right\}^{p/q} = \|Tf\|^{p/q}_{p,\omega}. \end{aligned} \tag{4.5}$$

Hence, by (4.4) and (4.5), we get

$$\|Tf\|^p_{p,\omega} \le \frac{c}{k\Gamma_k(\alpha)} c_1^{1/p} c_2^{1/q}\|f\|_{p,\omega_1}\|Tf\|^{p/q}_{p,\omega}.$$

It follows that

$$\|Tf\|_{p,\omega} \leq \frac{c}{k\Gamma_k(\alpha)} c_1^{1/p} c_2^{1/q} \|f\|_{p,\omega_1}. \tag{4.6}$$

The proof is completed. □

Corollary 4.2. *Under the assumptions of Corollary* 3.4*, then integral operator T is defined by* (2.2)*:*

$$T : L^p_{\omega_1}(0,\infty) \to L^p_{\omega}(0,\infty)$$

exists as a bounded operator and

$$\|Tf\|_{p,\omega} \leq c(p)\|f\|_{p,\omega_1}, \tag{4.7}$$

where

$$c(p) = \|T\| = \frac{1}{\lambda_1}\frac{c}{k\Gamma_k(\alpha)} B\left(\frac{\lambda}{p}\left(\frac{\alpha}{k}-1\right), \frac{\lambda}{q}\left(\frac{\alpha}{k}-1\right)\right), \tag{4.8}$$

is the best constant factor and ω_1 and ω are defined by (3.1) *and* (4.1)*, respectively.*

Remark 4.3. If we take $\alpha = 2, k = c = 1$, then the above results reduce to the classical case, that is,

Case I: (3.15) reduces to

$$\int_0^\infty \int_0^\infty \left\{\frac{1}{(x^{\lambda_1} + y^{\lambda_2})^{\lambda}}\right\} f(x)g(y)dxdy \leq \frac{1}{\lambda_1} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \|f\|_{p,\omega_1}\|g\|_{q,\omega_2}, \tag{4.9}$$

where $\omega_1(x) = x^{(\lambda\lambda_1+1)(p-1)}$, and $\omega_2(x) = x^{(\lambda\lambda_1+1)(q-1)}$ and the constant factor $\frac{1}{\lambda_1}B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible.

Case II: (4.7) reduces to

$$\|Tf\|_{p,\omega} \leq c(p)\|f\|_{p,\omega_1}, \tag{4.10}$$

where $\omega(x) = x^{\lambda\lambda_1 - 1}$ and

$$c(p) = \|T\| = \frac{1}{\lambda_1} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) = \frac{1}{\lambda_1}\frac{\Gamma(\lambda/p)\Gamma(\lambda/q)}{\Gamma(\lambda)} \tag{4.11}$$

is the best constant factor.

Case III: If we take $\lambda = \lambda_1 = 1$ in (4.9), then (4.9) reduces to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq B\left(\frac{1}{p}, \frac{1}{q}\right) \|f\|_p \|g\|_{q,\omega_2}, \tag{4.12}$$

where $\omega_2(x) = x^{2(q-1)}$, and the constant factor $B\left(\frac{1}{p}, \frac{1}{q}\right) = B\left(\frac{1}{p}, 1-\frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)}$ is the best possible. By comparing (4.12) and (1.20), the author finds out unexpected the constant factor

$$B\left(\frac{1}{p}, \frac{1}{q}\right) = B\left(\frac{1}{p}, 1-\frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)}$$

is still the best possible after $\|g\|_q$ is replaced by $\|g\|_{q,\omega_2}$ in (1.20).

So the above classical results are also new.

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