Montes Taurus J. Pure Appl. Math. 4 (2), 11–17, 2022

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Geometry of contrapedal curves of Bézier curves

Ayşe Yılmaz Ceylan [a]

[a]Department of Mathematics, Faculty of Science, Akdeniz University, 07058 Antalya, Turkey

Abstract

The scope of this paper is to study the geometric structures of contrapedal curves of Bézier curves which has many applications in computer graphics and related areas. Especially, the curvature of a contrapedal curve of a planar Bézier curve are examined. Moreover, the curvature of this curve couple is handled with the origin pedal point. In addition, the curvatures are investigated at the end points.

Keywords: Bézier curve, curvature, contrapedal curve

2010 MSC: 65D17, 14H50, 53A04

1. Introduction

Bézier curves are the most important mathematical representations of curves which are applied to computer graphics and related areas. See more for details in [4, 11]. Recently, the geometry of Bézier curves have been studied by many authors due to the fact that they have several important properties. Incesu and Gürsoy studied the curvatures and principal form of the Bézier curve in [8]. Georgiev worked on the shapes of planar and cubic Bézier curve in [5, 6].
The geometry of curves in the Euclidean space are one of the most important subjects in differential geometry. For that reason, the curve couples of Bézier curves have attracted attention by many mathematicians. In the recent studies, Kılıçoğlu and Şenyurt studied the involute of the cubic Bézier curve in Euclidean 3-space [9]. In [3], the evolute-involute curve couples of Bézier curves in Euclidean 3-space were investigated. In [1] evolute, involute, parallel curve couples of Bézier curves were constructed and their curvatures were calculated in the Euclidean 2-space in which the Bézier curve couples need not to be unit speed. Among the curve couples, examining the contrapedal curves is a fundamental issue. A contrapedal curve of a regular plane curve is the locus of the feet of the perpendiculars from a point to the normals to the curve. [10] and [13] studied pedal and contrapedal curves of fronts in the Euclidean plane. In the CAGD field, a classical family of sinusoidal spirals was introduced by Ueda [14] and [15] via a pedal-point construction, and later identified as belonging to the special subset of rational Bézier curves called p-Bézier curves [12]. Ceylan and Kara characterized the pedal and contrapedal curves of Bézier curves in [2]. In the recent studies, there is a lack of knowledge related to geometry of contrapedal curves. For that reason, our motivation in this paper is to examine the curvature of contrapedal curves of Bézier curves.
The rest part of the paper is given as follows: Section 2 gives some basic notations and definitions for needed throughout the study. Section 3 gives the characterization of contrapedal curves of Bézier curve. Section 4 examines the

†*Article ID:* MTJPAM-D-21-00072
Email address: ayilmazceylan@akdeniz.edu.tr (Ayşe Yılmaz Ceylan)
Received:29 November 2021, Accepted:4 March 2022, Published:29 April 2022
⋆*Corresponding Author:* Ayşe Yılmaz Ceylan

curvature of a contrapedal curve of a planar Bézier curve and investigate this curvature at end points. In addition, the origin is taken as a pedal point. In the final section, we conclude our work.

2. Preliminaries

A classical Bézier curve of degree m with control points p_i is defined as

$$B(t) = \sum_{i=0}^{m} B_i^m(t) p_i, \quad t \in [0,1] \tag{2.1}$$

$$B_{i,m}(t) = \begin{cases} \frac{m!}{(m-i)!i!}(1-t)^{m-i}t^i, & \text{if } 0 \le i \le m \\ 0, & \text{otherwise} \end{cases}$$

are called the Bernstein basis functions of degree m.

Lemma 2.1 (*cf.* [11])**.** *The first and second derivatives of a degree-m Bézier curve $B(t)$ is clearly degree $m-1$ and $m-2$ curve respectively. These curves can be written in Bézier form as*

$$B'(t) = m \sum_{i=0}^{m-1} B_i^{m-1}(t) \triangle p_i \tag{2.2}$$

and

$$B''(t) = m(m-1) \sum_{i=0}^{m-2} B_i^{m-2}(t) \triangle^2 p_i \tag{2.3}$$

where $\triangle p_i = p_{i+1} - p_i$, $i = 0, 1, ..., m-1$ *are the control points of* $B^{'}(t)$ *and* $\triangle^2 p_i = \triangle p_{i+1} - \triangle p_i = p_{i+2} - 2p_{i+1} + p_i$ *are the control points of* $B^{''}(t)$*.*

Definition 2.2 (*cf.* [7])**.** Let $\alpha : I \to E^2$ be a non-unit speed planar curve. The Serret-Frenet frame $\{\mathbf{T}, \mathbf{N}\}$ and curvature κ of α for $\forall t \in I$ are defined by

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad N(t) = \frac{J\alpha'(t)}{\|\alpha'(t)\|}, \quad \kappa(t) = \frac{<\alpha''(t), J\alpha'(t)>}{\|\alpha'(t)\|^3} \tag{2.4}$$

where $J : E^2 \to E^2$ is a linear transformation defined by the following equation:

$$J(p_1, p_2) = (-p_2, p_1).$$

Definition 2.3 (*cf.* [7])**.** The contrapedal curve of a regular curve $\beta : (a,b) \to R^2$ with respect to a point $p \in R^2$ is defined by

$$\beta^*[\beta, p](t) = p + \frac{<\beta(t) - p, \beta'(t)>}{\|\beta'(t)\|^2} \beta'(t). \tag{2.5}$$

From now on, we will say a Bézier curve instead of a non-unit speed planar Bézier curve of degree n with control points $p_0, p_1, ..., p_n$ throughout the paper.

3. Contrapedal curves of Bézier curves

In this section, we characterize contrapedal curve of a planar Bézier curve and investigate this curve at $t = 0$ and $t = 1$.

Theorem 3.1 (*cf.* [2])**.** *The contrapedal curve $B^*(t)$ of a Bézier curve defined by* (2.1) *for* $\forall t \in [0,1]$ *and pedal point p is*

$$B^*[B,p](t) = p + \frac{\langle \sum_{j=0}^{m} B_j^m(t)\, p_j - p, \sum_{i=0}^{m-1} B_i^{m-1}(t) \triangle p_i \rangle}{\sum_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) \langle \triangle p_j, \triangle p_i \rangle} \sum_{k=0}^{m-1} B_k^{m-1}(t) \triangle p_k. \tag{3.1}$$

Corollary 3.2 (*cf.* [2])**.** *The contrapedal curve couples $B^*(t)$ of a Bézier curve which is defined by* (2.1) *and pedal point p are*

$$B^*[B,p](0) = p + \frac{\langle p_0 - p, \triangle p_0 \rangle}{\langle \triangle p_0, \triangle p_0 \rangle} \triangle p_0$$

and

$$B^*[B,p](1) = p + \frac{\langle p_m - p, \triangle p_{m-1} \rangle}{\langle \triangle p_{m-1}, \triangle p_{m-1} \rangle} \triangle p_{m-1}$$

at $t = 0$ and $t = 1$, respectively.

Theorem 3.3 (*cf.* [2])**.** *The contrapedal curve $B^*(t)$ of a Bézier curve defined by* (2.1) *for* $\forall t \in [0,1]$ *and pedal point $p = (0,0) = \boldsymbol{0}$ is*

$$B^*[B,\boldsymbol{0}](t) = \frac{\sum_{i=0}^{m} B_i^n(t) \sum_{j=0}^{m-1} B_j^{m-1}(t) \langle p_i, \triangle p_j \rangle}{\sum_{i,j=0}^{m-1} B_i^{m-1}(t) B_j^{m-1}(t) \langle \triangle p_i, \triangle p_j \rangle} \sum_{k=0}^{m-1} B_k^{m-1}(t) \triangle p_k.$$

Corollary 3.4 (*cf.* [2])**.** *The contrapedal curve couples $B^*(t)$ of a Bézier curve which is defined by* (2.1) *and pedal point $p = (0,0) = \boldsymbol{0}$ are*

$$B^*[B,\boldsymbol{0}](0) = \frac{\langle p_0, \triangle p_0 \rangle}{\langle \triangle p_0, \triangle p_0 \rangle} \triangle p_0$$

and

$$B^*[B,\boldsymbol{0}](1) = \frac{\langle p_m, \triangle p_{m-1} \rangle}{\langle \triangle p_{m-1}, \triangle p_{m-1} \rangle} \triangle p_{m-1}$$

at $t = 0$ and $t = 1$, respectively.

4. Curvature of contrapedal curves of Bézier curves

In this section, we examine the curvature of a contrapedal curve of a planar Bézier curve and investigate this geometric structure at the starting and the ending points.

Theorem 4.1. *The curvature of contrapedal curve couple $B^*(t)$ of a Bézier curve defined by* (3.1) *with pedal point p is*

$$\kappa^*_{B^*[B,p]}(t) = t_1 + t_2 \tag{4.1}$$

where t_1 and t_2 are given by the following equations:

$$t_1 = \frac{\begin{array}{c} m\langle \sum_{i=0}^{m} B_i^m(t)p_i - Q, \sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\rangle \cdot \left(\sum_{i,j=0}^{n-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right) \\ \cdot\Big[(m-1)(m-2)\sum_{i=0}^{m-3} B_i^{m-3}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^3 p_i, J\triangle p_j\rangle \\ \cdot\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right) - 3(m-1)^2 \cdot \\ \left(\left(\sum_{i=0}^{m-1} B_i^{m-1}(t)\triangle_x p_i\right)\cdot\left(\sum_{j=0}^{m-2} B_j^{m-2}(t)\triangle_x^2 p_j\right) + \left(\sum_{i=0}^{m-1} B_i^{m-1}(t)\triangle_y p_i\right)\cdot \right. \\ \left.\left(\sum_{j=0}^{m-2} B_j^{m-2}(t)\triangle_y^2 p_j\right)\right)\cdot \sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\Big] \end{array}}{\begin{array}{c}\Big[\left((m-1)\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)^2 \left(\sum_{i,j=0}^{m} B_i^m(t)B_j^m(t)\langle p_i, p_j\rangle\right. \\ \left. -2\sum_{i=0}^{m} B_i^m(t)\langle p_i, Q\rangle + \|Q\|^2\right) + 2m(n-1)\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)\cdot \\ \left(\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)\left(\langle\sum_{i=0}^{m} B_i^m(t)p_i - Q, \sum_{j=0}^{m-1} B_j^{m-1}(t)J\triangle p_j\rangle\right) \\ + m^2\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)^3\Big]^{\frac{3}{2}}\end{array}}$$

$$t_2 = \frac{2(m-1)\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle}{\begin{array}{c}\Big[\left((m-1)\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)^2 \left(\sum_{i,j=0}^{m} B_i^m(t)B_j^m(t)\langle p_i, p_j\rangle\right. \\ \left. -2\sum_{i=0}^{m} B_i^m(t)\langle p_i, Q\rangle + \|Q\|^2\right) + 2m(m-1)\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)\cdot \\ \left(\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)\left(\langle\sum_{i=0}^{m} B_i^m(t)p_i - Q, \sum_{j=0}^{m-1} B_j^{m-1}(t)J\triangle p_j\rangle\right) \\ + m^2\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)^3\Big]^{\frac{1}{2}}\end{array}}$$

In the above equations, $\triangle_x p_i = (p_{i+1})_x - (p_i)_x$, $\triangle_y p_i = (p_{i+1})_y - (p_i)_y$, $\triangle_x^2 p_j = \triangle_x p_{j+1} - \triangle_x p_j$ *and* $\triangle_y^2 p_j = \triangle_y p_{j+1} - \triangle_y p_j$.

Proof. By using the equation (2.4), the equation (2.5) can be written as

$$B^*[B,p](t) = p + < B(t) - p, T(t) > T(t)$$

where $B(t)$ is Bézier curve defined by (2.1). From the curvature formula given by (2.4), the following equation is handled:

$$\kappa^*_{B^*[B,p]}(t) = \frac{\langle (B^*[B,p](t))'', J(B^*[B,p](t))'\rangle}{\|(B^*[B,p](t))'\|^3}. \tag{4.2}$$

If the releveant derivatives are calculated in the equation (4.2) by using the equations (2.2) and (2.3), the equation (4.1) is obtained. □

Corollary 4.2. *The curvature of contrapedal curve couple $B^*(t)$ of a Bézier curve defined by* (3.1) *with pedal point p is defined by*

$$\kappa^*_{B^*[B,p]}(0) = \frac{\begin{array}{c} m(m-1)\langle p_0 - p, \triangle p_0\rangle\|\triangle p_0\|^2[(m-2)\langle \triangle^3 p_0, J\triangle p_0\rangle\|\triangle p_0\|^2 \\ -3(n-1)\left(\triangle_x p_0 \cdot \triangle_x^2 p_0 + \triangle_y p_0 \cdot \triangle_y^2 p_0\right)\langle \triangle^2 p_0, J\triangle p_0\rangle] \end{array}}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_0, J\triangle p_0\rangle^2\left(\|p_0\|^2 - 2\langle p_0, p\rangle + \|p\|^2\right) + 2m(m-1) \\ \cdot\|\triangle p_0\|^2\langle \triangle^2 p_0, J\triangle p_0\rangle\langle p_0 - p, J\triangle p_0\rangle + n^2\|\triangle p_0\|^6]^{\frac{3}{2}} \end{array}}$$

$$+\frac{2(m-1)\langle \triangle^2 p_0, J\triangle p_0\rangle}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_0, J\triangle p_0\rangle^2\left(\|p_0\|^2 - 2\langle p_0, p\rangle + \|p\|^2\right) + 2n\cdot \\ (m-1)\|\triangle p_0\|^2\langle \triangle^2 p_0, J\triangle p_0\rangle\langle p_0 - p, J\triangle p_0\rangle + m^2\|\triangle p_0\|^6]^{\frac{1}{2}} \end{array}}$$

at $t = 0$ and

$$\kappa^*_{B^*[B,p]}(1) = \frac{2(m-1)\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle^2\left(\|p_m\|^2 - 2\langle p_m, p\rangle + \|p\|^2\right) \\ +2m(m-1)\|\triangle p_{m-1}\|^2\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle \\ \cdot\langle p_m - p, J\triangle p_{m-1}\rangle + m^2\|\triangle p_{n-1}\|^6]^{\frac{1}{2}} \end{array}}$$

$$+\frac{\begin{array}{c} m(m-1)\langle p_m - p, \triangle p_{m-1}\rangle\|\triangle p_{m-1}\|^2 \\ \cdot[(m-2)\langle \triangle^3 P_{m-3}, J\triangle p_{m-1}\rangle\|\triangle p_{m-1}\|^2 - 3(m-1)\cdot \\ \left(\triangle_x p_{m-1} \cdot \triangle_x^2 p_{m-2} + \triangle_y p_{m-1} \cdot \triangle_y^2 p_{m-2}\right)\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle] \end{array}}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle^2\left(\|p_m\|^2 - 2\langle p_m, p\rangle + \|p\|^2\right) \\ +2m(m-1)\|\triangle p_{m-1}\|^2\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle \\ \cdot\langle p_m - p, J\triangle p_{m-1}\rangle + m^2\|\triangle p_{m-1}\|^6]^{\frac{3}{2}} \end{array}}$$

at $t = 1$. In addition, the curvature of contrapedal curve couple $B^*(t)$ with the pedal point $p = p_0$ at $t = 0$ is

$$\kappa^*_{B^*[B,p_0]}(0) = \frac{2(n-1)\langle \triangle^2 p_0, J\triangle p_0\rangle}{n\|\triangle p_0\|^3}$$

and the curvature of contrapedal curve couple $B^*(t)$ with the pedal point $p = p_n$ at $t = 1$ is

$$\kappa^*_{B^*[B,p_m]}(1) = \frac{2(m-1)\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle}{m\|\triangle p_{m-1}\|^3}.$$

Proof. Put the value $t = 0$ and $t = 1$ in the equation (4.1) respectively, it can be seen easily. □

Corollary 4.3. *The curvature of contrapedal curve couple $B^*(t)$ of a Bézier curve defined by* (3.1) *with pedal point $p = (0,0) = \mathbf{0}$ is given by*

$$\kappa^*_{B^*[B,\mathbf{0}]}(t) = k_1 + k_2,$$

where k_1 and k_2 are defined by the following equations:

$$k_1 = \frac{\begin{array}{c} m\sum_{i=0}^{m} B_i^m(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle p_i, \triangle p_j\rangle \cdot \sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle \\ \cdot\Bigg[(m-1)(m-2)\sum_{i=0}^{m-3} B_i^{m-3}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^3 p_i, J\triangle p_j\rangle \\ \cdot\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right) - 3(m-1)^2\cdot \\ \left(\left(\sum_{i=0}^{m-1} B_i^{m-1}(t)\triangle_x p_i\right)\cdot\left(\sum_{j=0}^{m-2} B_j^{m-2}(t)\triangle_x^2 p_j\right) + \left(\sum_{i=0}^{m-1} B_i^{m-1}(t)\triangle_y p_i\right)\cdot\right. \\ \left.\left(\sum_{j=0}^{m-2} B_j^{m-2}(t)\triangle_y^2 p_j\right)\right)\cdot\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\Bigg] \end{array}}{\begin{array}{c} \Bigg[\left((m-1)\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)^2 \sum_{i,j=0}^{m} B_i^m(t)B_j^m(t)\langle p_i, p_j\rangle \\ +2m(m-1)\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)\cdot \\ \left(\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)\left(\sum_{i=0}^{m} B_i^m(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle p_i, J\triangle p_j\rangle\right) \\ + m^2\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)^3\Bigg]^{\frac{3}{2}} \end{array}}$$

$$k_2 = \frac{2(m-1)\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle}{\begin{array}{c} \Bigg[\left((m-1)\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)^2 \sum_{i,j=0}^{m} B_i^m(t)B_j^m(t)\langle p_i, p_j\rangle \\ +2m(m-1)\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)\cdot \\ \left(\sum_{i=0}^{m-2} B_i^{m-2}(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle \triangle^2 p_i, J\triangle p_j\rangle\right)\left(\sum_{i=0}^{m} B_i^m(t)\sum_{j=0}^{m-1} B_j^{m-1}(t)\langle p_i, J\triangle p_j\rangle\right) \\ + m^2\left(\sum_{i,j=0}^{m-1} B_i^{m-1}(t)B_j^{m-1}(t)\langle \triangle p_i, \triangle p_j\rangle\right)^3\Bigg]^{\frac{1}{2}} \end{array}}$$

Proof. If the pedal point is chosen as the origin $p = \mathbf{0}$ in the equation (4.1), it can be proved after some simplifications. □

Corollary 4.4. *The curvature of contrapedal curve couple $B^*(t)$ of a Bézier curve defined by* (3.1) *with pedal point $p = (0,0) = \mathbf{0}$ is defined by*

$$\kappa^*_{B^*[B,\mathbf{0}]}(0) = \frac{\begin{array}{c} m(m-1)\langle p_0, \triangle p_0\rangle\|\triangle p_0\|^2[(n-2)\langle \triangle^3 p_0, J\triangle p_0\rangle\|\triangle p_0\|^2 \\ -3(m-1)\left(\triangle_x p_0\cdot\triangle_x^2 p_0 + \triangle_y p_0\cdot\triangle_y^2 p_0\right)\langle \triangle^2 p_0, J\triangle p_0\rangle] \end{array}}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_0, J\triangle p_0\rangle^2\|p_0\|^2 + 2m(m-1)\|\triangle p_0\|^2 \\ \langle \triangle^2 p_0, J\triangle p_0\rangle\langle p_0, J\triangle p_0\rangle + m^2\|\triangle p_0\|^6]^{\frac{3}{2}} \end{array}}$$

$$+\frac{2(m-1)\langle \triangle^2 p_0, J\triangle p_0\rangle}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_0, J\triangle p_0\rangle^2\|p_0\|^2 + 2m(m-1)\|\triangle p_0\|^2 \\ \langle \triangle^2 p_0, J\triangle p_0\rangle\langle p_0, J\triangle p_0\rangle + m^2\|\triangle p_0\|^6]^{\frac{1}{2}} \end{array}}$$

at $t = 0$ *and*

$$\kappa^*_{B^*[B,\mathbf{0}]}(1) = \frac{\begin{array}{c} m(m-1)\langle p_m, \triangle p_{m-1}\rangle \|\triangle p_{m-1}\|^2 \\ \cdot [(m-2)\langle \triangle^3 P_{m-3}, J\triangle p_{m-1}\rangle \|\triangle p_{m-1}\|^2 - 3(m-1)\cdot \\ \left(\triangle_x p_{m-1}\cdot \triangle_x^2 p_{m-2} + \triangle_y p_{m-1}\cdot \triangle_y^2 p_{m-2}\right)\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle] \end{array}}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle^2 \|p_m\|^2 + 2m(m-1) \\ \|\triangle p_{m-1}\|^2 \langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle \langle p_m, J\triangle p_{m-1}\rangle + m^2 \|\triangle p_{m-1}\|^6]^{\frac{3}{2}} \end{array}}$$

$$+ \frac{2(m-1)\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle}{\begin{array}{c} [(m-1)^2\langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle^2 \|p_m\|^2 + 2m(m-1) \\ \|\triangle p_{m-1}\|^2 \langle \triangle^2 p_{m-2}, J\triangle p_{m-1}\rangle \langle p_m, J\triangle p_{m-1}\rangle + m^2 \|\triangle p_{m-1}\|^6]^{\frac{1}{2}} \end{array}}$$

at $t = 1$.

Proof. Put the value $t = 0$ and $t = 1$ in the equation (4.3) respectively, it can be seen easily. □

5. Conclusion

In this paper, the curvature of a contrapedal curve of a planar Bézier curve is studied with regular and origin pedal points. Moreover, the curvatures are shown at the starting and the ending points.

Acknowledgments

The author would like to express her sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author Contributions: The author conceived and designed the work.

Conflicts of Interest: This work does not have any conflicts of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] A. Y. Ceylan, *Curve couples of Bézier curves in Euclidean* 2*-space*, Fundam. J. Math. Appl. **4** **(4)**, 245–250, 2021.

[2] A. Y. Ceylan and M. Kara, *On pedal and contrapedal curves of Bézier curves*, Konuralp J. Math. **9** **(2)**, 217–221, 2021.

[3] Z. Duman, *Bézier eğrilerinin involüt-evolüt eğri çiftleri*, Master Thesis, Sakarya University, Sakarya, 2021.

[4] G. Farin, *A history of curves and surfaces*, In: Handbook of Computer Aided Geometric Design (Ed. by G. Farin, J. Hoschek and M.-S. Kim), North Holland, 1–21, 2002.

[5] G. H. Georgiev, *Shapes of plane Bézier curves*, In: Curve and Surface Design-Avignon 2006 (Ed. by P. Chenin, T. Lyche and L. L. Schumaker), Nashboro Prees, Brentwood, 143–152, 2006.

[6] G. H. Georgiev, *On the shape of the cubic Bézier curve*, Proc. of International Congress Pure and Applied Differential Geometry-Brussels (Ed. by F. Dillen and I. Van de Woestyne), Shaker Verlag, Aachen, 98–106, 2007.

[7] A. Gray, E. Abbena and S. Salamon, *Modern differential geometry of curves and surfaces with mathematica*, Chapman and Hall/CRC, Boca Raton, FL, USA, 2016.

[8] M. Incesu and O. Gürsoy, *Bézier eğrilerinde esas formlar ve eğrilikler*, XVII Ulusal Matematik Sempozyumu, Bildiriler, Abant İzzet Baysal Üniversitesi, 146–157, 2004.

[9] Ş. Kılıçoğlu and S. Şenyurt, *On the involute of the cubic Bézier curve by using matrix representation in* E^3, Eur. J. Pure Appl. Math. **13** **(2)**, 216–226, 2020.

[10] Y. Li and D. Pei, *Pedal curves of frontals in the Euclidean plane*, Math. Methods Appl. Sci. **41**, 1988–1997, 2018.

[11] D. Marsh, *Applied geometry for computer graphics and CAD*, Springer, 2006.

[12] J. Sánchez-Reyes, *p-Bézier curves, spirals, and sectrix curves*, Comput. Aided Geom. Des. **19** **(6)**, 445–464, 2002.

[13] O. O. Tuncer, H. Ceyhan, İ. Gök and F. N. Ekmekci, *Notes on pedal and contrapedal curves of fronts in the Euclidean plane*, Math. Methods Appl. Sci. **41**, 5096–5111, 2018.

[14] K. Ueda, *A sequence of Bézier curves generated by successive pedal-point constructions*, In: Curves and Surfaces with Applications in CAGD (Ed. by A. Le Méhauté, C. Rabut and L. Schumaker), Vanderbilt University Press, 427–434, 1997.

[15] K. Ueda, *Pedal curves and surfaces*, In: Mathematical Methods for Curves and Surfaces–Oslo 2000 (Ed. by T. Lyche and L. L. Schumaker), Vanderbilt University Press, 497–507, 2001.