Montes Taurus J. Pure Appl. Math. 4 (2), 28–36, 2022

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Nonexistence of global weak solutions of semilinear degenerate hyperbolic equation of the second kind

Kangqun Zhang [a]

[a]Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing, 211167, China

Abstract

In the present paper, we investigate the existence of weak solutions of semilinear degenerate hyperbolic equation of the second kind in $C^1((0,T), L^p(\Omega))$ for any $p \in [1, +\infty]$, $n \in \mathbb{Z}^+$. Our approach is based on analyzing the first Fourier coefficient of solution to establish a scope of lifespan. Finally, we obtain nonexistence of global weak solution of a semilinear hyperbolic equation of the second kind with positive initial data.

Keywords: 35L70, 35L15, 35L20

2010 MSC: Hyperbolic equation, characteristic degeneration, the first Fourier coefficient, test function, nonexistence

1. Introduction

In this paper we investigate the semilinear hyperbolic equation

$$Ku = f(u) \quad \text{in} \quad (0,T) \times \Omega, \tag{1.1}$$

whose characteristics degenerate on the initial hyperplane $t = 0$, where $K = t^\alpha \partial_t^2 - \triangle$ is a Keldysh type operator with $\alpha \in \mathbb{R}^+/\{2\}$, $\triangle = \sum_{i=1}^n \partial_{x_i}^2$ is the Laplace operator. The variable coefficient leads to the propagation speed of wave in physics. The Cauchy problem of Eq. (1.1) for the case $\alpha \in (0,1)$, $\Omega = \mathbb{R}^n$ has been studied in [20] and the existence of local weak solution was established. For a bounded domain $\Omega \subseteq \mathbb{R}^n$ with a piecewise smooth boundary, given initial boundary conditions

$$\begin{cases} u(0,x) = \varphi_1(x), \quad \partial_t u(0,x) = \varphi_2(x) \quad \text{in} \quad \Omega, \\ u(t,x) = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \end{cases} \tag{1.2}$$

the local existence of weak solution of problem is also confirmed by use of the finite propagation speed of second order hyperbolic equation if initial data with compact support.

Taking $y + 1 = \frac{2}{2-\alpha} t^{\frac{2-\alpha}{2}}$, (1.1) is changed into

$$\Box u + \frac{\beta}{1+y} \partial_y u = f(u), \tag{1.3}$$

†*Article ID:* MTJPAM-D-21-00043

Email address: chkqnju@hotmail.com (Kangqun Zhang)

Received:16 June 2021, Accepted:14 April 2022, Published:27 May 2022

⋆*Corresponding Author:* Kangqun Zhang

where $\Box = \partial_y^2 - \triangle$ is wave operator and $\beta = \frac{\alpha}{\alpha-2}$. The corresponding Cauchy problem and initial boundary value problem of Eq. (1.3) with the initial data given on $y = 0$ have been extensively considered. There are many well-known results for solutions of wave equation which is the special case $\beta = 0$, including existence [1, 4, 6, 7] and blow up [8, 10]. The case for $\beta > 0$ has also been studied. In [18], Wirth proved that the solution of homogeneous equation (1.3) with initial data $(u, \partial_y u)(0, x) = (\varphi_1(x), \varphi_2(x))$ satisfy

$$\|u(t)\| \le C(\|\varphi_1\|_{L^2} + \|\varphi_2\|_{H^{-1}}) \begin{cases} (1+t)^{1-\beta} & 0 < \beta < 1, \\ \ln(e+t) & \beta = 1, \\ 1 & \beta > 1, \end{cases}$$

$$\|(\partial_t, \Delta)u(t)\| \le C(\|\varphi_1\|_{H^1} + \|\varphi_1\|_{L^2})(1+t)^{\max\{-\frac{\beta}{2}, -1\}}.$$

Recently, for semilinear equation with $f(u) = |u|^p$ in [17], Wakasugi found the critical exponent p_c for sufficiently large β. D'Abbicco [3] established the global existence of solutions under the assumption $k > n + 2$ and $p > p_c$ with small data. The case for $-1 < \beta < 0$ is corresponding to Eq. (1.1) with $0 < \alpha < 2$. There is a few results on the global solution as far as we know. In order to formulate the nonexistence of global weak solutions of problem (1.1)-(1.2), we give assumption on the source term $f(u)$ which is a convex function and satisfies

$$f(u) - \mu u \ge \nu u^{1+q}, \quad q > 0, \tag{1.4}$$

for $u > \epsilon_0$ with a number $\epsilon_0 > 0$, μ and ν are positive constants. For $n = 1$ and $f(u) = |u|^{q-1}u$ with $q > 1$ and $\alpha = 2$, Li applied Banach fixed point theorem and established the unique existence of local weak solutions of problem (1.1)-(1.2) in [14] and showed nonexistence of global solution with non-positive energy in [15]. In the present paper, we investigate the problem with a nonlinear term given in (1.4) based on analyzing the first Fourier coefficient of its solution. The technique was first introduced by Kaplan [9] for considering a nonlinear parabolic equation of second order, later be generalized by Glassey [5], Levine [12, 13] for studying classical or weak solutions of nonlinear wave equations. Recently, Zhang [21] have used this idea to solve with a singular Cauchy problem and a singular initial boundary value problem of semilinear Keldysh type equation, which is equivalent to a special case of Cauchy problem of Eq. (1.1). In [22], a generalized linear fractional model also has been studied.

In this work, the problem is more complicated and the calculation is more delicate, then based on establishing the inequality

$$G''(t) \ge ct^{-\alpha}G^{1+q}(t), \quad q > 0, \tag{1.5}$$

constant $c > 0$, we derive that the first Fourier coefficient of the solution becomes to infinity at finite time. Then, we confirm blow up of global weak solution of problem (1.1)-(1.2), although the method is independent of the spatial dimension and the weak conservation of energy law. Moreover, the obtained results for weak solution only depend on one t-derivative. Besides $\alpha \in (0, 2)$, the results also are holding for $\alpha \in [2, +\infty)$ by taking a similar procedure for $[1, 2)$.

2. Initial boundary value problem

Here we introduce $v(x)$ as the first eigenfunction of the eigenvalue problem

$$\begin{cases} \triangle v + \lambda v = 0, & \text{in} \quad \Omega, \\ v(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is the bounded domain with a piecewise smooth boundary, λ is the first eigenvalue. In terms of the famous Theorem [2, p. 451-455], the normalized positive eigenfunction $v(x)$ is unique that satisfying

$$\int_\Omega v(x)dx = 1. \tag{2.2}$$

Now we consider initial boundary value problem (1.1)-(1.2) in Ω. Assume that

a) for all $x \in \Omega$, nonnegative functions $v_i(x)$, and there exist $x_i \in \Omega$ such that $v_i(x_i) > 0, \quad i = 1, 2$.

b) $\int_\Omega v(x)v_i(x)dx = \zeta_i > 0, \quad i = 1, 2$.

For integral functions $u(t, x)$ and $v(t, x)$, take inner product by

$$(v(t, \cdot), u(t, \cdot)) = \int_\Omega v(t, x)u(t, x)dx. \tag{2.3}$$

Definition 2.1 (*cf.* [12, 21])**.** Call $u(t, x)$ a weak solution of the initial boundary value problem (1.1)-(1.2) if for all twice continuously differentiable functions $v(t, x)$ with respect to t satisfies

$$\begin{aligned} t^\alpha (v(t, \cdot), \partial_t u(t, \cdot)) = \int_0^t [\alpha\tau^{\alpha-1} (v(\tau, \cdot), \partial_\tau u(\tau, \cdot)) + \tau^\alpha(\partial_\tau v(\tau, \cdot), \partial_\tau u(\tau, \cdot)) \\ + (\triangle v(\tau, \cdot), u(\tau, \cdot)) + (v(\tau, \cdot), f(u(\tau, \cdot)))]d\tau \end{aligned} \tag{2.4}$$

and $u(t, \cdot)$, $\partial_t u(t, \cdot)$ are continuous with respect to t.

Theorem 2.2 (*cf.* [12, 21])**.** *Prescribed $u(t, x)$ is a weak solution in the sense of Definition* 2.1 *to the initial boundary value problem* (1.1)-(1.2) *with nonlinear term satisfies* (1.4)*, then there exists*

$$\lim_{t\to T^-} \|u(t, \cdot)\|_{L^p(\Omega)} = +\infty, \quad p \in [1, +\infty], \tag{2.5}$$

for some positive number T.

Proof. Given $v(x)$ defined by (2.1), let

$$G(t) = (v(x), u(t, x)) = \int_\Omega v(x)u(t, x)dx, \tag{2.6}$$

then there is

$$G'(t) = (v(x), \partial_t u(t, x)) = \int_\Omega v(x)\partial_t u(t, x)dx \tag{2.7}$$

which satisfies $G(0) = \zeta_1$ and $G'(0) = \zeta_2$.

According to Definition 2.1, $u(t, x)$ has not differentiability of second order with respect to t, then we should use (2.4) to obtain the second order derivative $G''(t)$. Multiply t^α on (2.7), applying (2.1) and (2.3), we derive

$$\begin{aligned} G''(t) &= (t^{-\alpha}t^\alpha G'(t))' = -\alpha t^{-1}G'(t) + t^{-\alpha}(t^\alpha G'(t))' \\ &= t^{-\alpha}((v(x), f(u(t, x))) - \lambda(v(x), u(t, x))). \end{aligned} \tag{2.8}$$

Since function $f(\cdot)$ is convex, then in terms of Jensen inequality, one has

$$(v(x), f(u(t, x)) \geq f((v(x), u(t, x))) = f(G(t)). \tag{2.9}$$

Then set $\lambda = \mu$ in (1.4), substitute (2.9) into (2.8), we obtain

$$G''(t) \geq t^{-\alpha}(f(v(x), u(t, x)) - \lambda(v(x), u(t, x))). \tag{2.10}$$

This yields (1.5).

Since $\zeta_i > 0$, $i = 1, 2$, then there exists an interval $(0, \eta)$ such that $G(t)$ is monotonic increasing and $G(t) > \zeta_1$. Furthermore, it derives

$$G''(t) \geq \nu t^{-m}G^{1+q}(t) > 0, \quad t \in (0, \eta). \tag{2.11}$$

This means that $G'(t)$ is monotonic increasing on $(0, \eta)$. On the other hand, $G'(0) = \zeta_2 \geq 0$. Then we obtain $G'(t) > \zeta_2 > 0$ on $(0, \eta)$. Multiply $G'(t)$ on (1.5) and then integrate it over (ε, t) yields

$$(G'(t))^2 \geq (G'(\varepsilon))^2 + 2\nu \int_\varepsilon^t \tau^{-\alpha}G^{1+q}(\tau)G'(\tau)d\tau. \tag{2.12}$$

Then (2.11) and (2.12) confirm that $G'(t) > \zeta_2$ and $G(t) > \zeta_1$ on the whole existence interval. Hence, $G(t)$ is positive.

Now for $\varepsilon > 0$, we set

$$H(t) = \int_{\varepsilon}^{t} \tau^{-\alpha} G^{1+q}(\tau) G'(\tau) d\tau, \tag{2.13}$$

$$F(t) = (2+p)^{-1} G^{2+q}(t). \tag{2.14}$$

Then it is easy to verify that $H(t)$, $F(t)$ are positive and $H'(t) = t^{-\alpha} F'(t)$. Combining to (2.11)-(2.14), we obtain

$$(H'(t))^2 \geq (2+q)^{1+\frac{q}{2+q}} \left((G'(\varepsilon))^2 + 2\nu H(t)\right) t^{-2\alpha} \left(\varepsilon^{\alpha} H\right)^{1+\frac{q}{2+q}}(t). \tag{2.15}$$

Set $d = \frac{1+q}{2+q}$, $\varepsilon_0 = G'(\varepsilon)$ and $p_\varepsilon = (2+q)^d \varepsilon^{d\alpha}$, (2.15) yields

$$\left(\varepsilon_0^2 + 2\nu H(t)\right)^{-\frac{1}{2}} H^{-d}(t) H'(t) \geq p_\varepsilon t^{-\alpha}. \tag{2.16}$$

In the following, a simple proof for the case $\alpha \in (0,1)$ is given since the arguments are standard, for more details one can find in [11, 16]. Integrate above inequality (2.16) over (ε, t), then

$$\int_{H(\varepsilon)}^{H(t)} \left(\varepsilon_0^2 + 2\nu H(\tau)\right)^{-\frac{1}{2}} H^{-d}(\tau) dH(\tau) \geq \int_{\varepsilon}^{t} p_\varepsilon \tau^{-\alpha} d\tau \geq (1-\alpha)^{-1} p_\varepsilon \left(t^{1-\alpha} - \varepsilon^{1-\alpha}\right). \tag{2.17}$$

It is easy to verify that the left integral in (2.17) is convergent as t tends to zero. Set

$$T_0 = \left(\varepsilon^{1-\alpha} + (1-\alpha) p_\varepsilon^{-1} \int_{H(\varepsilon)}^{+\infty} \left(\varepsilon_0^2 + 2\nu t\right)^{-\frac{1}{2}} t^{-d} dt\right)^{\frac{1}{1-\alpha}}, \quad 0 < \alpha < 1. \tag{2.18}$$

Then (2.17)-(2.18) imply that $H(t)$ has a singularity at some point $t_0 \leq T_0$. Otherwise, the right part will go to positive infinity when $t \to +\infty$.Then, we confirm that

$$\lim_{t \to T^-} H(t) = +\infty, \tag{2.19}$$

for some $T \leq T_0$.

On the other hand, note that $G(t)$ is a monotonic increasing function, then (2.14) yields $F(t)$ also is. By a direct computation with (2.13)-(2.14), we have

$$\begin{aligned} H(t) &= \int_{\varepsilon}^{t} \tau^{-\alpha} F'(\tau) d\tau \\ &= t^{-\alpha} F(t) - \varepsilon^{-\alpha} F(\varepsilon) + \alpha \int_{\varepsilon}^{t} \tau^{-\alpha-1} F(\tau) d\tau \\ &< t^{-\alpha} F(t) - \varepsilon^{-\alpha} F(\varepsilon) + F(t)(\varepsilon^{-\alpha} - t^{-\alpha}) \\ &< \varepsilon^{-\alpha} F(t). \end{aligned} \tag{2.20}$$

Hence, combing (2.14), (2.19) and (2.20), we confirm that

$$\lim_{t \to T^-} G(t) = +\infty. \tag{2.21}$$

Then, by use of Hölder's inequality, we arrive at

$$\begin{aligned} G(t) &= |G(t)| = |(v(x), u(t,x))| \\ &= \|v(x) u(t,x)\|_{L^1(\Omega)} \leq \|v(x)\|_{L^q(\Omega)} \|u(t,x)\|_{L^p(\Omega)}. \end{aligned} \tag{2.22}$$

Hence, (2.21) and (2.22) imply (2.5) holding for $p \in (1, +\infty)$. Especially, by use of (2.3) and (2.22), we obtain

$$G(t) \leq \|u(t,x)\|_{L^\infty(\Omega)}. \tag{2.23}$$

Then (2.5) is confirmed for $p = +\infty$.

Next, we consider the case for $\alpha \in [1, 2)$. For $0 < \varepsilon < t$, It follows (2.12) that

$$(G'(t))^2 \geq t^{-\alpha}\Big(\varepsilon^{\alpha}(G'(\varepsilon))^2 + \frac{2\nu}{2+q}\left(G^{2+q}(t) - G^{2+q}(\varepsilon)\right)\Big). \tag{2.24}$$

Since $G'(t) > 0$ is holding in the whole existence interval, then (2.24) becomes

$$\Big(\varepsilon^{\alpha}(G'(\varepsilon))^2 + \frac{2\nu}{2+q}\left(G^{2+q}(t) - G^{2+q}(\varepsilon)\right)\Big)^{-\frac{1}{2}} G'(t) \geq t^{-\frac{\alpha}{2}}. \tag{2.25}$$

Fix ε such that $2\varepsilon < t$, then for $\alpha \in [1, 2)$, integrate above inequality (2.25) over (ε, t), we derive

$$\begin{aligned} +\infty > & \int_{G(\varepsilon)}^{G(t)} \left(\varepsilon^{\alpha}(G'(\varepsilon))^2 + \frac{2\nu}{2+q}\left(G^{2+q}(t) - G^{2+q}(\varepsilon)\right)\right)^{-\frac{1}{2}} dG(t) \\ > & \frac{2-\alpha}{2}\left(t^{\frac{2-\alpha}{2}} - \varepsilon^{\frac{2-\alpha}{2}}\right). \end{aligned} \tag{2.26}$$

This implies that $G(t)$ has a singularity at some point $T \leq T_1$, where

$$T_1 = \left(\varepsilon^{\frac{2-\alpha}{2}} + \frac{2}{2-\alpha}\int_{G(\varepsilon)}^{+\infty}\left(\varepsilon^{\alpha}(G'(\varepsilon))^2 + \frac{2\nu}{2+q}\left(t^{2+q} - G^{2+q}(\varepsilon)\right)\right)^{-\frac{1}{2}} dt\right)^{\frac{2}{2-\alpha}}. \tag{2.27}$$

Hence, (2.22) is also holding for $\alpha \in [1, 2)$. It follows that (2.5) are right by use of Hölder's inequality for $p \in [1, +\infty)$. □

3. Cauchy problem

In this section, we let $v(x)$ is the first eigenfunction of the eigenvalue problem (2.1), then (2.2) has only one positive eigenfunction. Motivated by the idea used in [12], set $\Omega = S_r = \{x : |x| < r\}$ in (2.2), and denote $v_r(x)$ as the only positive eigenfunction of the first eigenvalue λ_r, then $v_r(x)$ is also the function of $|x|$ only and λ_r decrease with r.

Assume that

1) $v_i(x) \geq 0$, $i = 1, 2$ has support in $S_{R-\delta}$ for some constant $\delta \in (0, R)$ and $v_i(x_i) > 0$ for some $x_i \in S_{R-\delta}$, $i = 1, 2$.

2) $\int_{S_{R-\delta}} v_r(x)v_i(x)dx = \zeta_i > 0$, $i = 1, 2$.

Set

$$\left(v(t,\cdot), u(t,\cdot)\right) = \int_{\mathbb{R}^n} v(t,x)u(t,x)dx. \tag{3.1}$$

Under the assumption 1), the solution $u(t,\cdot)$ is support in $S_{R-\delta+\gamma(t)}$ in view of the finite propagation property of hyperbolic equation [20], where

$$\gamma(t) = \frac{2}{2-\alpha}t^{\frac{2-\alpha}{2}}. \tag{3.2}$$

Now given $r > R$ and the exact value will be fixed later. Set $t \leq \gamma^{-1}(r - R)$, then $u(t, x)$ has support in $S_{R-\delta+\gamma(t)} \times \{t \leq \gamma^{-1}(r-R)\}$, which is contained in $S_{r-\delta} \times \{t \leq \gamma^{-1}(r-R)\}$. The condition $u(t, x) = 0$ given on the boundary $\partial\Omega$ in (1.2) is unnecessary in Cauchy problem.

Definition 3.1 (*cf.* [12, 21])**.** Call $u(t, x)$ is called a weak solution of Cauchy problem of Eq. (1.1) if for all twice continuously differentiable function $v(t, x)$ with respect to t satisfies

$$\begin{aligned} t^{\alpha}(v(t,\cdot), \partial_t u(t,\cdot)) = \int_0^t & [\alpha\tau^{\alpha-1}(v(\tau,\cdot), \partial_\tau u(\tau,\cdot)) + \tau^{\alpha}(\partial_\tau v(\tau,\cdot), \partial_\tau u(\tau,\cdot)) \\ & + (\triangle v(\tau,\cdot), u(\tau,\cdot)) + (v(\tau,\cdot), f(u(\tau,\cdot)))]d\tau \end{aligned} \tag{3.3}$$

and $u(t,\cdot)$ and $\partial_t u(t,\cdot)$ are continuous with respect to t.

Theorem 3.2 (*cf.* [12, 21])**.** *Prescribed $u(t,x)$ is a weak solution in the sense of Definition 3.1 of Cauchy problem* (1.1)-(1.2) *with nonlinear term satisfies* (1.4)*, then there exists*

$$\lim_{t\to T^-} \|u(t,\cdot)\|_{L^q(\mathbb{R}^n)} = +\infty, \quad q \in [1,+\infty], \tag{3.4}$$

for some positive number T.

Proof. Define a function $V_r(x)$, that is

$$V_r(x) = \begin{cases} v_r(x), & |x| \le r - \frac{\delta}{2} \\ v_0(x), & r - \frac{\delta}{2} \le |x| \le r \\ 0, & |x| \ge r, \end{cases} \tag{3.5}$$

where v_r is the first eigenfunction of eigenvalue problem (6), v_0 is a function of $r = |x|$ and twice continuously differentiable, which satisfies $v_0^i(r - \frac{\delta}{2}) = v_r^i(r - \frac{\delta}{2})$ and $v_0^i(r) = 0$ for $i = 0, 1, 2$. Since the support of $u(t,x)$ contained in $S_{r-\delta} \times [0, \gamma^{-1}(r-R))$ for $t < \gamma^{-1}(r-R)$, then substitute $v(t,x)$ by $V_r(t,x)$ in (3.3), we obtain

$$t^\alpha(v_r(\cdot), \partial_t u(t,\cdot)) = \int_0^t \Big(\alpha\tau^{\alpha-1}(v_r(\cdot), \partial_\tau u(\tau,\cdot)) + (\triangle v_r(\cdot), u(\tau,\cdot)) + (v_r(\cdot), f(u(\tau,\cdot)))\Big)\, d\tau. \tag{3.6}$$

Under the assumption, set

$$G_r(t) = (v_r(x), u(t,x)) = \int_{\mathbb{R}^n} v_r(x)u(t,x)dx, \tag{3.7}$$

then we have

$$G_r(t) = \int_{B_{r-\delta}} v_r(x)u(t,x)dx, \tag{3.8}$$

$$G_r'(t) = (\psi_r(x), \partial_t u(t,x)) = \int_{B_{r-\delta}} \psi_r(x)\partial_t u(t,x)dx. \tag{3.9}$$

According to (3.6) and (3.9), it yields that

$$t^\alpha G_r'(t) = \int_0^t \Big(\alpha\tau^{\alpha-1}(v_r(\cdot), \partial_\tau u(\tau,\cdot)) + (\triangle v_r(\cdot), u(\tau,\cdot)) + (v_r(\cdot), f(u(\tau,\cdot)))\Big)\, d\tau. \tag{3.10}$$

Therefore, we obtain

$$\begin{aligned} G_r''(t) &= (t^{-\alpha}(t^\alpha G_r'(t)))' \\ &= t^{-\alpha}((v_r(x), f(u(t,x))) - \lambda_r(v_r(x), u(t,x))). \end{aligned} \tag{3.11}$$

By use of Jensen inequality and the convexity of $f(\cdot)$, we arrive at

$$\begin{aligned} (v_r(x), f(u(t,x))) &= \int_{\mathbb{R}^n} v_r(x) f(u(t,x))dx \\ &\ge f\left(\int_{\mathbb{R}^n} v_r(x)u(t,x)dx\right) \\ &= f(G_r(t)). \end{aligned} \tag{3.12}$$

Then substitute (3.12) into (3.11), together with (4) for $\lambda_r = \mu$, we obtain

$$\begin{aligned} G_r''(t) &\ge t^{-\alpha}\left(f(\psi_r(x), u(t,x)) - \lambda_r(\psi_r(x), u(t,x))\right) \\ &\ge \nu t^{-\alpha} G_r^{1+q}(t). \end{aligned} \tag{3.13}$$

Note that $G_r(0) = \zeta_1 > 0$ and $G_r'(0) = \zeta_2 > 0$, then the monotonicity confirm that $G_r(t) > \zeta_1$ in some interval $(0, \eta)$. Furthermore, we have

$$G_r''(t) \geq \nu t^{-\alpha} G_r^{1+q}(t) > 0, \quad t \in (0, \eta). \tag{3.14}$$

This means that $G_r'(t)$ is strictly monotonic increasing on $(0, \eta)$. On the other hand, $G_r'(0) = \zeta_2 > 0$. Then we obtain $G_r'(t) > \zeta_2 > 0$ on $(0, \eta)$. Multiply $G_r'(t)$ and then integrate (3.14) over (ε, t) for $t < \eta$

$$(G_r'(t))^2 \geq (G_r'(\varepsilon))^2 + 2\nu \int_\varepsilon^t \tau^{-\alpha} G_r^{1+q}(\tau) G_r'(\tau) d\tau. \tag{3.15}$$

It follows from (3.14) and (3.15) that $G_r'(t) > \zeta_2$ and $G_r(t) > \zeta_1$ on the whole existence interval.

Now for $\varepsilon > 0$, we set

$$H_r(t) = \int_\varepsilon^t \tau^{-\alpha} G_r^{1+q}(\tau) G_r'(\tau) d\tau, \tag{3.16}$$

$$F_r(t) = (2+q)^{-1} G_r^{2+q}(t). \tag{3.17}$$

Then there $H_r'(t) = t^{-\alpha} F_r'(t)$ holds and $H_r(t)$, $F_r(t)$ are both positive. Combining to (3.13)-(3.15), we derive that

$$(H_r'(t))^2 \geq p_\varepsilon^2 \left((G_r'(\varepsilon))^2 + 2\nu H_r(t) \right) t^{-2\alpha} H_r^{2d}(t) \tag{3.18}$$

which implies

$$\left(\varepsilon_1^2 + 2\nu H(t) \right)^{-\frac{1}{2}} H^{-d}(t) H'(t) \geq p_\varepsilon t^{-\alpha} \tag{3.19}$$

with $\varepsilon_1 = \sup \left\{ G_r'(t) | t < \gamma^{-1}(r - R), t = \varepsilon \right\}$.

Then, for $\alpha \in (0, 1)$, we obtain

$$\int_{H_r(\varepsilon)}^{H_r(t)} \left(\varepsilon_1^2 + 2\nu H(\tau) \right)^{-\frac{1}{2}} H^{-d}(\tau) dH_r(\tau) \geq p_\varepsilon \left(t^{1-\alpha} - \varepsilon^{1-\alpha} \right). \tag{3.20}$$

By a direct computation yields

$$\int_{H_r(\varepsilon)}^{H_r(t)} \left(\varepsilon_1^2 + 2\nu H_r(\tau) \right)^{-\frac{1}{2}} H_r^{-d}(\tau) dH_r(\tau) < \int_0^{+\infty} \left(\varepsilon_1^2 + 2\nu t \right)^{-\frac{1}{2}} t^{-d} dt. \tag{3.21}$$

Set $C_0 = \int_0^{+\infty} \left(\varepsilon_1^2 + 2\nu t \right)^{-\frac{1}{2}} t^{-d} dt$, then a finite positive number C_0 is independent of r, and

$$p_\varepsilon \left(t^{1-\alpha} - \varepsilon^{1-\alpha} \right) < C_0. \tag{3.22}$$

Now fix ε, we choose

$$r = r_0 = \left(C_0 (2+q)^{-d} + \varepsilon^{1-\alpha} \right)^{\frac{1}{1-\alpha}} + 2R, \tag{3.23}$$

then for all $t \in (\varepsilon, T) \cap (0, r_0 - R)$, we obtain

$$t < r_0 - 2R < r_0 - R, \tag{3.24}$$

where (0, T) is the whole existence interval of solution of problem (1.1)-(1.2). This means $T \leq r_0 - 2R$ and

$$\lim_{t \to T^-} H(t) = +\infty. \tag{3.25}$$

Similarly as deriving (2.21), we have

$$\lim_{t \to T^-} G(t) = +\infty. \tag{3.26}$$

Hence, by Hölder inequality, we derive (3.4) for $\alpha \in (0, 1)$.

For $\alpha \in [1, 2)$, by taking a similar procedure in proof of (2.5), we also obtain (2.25), then integrate over (ε, t)

$$\int_{G_r(\varepsilon)}^{G_r(t)} \left(\varepsilon^\alpha (G_r'(\varepsilon))^2 + \frac{2\nu}{2+q} \left(G_r^{2+q}(t) - G_r^{2+q}(\varepsilon) \right) \right)^{-\frac{1}{2}} dG_r(t) > \frac{2-\alpha}{2} \left(t^{\frac{2-\alpha}{2}} - \varepsilon^{\frac{2-\alpha}{2}} \right). \tag{3.27}$$

Set $C_1 = \int_{\zeta_1}^{+\infty} \left(\varepsilon^{\alpha}\zeta_2^2 + \frac{2\nu}{2+q}\left(t^{2+q} - \zeta_1^{2+q}\right)\right)^{-\frac{1}{2}} dt$, C_1 is a finite positive constant that independent of r and ε, then it follows (3.27) that

$$C_1 > \frac{2-\alpha}{2}\left(t^{\frac{2-\alpha}{2}} - \varepsilon^{\frac{2-\alpha}{2}}\right), \tag{3.28}$$

for $\alpha \in [1, 2)$. Then, the inequality (3.28) implies that there is a $T \in (0, r-2R) \subset (0, r-R)$ such that (3.28) is holding for

$$r = r_1 = \left(\frac{2}{2-\alpha}C_1 + \varepsilon^{\frac{2-\alpha}{2}}\right)^{\frac{2}{2-\alpha}} + 2R,$$

then for all $t \in (\varepsilon, T) \cap (0, r_1 - R)$, we obtain

$$t < r_1 - 2R < r_1 - R,$$

where (0, T) is the whole existence interval of solution of problem (1.1)-(1.2). This implies $T \leq r_1 - 2R$ and

$$\lim_{t \to T^-} H(t) = +\infty.$$

Then, we obtain

$$\lim_{t \to T^-} G(t) = +\infty.$$

Hence, by Hölder inequality, we derive (3.4) for $\alpha \in [1, 2)$.

Finally, we proved Theorem 3.2. □

4. Conclusion

The author showed the local existence of weak solution of the second order hyperbolic equations with characteristic degeneration on the initial hyperplane and the global existence of solution if the source term with a decay at infinity in [20]. Without this constraint, we further obtain nonexistence of global weak solution of semilinear degenerate hyperbolic equation of the second kind based on choosing suitable test functions and analyzing the first Fourier coefficient of solution.

Acknowledgments

The author is grateful to the anonymous referees for their valuable comments and the support of Qinglan Project of Jiangsu Province.

Author Contributions: The author conceived and designed the work.

Conflict of Interest: The work does not have any conflicts of interests.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] P. Brenner and W. von Wahl, *Global classical solutions of nonlinear wave equations*, Math. Z. **176**, 87–121, 1981.
[2] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 1., CUP Archive, 1966.
[3] M. D'Abbicco, *The threshold of effective damping for semilinear wave equations*, Math. Methods Appl. Sci. **38 (6)**, 1032–1045, 2015.
[4] V. Georgiev, H. Lindblad and C. D. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **119 (6)**, 1291–1319, 1997.
[5] R. T. Glassey, *Blow-up theorems for nonlinear wave equations*, Math. Z. **132 (3)**, 183–203, 1973.
[6] M. G. Grillakis, *Regularity and asymptotic behavior of the wave equation with a critical nonlinearity*, Ann. of Math. **132 (3)**, 485–509, 1990.
[7] K. Jörgens, *Das Anfangswert problem in Grossen für eine klasse nichtlinearer Wellengleichungen*, Math. Z. **77 (1)**, 295–308, 1961.
[8] K. Jörgens, *Nonlinear wave equations*, University of Colorado, Department of Mathematics, 1970.

[9] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. **16 (3)**, 305–330, 1963.
[10] J. B. Keller, *On solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **10 (4)**, 523–530, 1957.
[11] P. Korman, *On blow up of solutions of nonlinear evolution equations*, Proc. Amer. Math. Soc. **103 (1)**, 189–197, 1988.
[12] H. A. Levine, *On the nonexistence of global solutions to a nonlinear Euler-Poisson-Darboux equation*, J. Math. Anal. Appl. **48 (3)**, 646–651, 1974.
[13] H. A. Levine, *Growth of solutions of generalized nonlinear Euler-Poisson-Darboux equations*, Arch. Ration. Mech. Anal. **61 (1)**, 77–89, 1976.
[14] M.-R. Li, *Existence and uniqueness of local weak solutions for the Emden-Fowler wave equation in one dimension*, Electron. J. Differ. Equ. **2015 (145)**, 1–10, 2015.
[15] M.-R. Li, *Nonexistence of global solutions of Emden-Fowler type semilinear wave equations with nonpositive energy*, Electron. J. Differ. Equ. **2016 (93)**, 1–10, 2016.
[16] D. Li and H. Huang, *Blow-up phenomena of second-order nonlinear differential equations*, J. Math. Anal. Appl. **276 (1)**, 184–195, 2002.
[17] Y. Wakasugi, *Critical exponent for the semilinear wave equation with scale invariant damping*, In: Fourier Analysis, Trends in Mathematics (Ed. by M. Ruzhansky and V. Turunen), Birkhäuser, Cham., 375–390, 2014.
[18] J. Wirth, *Solution representations for a wave equation with weak dissipation*, Math. Methods Appl. Sci. **27 (1)**, 101–124, 2004.
[19] B. Yordanov and Q. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. **231 (2)**, 361–374, 2006.
[20] K. Zhang, *The Cauchy problem for semilinear hyperbolic equation with characteristic degeneration on the initial hyperplane*, Math. Methods Appl. Sci. **41 (6)**, 2429–2441, 2018.
[21] K. Zhang, *Nonexistence of global weak solutions of nonlinear Keldysh type equation with one derivative term*, Adv. Math. Phys. **2018**, 2018; Article ID 3931297.
[22] K. Zhang, *Applications of Erdélyi-Kober fractional integral for solving time-fractional Tricomi-Keldysh type equation*, Fract. Calc. Appl. Anal. **23 (5)**, 1381–1400, 2020.