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Uniform Convexity, N-quasisuperquadracity, N-quasiconvexity and Extensions of the Euler-Lagrange Identity

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Abstract

Using convexity, ψ-uniformly convexity, N-quasiconvexity and N-quasisuperquadracity we extend and refine inequalities related to the Euler-Lagrange identity.

Keywords: Euler-Lagrange identity, Convexity, ψ-uniformly convexity, N-quasiconvexity, Superquadracity, N-quasisuperquadracity

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1. Introduction

In [3, Theorem 3] inequalities related to the Euler-Lagrange identity are proved. The first of which is the inequality

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \frac{\left(\sum_{i=1}^{n} a_i x_i\right)^p}{\lambda}, \quad \lambda \geq \overline{\lambda} = \left(\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q\right)^{p-1}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.1}$$

where $x_i \geq 0$, $\mu_i > 0$, $a_i \geq 0$, $i = 1, ..., n$, $p > 1$, $\lambda > 0$, $\sum_{i=1}^{n} a_i > 0$. This inequality is proved there by using the convexity of the function $f(x) = x^p$, $x \geq 0$, $p \geq 1$. The same inequality was proved earlier in [6] without using the convexity of $f(x) = x^p$, $x \geq 0$, $p > 1$.

Here we prove generalisations of Inequality (1.1), and obtain the Euler-Lagrange identity and other similar identities as special cases. We get these inequalities and identities by using the so called ψ-uniformly convex functions, N-quasiconvex functions and N-quasisuperquadratic functions which are closely related to convex functions and superquadratic functions, and by dealing with other convex functions besides $f(x) = x^p$.

We start with some definitions, theorems and corollaries used in the sequel.

Definition 1.1 (*cf.* [5, 7])**.** Let $I = [a, b] \subset \mathbb{R}$ be an interval and $\psi : [0, b-a] \to \mathbb{R}$ be a function. A function $f : [a, b] \to \mathbb{R}$ is said to be **generalized ψ-uniformly convex** if:

$$tf(x) + (1-t) f(y) \geq f(tx + (1-t) y) + t(1-t) \psi(|x-y|)$$
$$\text{for } x, y \in I \text{ and } t \in [0, 1].$$

If in addition $\psi \geq 0$, then f is said to be **ψ-uniformly convex**.

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Remark 1.2 (*cf.* [5]). The inequality

$$f(y) \geq f(x) + f'(x)(y-x) + \psi(|y-x|)$$

holds for ψ-uniformly convex function f, as well as the inequality

$$\sum_{i=1}^{m} p_i f(\xi_i) - f(\bar{\xi}) \geq \sum_{i=1}^{m} p_i \psi\left(\left|\xi_i - \bar{\xi}\right|\right),$$

when $a \leq \xi_i \leq b$, $i = 1, ..., m$, and $\bar{\xi} = \sum_{i=1}^{m} p_i \xi_i$, $p_i \geq 0$, $i = 1, ..., m$, and $\sum_{i=1}^{m} p_i = 1$.

Lemma 1.3 (*cf.* [2, Lemma 3]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function and let $0 \leq a < b < \infty$. Then for $m = 2, 3, ...,$ and $0 \leq t \leq 1$ we get that*

$$(1-t) f^m(a) + t f^m(b) - ((1-t) f(a) + t f(b))^m \geq t(1-t)(f(b) - f(a))^m \geq 0.$$

In particular if $f(x) = x$ the inequalities

$$(1-t) a^m + t b^m - ((1-t) a + t b)^m \geq t(1-t)(b-a)^m \geq 0 \tag{1.2}$$

hold. Hence x^m, $x \geq 0$, $m = 2, 3, ...$ is ψ-uniformly convex on $[a, b]$ where $\psi = f$.

Definition 1.4 (*cf.* [4, Definition 2.1]). A function $\varphi : [0, B) \to \mathbb{R}$ is superquadratic provided that for all $x \in [0, B)$ there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that the inequality

$$\varphi(y) \geq \varphi(x) + C_\varphi(x)(y-x) + \varphi(|y-x|) \tag{1.3}$$

holds for all $y \in [0, B)$, (see [4, Definition 2.1], there $[0, \infty)$ instead $[0, B)$).

φ is called subquadratic if $-\varphi$ is superquadratic.

Corollary 1.5 (*cf.* [4, Lemma 2.1]). *When φ is a differentiable, non-negative superquadratic function, it is also convex, $C_\varphi(x) = \varphi'(x)$ and $\varphi'(0) = \varphi(0) = 0$. In particular, the functions $\varphi(x) = x^p$, $x \geq 0$ are superquadratic for $p \geq 2$, subquadratic for $0 < p \leq 2$, for which $C_\varphi(x) = p x^{p-1} = \varphi'(x)$. When $p = 2$, (1.3) is an equality.*

Corollary 1.6 (*cf.* [4]). *Suppose that f is superquadratic. Let $0 \leq \xi_i \leq B$, $i = 1, ..., m$, and let $\bar{\xi} = \sum_{i=1}^{m} p_i \xi_i$ where $p_i \geq 0$, $i = 1, ..., m$, and $\sum_{i=1}^{m} p_i = 1$. Then*

$$\sum_{i=1}^{m} p_i f(\xi_i) - f(\bar{\xi}) \geq \sum_{i=1}^{m} p_i f\left(\left|\xi_i - \bar{\xi}\right|\right),$$

and in the special case that $m = 2$, $0 \leq \lambda \leq 1$ and $0 \leq x, y < B \leq \infty$

$$(1-\lambda) f(y) + \lambda f(x) \geq f((1-\lambda) y + \lambda x) + (1-\lambda) f(\lambda |x-y|) + \lambda f((1-\lambda)|x-y|)$$

hold.

Definition 1.7 (*cf.* [1, Definition 3]). A function $\phi_N : [0, B) \to \mathbb{R}$ is N-quasiconvex, where $N \in \mathbb{R}$, provided that for all $x \in [0, B)$, $\phi_N(x) = x^N \varphi(x)$, where φ is a convex function on $[0, B)$.

Definition 1.8 (*cf.* [1, Definition 2]). A function $\phi_N : [0, B) \to \mathbb{R}$ is N-quasisuperquadratic, where $N \in \mathbb{R}$, provided that for all $x \in [0, B)$, $\phi_N(x) = x^N \varphi(x)$, where φ is a superquadratic function on $[0, B)$.

Euler-Lagrange type inequalities and identities are proved in [3] by using the superquadracity of $f(x) = x^p$ when $p \geq 2$, and by using the 1-quasisuperquadracity when $p \geq 3$:

Theorem 1.9 (*cf.* [3, Theorem 4 and Corollary 1]). *Let $x_i \geq 0$, $a_i > 0$ and $\mu_i > 0$, $i = 1, ..., n$. Let $p, q \in \mathbb{R}$ with $p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \frac{\left(\sum_{i=1}^{n} a_i x_i\right)^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} + \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \left(\left|\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \sum_{j=1}^{n} a_j x_j\right|\right)^p. \tag{1.4}$$

If $1 < p \leq 2$, the reverse of (1.4) holds.

In particular, for $n = 2$, $p = 2$, the equality

$$\frac{x^2}{\mu} + \frac{y^2}{\nu} = \frac{(ax + by)^2}{\mu a^2 + \nu b^2} + \frac{(\nu bx - a\mu y)^2}{\mu\nu\left(\mu a^2 + \nu b^2\right)}, \tag{1.5}$$

is obtained, which is the Euler-Lagrange identity.

Theorem 1.10 (*cf.* [3, Theorem 4 and Corollary 3]). *Let $x_i \geq 0$, $a_i > 0$ and $\mu_i > 0$, $i = 1, ..., n$. Let $p, q \in \mathbb{R}$ with $p \geq 3$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq{}& \frac{\bar{x}^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} + \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \bar{x}\right)^2 (p-1)\bar{x}^{p-2} \\ &+ \frac{\sum_{i=1}^{n} a_i x_i}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \left(\left|\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \bar{x}\right|\right)^{p-1},\end{aligned} \tag{1.6}$$

where $\bar{x} = \sum_{j=1}^{n} a_j x_j$.

If $2 < p \leq 3$, then the reverse of (1.6) holds.

In particular, for $p = 3$ the Euler-Lagrange type identity:

$$\frac{x^3}{\mu} + \frac{y^3}{\nu} = \frac{(ax + by)^3}{\left(\mu^{\frac{1}{2}} a^{\frac{3}{2}} + \nu^{\frac{1}{2}} b^{\frac{3}{2}}\right)^2} + \left(\frac{ab}{\mu\nu}\right)^{\frac{1}{2}} \left(\frac{\nu^{\frac{1}{2}} b^{\frac{1}{2}} x - \mu^{\frac{1}{2}} a^{\frac{1}{2}} y}{\mu^{\frac{1}{2}} a^{\frac{3}{2}} + \nu^{\frac{1}{2}} b^{\frac{3}{2}}}\right)^2 2(ax + by) + \frac{\nu b^2 x + \mu a^2 y}{\mu\nu} \left(\frac{\nu^{\frac{1}{2}} b^{\frac{1}{2}} x - \mu^{\frac{1}{2}} a^{\frac{1}{2}} y}{\mu^{\frac{1}{2}} a^{\frac{3}{2}} + \nu^{\frac{1}{2}} b^{\frac{3}{2}}}\right)^2 \tag{1.7}$$

is obtained.

In Section 2, we prove more Euler-Lagrange type inequalities and show examples to illuminate the results:

In Subsection 2.1, using ψ-uniformly convexity, we extend the Euler-Lagrange type inequalities and obtain as a special case the Euler-Larange Identity (1.5).

In Subsection 2.2, using N-quasiconvexity, we extend the Euler-Lagrange inequality of (1.1).

In Subsection 2.3 using N-quasisuperquadracity, we extend the Euler-Lagrange type inequalities and identities of Theorem 1.9 and Theorem 1.10.

In Subsection 2.4, we deal with examples of convex functions other than power functions satisfying Euler-Lagrange type inequalities.

In Section 3 we present Euler-Lagrange and Jensen type inequality for non-convex and non-superquadratic functions.

2. More on Euler-Lagrange type inequalities

2.1. Euler-Lagrange type inequalities through ψ-uniformly convex functions

In this subsection we deal with Euler-Lagrange type inequalities for ψ-uniformly convex functions, that is, with the functions f which satisfy Definition 1.1 where $\psi \geq 0$ on $x \geq 0$.

Theorem 2.1. *Let $x_i \geq 0$, $a_i > 0$ and $\mu_i > 0$, $i = 1, 2$ be given. Let f be a non-negative strictly increasing function on $[0, \infty)$, and let the inverse of f satisfy*

$$f^{-1}(A) f^{-1}(B) \leq f^{-1}(AB), \quad A, B, AB \geq D \geq 0. \tag{2.1}$$

Let f be a ψ-uniformly convex function. Then

$$\frac{f(x_1)}{\mu_1} + \frac{f(x_2)}{\mu_2} \geq \frac{f(a_1 x_1 + a_2 x_2)}{\lambda} + \frac{Q_1 Q_2}{Q_1 + Q_2} \psi\left(\left| x_1 f^{-1}\left(\frac{1}{\mu_1 Q_1}\right) - x_2 f^{-1}\left(\frac{1}{\mu_2 Q_2}\right)\right|\right), \tag{2.2}$$

where Q_1 and Q_2 are the solutions of:

$$a_1 = \frac{Q_1}{Q_1 + Q_2} f^{-1}\left(\frac{1}{\mu_1 Q_1}\right), \qquad a_2 = \frac{Q_2}{Q_1 + Q_2} f^{-1}\left(\frac{1}{\mu_2 Q_2}\right), \qquad \text{and} \qquad \lambda = (Q_1 + Q_2)^{-1}. \tag{2.3}$$

Proof. The function f is strictly increasing, therefore we can rewrite $\sum_{i=1}^{2} \frac{f(x_i)}{\mu_i}$ as

$$\sum_{i=1}^{2} \frac{f(x_i)}{\mu_i} = \sum_{i=1}^{2} Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right). \tag{2.4}$$

Because f satisfies (2.1) and is increasing, we get

$$Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right) \geq Q_i f\left(x_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right), \qquad i = 1, 2, \tag{2.5}$$

holds. Then by (2.5) and by the ψ-uniformly convexity of f, as defined in Definition 1.1, we get that

$$\begin{aligned}\sum_{i=1}^{2} Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right) \quad &\geq \quad \left(\sum_{j=1}^{2} Q_j\right) f\left(\frac{\sum_{i=1}^{2} x_i Q_i f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)}{\sum_{j=1}^{2} Q_j}\right) \\ &+ \quad \frac{Q_1 Q_2}{Q_1 + Q_2} \psi\left(\left| x_1 f^{-1}\left(\frac{1}{\mu_1 Q_1}\right) - x_2 f^{-1}\left(\frac{1}{\mu_2 Q_2}\right)\right|\right).\end{aligned} \tag{2.6}$$

Therefore, from (2.4), (2.5) and (2.6) it is enough to solve the equality

$$\left(\sum_{j=1}^{2} Q_j\right) f\left(\frac{\sum_{i=1}^{2} x_i Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{2} Q_j}\right) = \frac{f\left(\sum_{i=1}^{2} a_i x_i\right)}{\lambda},$$

in other words to solve

$$\frac{Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{2} Q_j} = a_i, \qquad i = 1, 2 \tag{2.7}$$

and then insert

$$\lambda = \left(\sum_{j=1}^{2} Q_j\right)^{-1} \tag{2.8}$$

and get that Inequality (2.2) holds when (2.3) is satisfied.

The proof of the theorem is complete. □

For the special case when $f(x) = x^m$, $x \geq 0$, $m = 2, 3, ...$, which according to Lemma 1.3 f is ψ-uniformly convex function and $f = \psi$, we get the following corollary:

Corollary 2.2. *Let $f(x) = x^m$, $m = 2, 3, ...,$ $x \geq 0$. Then, from Theorem* 2.1 *we get that the inequality*

$$\frac{x_1^m}{\mu_1} + \frac{x_2^m}{\mu_2} \geq \frac{(a_1 x_1 + a_2 x_2)^m}{\lambda} + \frac{(\mu_1 \mu_2)^{-1}}{\left(\mu_1^{\frac{1}{m-1}} a_1^{\frac{m}{m-1}} + \mu_2^{\frac{1}{m-1}} a_2^{\frac{m}{m-1}}\right)} \left(\left| x_1 \mu_1^{\frac{1}{m-1}} a_1^{\frac{1}{m-1}} - x_2 \mu_2^{\frac{1}{m-1}} a_2^{\frac{1}{m-1}} \right|\right)^m \tag{2.9}$$

holds. In particular when $m = 2$ we get the Euler-Lagrange identity (1.5) *holds.*

Inequality (2.9) *is obtained by solving* (2.7) *from which we get that*

$$Q_i = \frac{\mu_i^{\frac{1}{m-1}} a_i^{\frac{m}{m-1}}}{\left(\sum_{j=1}^2 \mu_j^{\frac{1}{m-1}} a_j^{\frac{m}{m-1}}\right)^m}, \quad i = 1, 2, \tag{2.10}$$

and from (2.8) *that*

$$\overline{\lambda} = \left(\sum_{i=1}^2 Q_i\right)^{-1} = \left(\sum_{i=1}^2 \mu_i^{\frac{1}{m-1}} a_i^{\frac{m}{m-1}}\right)^{m-1}. \tag{2.11}$$

2.2. *Euler-Lagrange type inequalities through N-quasiconvex functions*

In this subsection we deal with Euler-Lagrange type inequalities for N-quasiconvex functions, that is, with the functions ϕ which satisfy $\phi(x) = x^N \varphi(x)$ where φ are convex on $x \geq 0$.

We start this subsection with a basic theorem on N-quasiconvex function:

Theorem 2.3 (*cf.* [1, Theorem 1])**.** *Let $\varphi : [a, b) \to \mathbb{R}$, $0 \leq a < b \leq \infty$ be a convex differentiable function, and let the k-quasiconvex functions $\psi_k(x)$ be $\psi_k(x) = x^k \varphi(x)$, $k = 0, 1, ..., N$, where $\psi_0 = \varphi$. Let $\theta_i \geq 0$, $A_i \in [a, b)$, $i = 1, ..., n$, $\sum_{i=1}^n \theta_i > 0$ and $\overline{A} = \sum_{i=1}^n \frac{\theta_i A_i}{\sum_{j=1}^n \theta_j}$. Then a Jensen's type inequality holds, where:*

$$\begin{aligned} \sum_{i=1}^n \theta_i \psi_N(A_i) - \sum_{j=1}^n \theta_j \psi_N\left(\overline{A}\right) &\geq \sum_{i=1}^n \sum_{k=1}^N \theta_i \left(A_i - \overline{A}\right)^2 A_i^{k-1} \left(\psi_{N-k}\left(\overline{A}\right)\right)' \\ &= \sum_{i=1}^n \theta_i \left(A_i - \overline{A}\right)^2 \frac{\partial}{\partial \overline{A}} \left(\frac{\overline{A}^N - A_i^N}{\overline{A} - A_i} \varphi\left(\overline{A}\right)\right). \end{aligned} \tag{2.12}$$

If φ is also non-negative and increasing then for $N = 2, ...,$ the above inequality refines Jensen's inequality.

For $N = 1$ we get

$$\sum_{i=1}^n \theta_i \psi_1(A_i) - \sum_{j=1}^n \theta_j \psi_1\left(\overline{A}\right) \geq \sum_{i=1}^n \theta_i \varphi'\left(\overline{A}\right) A_i \left(A_i - \overline{A}\right) = \sum_{i=1}^n \theta_i \varphi'\left(\overline{A}\right)\left(A_i - \overline{A}\right)^2. \tag{2.13}$$

In particular if $N = 1$, and $n = 2$:

$$\theta_1 \psi_1(A_1) + \theta_2 \psi_1(A_2) - (\theta_1 + \theta_2) \psi_1\left(\frac{\theta_1 A_1 + \theta_2 A_2}{\theta_1 + \theta_2}\right) \geq \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \varphi'\left(\frac{\theta_1 A_1 + \theta_2 A_2}{\theta_1 + \theta_2}\right)(A_2 - A_1)^2. \tag{2.14}$$

From Theorem 2.3 together with condition (2.15) we derive:

Theorem 2.4. *Let $\psi_N(x)$ be a non-negative strictly increasing function on $[0, \infty)$, and let the inverse of ψ_N be ψ_N^{-1}, defined on $[D, \infty)$, and satisfies*

$$\psi_N^{-1}(A) \psi_N^{-1}(B) \leq \psi_N^{-1}(AB), \quad A, B, AB \geq D \geq 0. \tag{2.15}$$

Let ψ_N be an N-quasiconvex function, $N = 0, 1, 2...$. Then for given $\mu_i > 0$, $a_i \geq 0$, $i = 1, ..., n$ when $\overline{A} = \sum_{i=1}^{n} a_i x_i$, the inequality

$$\sum_{i=1}^{n} \frac{\psi_N(x_i)}{\mu_i} - \frac{\psi_N\left(\sum_{i=1}^{n} a_i x_i\right)}{\lambda} \geq \sum_{i=1}^{n}\sum_{k=1}^{N} \theta_i \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right) - \sum_{i=1}^{n} a_i x_i\right)^2 \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right)^{k-1} \frac{d\left(\psi_{N-k}\left(\overline{A}\right)\right)}{d\overline{A}} \tag{2.16}$$

holds where $\lambda \geq \overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1}$, and Q_i, $i = 1, ..., n$ are the solutions of

$$\frac{Q_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j} = a_i, \qquad i = 1, ..., n. \tag{2.17}$$

Proof. Rewrite

$$\sum_{i=1}^{n} \frac{\psi_N(x_i)}{\mu_i} = \sum_{i=1}^{n} Q_i \psi_N\left(\psi_N^{-1}\left(\frac{\psi_N(x_i)}{\mu_i Q_i}\right)\right). \tag{2.18}$$

From (2.15) we get that $\psi_N^{-1}\left(\frac{\psi_N(x_i)}{\mu_i Q_i}\right) \geq x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)$ and then, as ψ_N is increasing, the inequality

$$\sum_{i=1}^{n} Q_i \psi_N\left(\psi_N^{-1}\left(\frac{\psi_N(x_i)}{\mu_i Q_i}\right)\right) \geq \sum_{i=1}^{n} Q_i \psi_N\left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right) \tag{2.19}$$

$$= \sum_{i=1}^{n} Q_i \psi_N(A_i), \quad A_i = x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right), \quad i = 1, ..., n$$

holds. The use of inequality (2.12), which ψ_N satisfies, leads to

$$\begin{aligned} &\sum_{i=1}^{n} Q_i \psi_N\left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right) - \left(\sum_{j=1}^{n} Q_j\right) \psi_N\left(\frac{\sum_{i=1}^{n} x_i Q_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right) \\ &\geq \sum_{i=1}^{n}\sum_{k=1}^{N} \theta_i \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right) - \frac{\sum_{i=1}^{n} x_i Q_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right)^2 \\ &\times \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right)^{k-1} \frac{d\left(\psi_{N-k}\left(\overline{A}\right)\right)}{d\overline{A}}, \qquad \overline{A} = \frac{\sum_{i=1}^{n} x_i Q_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}. \end{aligned} \tag{2.20}$$

Therefore, it is enough to solve the n equations in Q_i, for the given a_i and μ_i, $i = 1, ..., n$.

$$\frac{x_i Q_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j} = a_i x_i, \quad i = 1, ..., n. \tag{2.21}$$

In other words to solve the equations (2.21) in Q_i, $i = 1, ..., n$ and then insert

$$\overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1}. \tag{2.22}$$

From (2.18), (2.19), (2.20), (2.21) and (2.22) we get that (2.16) is satisfied. □

Example 2.5. When $\theta, \eta, A, B > 0$ the 1-quasiconvex function $\phi(x) = x^p$, $x \geq 0$, $p \geq 2$, according to (2.14), satisfies the inequality

$$\theta A^p + \eta B^p \geq (\theta + \eta)\left(\frac{\theta A + \eta B}{\theta + \eta}\right)^p + (p-1)(\theta + \eta)\left(\frac{\theta A + \eta B}{\theta + \eta}\right)^{p-2} \frac{\theta\eta(B-A)^2}{(\theta + \eta)^2}.$$

In other words, in the case that we deal with $\frac{\phi_1(x)}{\mu} + \frac{\phi_1(y)}{\nu}$ for the 1-quasiconvex function $\phi(x) = x^p$, $x \geq 0$, $p \geq 2$, we get that (2.15) is an equality and

$$\frac{x^p}{\mu} + \frac{y^p}{\nu} = \theta A^p + \eta B^p \geq (\theta+\eta)(ax+by)^p + (\theta+\eta)(ax+by)^{p-2}\left[(p-1)\frac{\theta\eta(B-A)^2}{(\theta+\eta)^2}\right]$$

where $A = x(\mu\theta)^{-\frac{1}{p}}$, $B = y(\nu\eta)^{-\frac{1}{p}}$.

To get θ and η we solve the two equations $a = \frac{\theta^{1-\frac{1}{p}}\mu^{-\frac{1}{p}}}{\theta+\eta}$, $b = \frac{\eta^{1-\frac{1}{p}}\nu^{-\frac{1}{p}}}{\theta+\eta}$, get that $\lambda = \left(\mu^{\frac{1}{p-1}}a^q + \nu^{\frac{1}{p-1}}b^q\right)^{1-p} = (\theta+\eta)^{-1}$ and that:

$$\frac{x^p}{\mu} + \frac{y^p}{\nu} \geq \frac{(ax+by)^p}{\lambda} + \frac{(ax+by)^{p-2}}{\lambda}\left[(p-1)\frac{\left(y(a\mu)^{\frac{q}{p}} - x(b\nu)^{\frac{q}{p}}\right)^2}{(ab)^{q-2}(\mu\nu)^{\frac{1}{p-1}}}\right].$$

In the special case where $p = 2$ we get the Euler-Lagrange identity:

$$\frac{x^2}{\mu} + \frac{y^2}{\nu} = \frac{(ax+by)^2}{\mu a^2 + \nu b^2} + \frac{(ya\mu - xb\nu)^2}{\mu\nu(\mu a^2 + \nu b^2)}.$$

Remark 2.6. It is obvious that we get an identity in the Euler-Lagrange type inequalities in case that the $(N-1)$-quasiconvex functions $\phi_{N-1}(x) = x^{N-1}\varphi(x)$ where $\varphi(x) = x$. In other words $\phi_{N-1}(x) = x^N$ for $N = 1, 2, \ldots$.

2.3. Euler-Lagrange type inequalities through N-quasisuperquadratic functions

In this subsection we deal with Euler-Lagrange type inequalities for N-quasisuperquadratic functions, that is, with the functions ψ_N which satisfy $\psi_N(x) = x^N\varphi(x)$ where φ are superquadratic functions on $x \geq 0$.

We start this subsection with a basic result on N-quasisuperquadratic function. This result is analogous to the result of Theorem 2.3 (see [1, Remark 1]):

Theorem 2.7. *Let φ be a superquadratic function and $\psi_N(x) = x^N\varphi(x)$, that is ψ_N is an N-quasisuperquadratic function and $N = 1, 2, \ldots$. Then,*

$$\begin{aligned}\sum_{i=1}^n \theta_i\psi_N(A_i) - \sum_{i=1}^n \theta_i\psi_N\left(\overline{A}\right) &\geq \sum_{i=1}^n\sum_{k=1}^N \theta_i\left(A_i - \overline{A}\right)^2 A_i^{k-1}\left(\psi_{N-k}\left(\overline{A}\right)\right)' + \sum_{i=1}^n \theta_i A_i^N\varphi\left(\left|A_i - \overline{A}\right|\right)\\ &= \sum_{i=1}^n \theta_i\left(A_i - \overline{A}\right)^2\frac{\partial}{\partial\overline{A}}\left(\frac{\overline{A}^N - A_i^N}{\overline{A} - A_i}\varphi\left(\overline{A}\right)\right) + \sum_{i=1}^n \theta_i A_i^N\varphi\left(\left|A_i - \overline{A}\right|\right),\end{aligned}$$

where A_i, $\theta_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n \theta_i > 0$, $\overline{A} = \sum_{i=1}^n \frac{\theta_i A_i}{\sum_{j=1}^n \theta_j}$. If φ is also non-negative then the above inequality refines Jensen's inequality.

In the special case that $n = 2$ we get

$$\begin{aligned}\theta\phi_N(A) + \eta\phi_N(B) - (\theta+\eta)\phi_N(\overline{x}) &\geq \theta\eta^2(A-B)^2\frac{\partial}{\partial\overline{x}}\left(\frac{\overline{x}^N - A^N}{\overline{x} - A}\varphi(\overline{x})\right) + \eta\theta^2(A-B)^2\frac{\partial}{\partial\overline{x}}\left(\frac{\overline{x}^N - B^N}{\overline{x} - B}\varphi(\overline{x})\right)\\ &+ \theta A^N\varphi(\eta|A-B|) + \eta B^N\varphi(\theta|A-B|),\end{aligned}$$

where $\theta \geq 0$, $\eta \geq 0$, $\theta + \eta > 0$, A, $B \geq 0$ and $\overline{x} = \frac{\theta A + \eta B}{\theta+\eta}$.

From Theorem 2.7 together with the additional condition (2.23), similar to Theorem 2.4, and therefore omiting the proof, we get:

Theorem 2.8. *Let $\psi_N(x)$ be a non-negative strictly increasing function on $[0, \infty)$, and let ψ_N^{-1} be defined on $[D, \infty)$ and satisfies*

$$\psi_N^{-1}(A)\,\psi_N^{-1}(B) \leq \psi_N^{-1}(AB), \quad A, B, AB \geq D \geq 0. \tag{2.23}$$

Let ψ_N be an N-quasisuperquadratic function. Then for given $\mu_i > 0$ and $a_i > 0$, $i = 1, ..., n$ the inequality

$$\sum_{i=1}^{n} \frac{\psi_N(x_i)}{\mu_i} - \frac{\psi_N\left(\sum_{i=1}^{n} a_i x_i\right)}{\lambda} \geq \sum_{i=1}^{n}\sum_{k=1}^{N} \theta_i \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right) - \sum_{i=1}^{n} a_i x_i\right)^2 \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right)^{k-1} \frac{d\left(\psi_{N-k}\left(\overline{A}\right)\right)}{d\overline{A}} + \sum_{i=1}^{n} \theta_i \left(x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)\right)^N \varphi\left(\left|x_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right) - \sum_{i=1}^{n} a_i x_i\right|\right), \tag{2.24}$$

where $\lambda \geq \overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1}$, $\overline{A} = \sum_{i=1}^{n} a_i x_i$, holds, and Q_i, $i = 1, ..., n$ are the solutions of

$$\frac{Q_i \psi_N^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j} = a_i, \qquad i = 1, ..., n. \tag{2.25}$$

When $N = 0$, the inequality becomes

$$\sum_{i=1}^{n} \frac{\varphi(x_i)}{\mu_i} - \frac{\varphi\left(\sum_{i=1}^{n} a_i x_i\right)}{\lambda} \geq \sum_{i=1}^{n} Q_i \varphi\left(\left|x_i \varphi^{-1}\left(\frac{1}{\mu_i Q_i}\right) - \sum_{i=1}^{n} a_i x_i\right|\right),$$

where

$$\frac{Q_i \varphi^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j} = a_i, \qquad i = 1, ..., n, \quad \lambda \geq \overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1}.$$

Example 2.9. *Let $x_i \geq 0$, $a_i \geq 0$ and $\mu_i > 0$, $i = 1, ..., n$. Let $p, q \in \mathbb{R}$ with $p \geq 3$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} \sum_{i=1}^{n} \frac{x_i^p}{\mu_i} - \frac{\bar{x}^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \geq & \sum_{i=1}^{n}\sum_{k=1}^{N} \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q x_i - \bar{x}\right)^2 \\ & \times \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q x_i\right)^{k-1} (p-k)\bar{x}^{p-1-k} \\ & + \frac{\sum_{i=1}^{n} a_i x_i}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q x_i\right)^{p-N} \\ & \times \left(\left|\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q x_i - \bar{x}\right|\right)^N, \end{aligned} \tag{2.26}$$

where $\bar{x} = \sum_{j=1}^{n} a_j x_j$.

In particular if $p = N + 2$ then (2.26) is an equality which can be considered as an extended Euler-Lagrange identity. If $N = 0$ we get Euler-Lagrange identity (1.5) and if $N = 1$ we get an extended Euler-Lagrange identity in (1.7).

Remark 2.10. It is obvious that we get an identity in the Euler-Lagrange type inequalities in case that the $(N-2)$-quasisuperquadratic functions $\psi_{N-2}(x) = x^{N-2}\varphi(x)$ where $\varphi(x) = x^2$. In other words $\psi_{N-2}(x) = x^N$ for $N = 2, 3, ...$ The identities are the same as the identities in Remark 2.6.

2.4. Examples of Euler-Lagrange type inequalities for other than power functions

In this subsection we deal with a generalization of Euler-Lagrange type inequality where instead of the power function in $\frac{x^p}{\mu}+\frac{y^p}{\nu}\geq\frac{(ax+by)^p}{\lambda}$ we take a convex and strictly increasing function f that under an additional condition on f an inequality of the type $\sum_{i=1}^{n}\frac{f(x_i)}{\mu_i}\geq\frac{f\left(\sum_{i=1}^{n}a_ix_i\right)}{\lambda}$ is obtained.

Theorem 2.11 appears in [3, inequalities (1.7)-(1.13)] and used there only for power functions.

Theorem 2.11 (*cf.* [3])**.** *Let $f(x)$ be a non-negative strictly increasing convex function on $[0,\infty)$, and let f^{-1} be defined on $[D,\infty)$ and satisfies*

$$f^{-1}(A)\,f^{-1}(B)\leq f^{-1}(AB),\quad A,B,AB\geq D>0. \tag{2.27}$$

Then for given $\mu_i>0$ and $a_i\geq 0$, $i=1,...,n$ the inequality

$$\sum_{i=1}^{n}\frac{f(x_i)}{\mu_i}\geq\frac{f\left(\sum_{i=1}^{n}a_ix_i\right)}{\overline{\lambda}},\quad \lambda\geq\overline{\lambda}=\left(\sum_{j=1}^{n}Q_j\right)^{-1} \tag{2.28}$$

holds, where Q_i, $i=1,...,n$ are the solutions of

$$\frac{Q_if^{-1}\left(\frac{1}{\mu_iQ_i}\right)}{\sum_{j=1}^{n}Q_j}=a_i,\qquad i=1,...,n. \tag{2.29}$$

Example 2.12. It is obvious that when $f(x)=x^p$, $x\geq 0$, $p\geq 1$ Inequality (1.1) is satisfied. In this case inequality (2.27) become equalities.

Example 2.13. Let $f(x)=\left(x^2+1\right)^{\frac{1}{2}}$. This function is convex and strictly increasing on $(0,\infty)$. Its inverse function $f^{-1}(x)=\left(x^2-1\right)^{\frac{1}{2}}$, is defined on $[1,\infty)$. For this function $f^{-1}(A)\,f^{-1}(B)\leq f^{-1}(AB)$ holds. Therefore, following the computation in Theorem 2.11, we get for $n=2$ that the inequality

$$\frac{f(x_1)}{\mu_1}+\frac{f(x_2)}{\mu_2}\geq\frac{f(a_1x_1+a_2x)}{\overline{\lambda}},\quad \overline{\lambda}=(Q_1+Q_2)^{-1}$$

holds, where Q_i satisfies

$$a_i=\frac{Q_i\left(\mu_i^{-2}Q_i^{-2}-1\right)^{\frac{1}{2}}}{\sum_{i=1}^{2}Q_i},\quad i=1,2, \tag{2.30}$$

and we get that

$$\frac{\left(x_1^2+1\right)^{\frac{1}{2}}}{\mu_1}+\frac{\left(x_2^2+1\right)^{\frac{1}{2}}}{\mu_2}\geq\frac{\left((a_1x_1+a_2x_2)^2+1\right)^{\frac{1}{2}}}{\overline{\lambda}},$$

for all $x_i\in\mathbb{R}_+$, $i=1,2$ where $\overline{\lambda}=(Q_1+Q_2)^{-1}$. Q_i are the solutions of (2.30) for given $\mu_1>0,\mu_2>0,a_1\geq 0,a_2\geq 0$.

In a specific case where $\mu_1=\frac{1}{2},\mu_2=3,a_1=\frac{2}{3},a_2=\frac{4}{27}$ we obtain that $Q_1=\frac{8}{5}$, $Q_2=\frac{1}{5}$ and get that

$$\frac{\left(x_1^2+1\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+\frac{\left(x_2^2+1\right)^{\frac{1}{2}}}{3}\geq\frac{\left(\left(\frac{2}{3}x_1+\frac{4}{27}x_2\right)^2+1\right)^{\frac{1}{2}}}{\left(\frac{5}{9}\right)},$$

for all x_1,$x_2\in\mathbb{R}_+$.

For a concave function we demonstrate the following example:

Example 2.14. Let the non-negative strictly increasing concave function be $\phi(x) = x(1+x)^{-1}$, $x \geq 0$. Therefore $\phi^{-1}(x) = x(1-x)^{-1}$, $0 \leq x < 1$. The function $\phi^{-1}(x)$ satisfies the reverse of (2.27), that is $\phi^{-1}(AB) \leq \phi^{-1}(A)\phi^{-1}(B)$. Therefore for $\theta > 0$ and $\eta > 0$ we get that

$$\frac{\phi(x)}{\mu} + \frac{\phi(y)}{\nu} \leq \theta\phi\left(x\phi^{-1}\left(\frac{1}{\theta\mu}\right)\right) + \eta\phi\left(y\phi^{-1}\left(\frac{1}{\eta\nu}\right)\right).$$

We want to find θ and η for which the inequality $\frac{\phi(x)}{\mu} + \frac{\phi(y)}{\nu} \leq \frac{\phi(ax+by)}{(\theta+\eta)^{-1}}$ holds for given $\mu > 0$, $\nu > 0$, $a \geq 0$ and $b \geq 0$. Because ϕ is increasing and concave on $x \geq 0$, the inequalities

$$\begin{aligned}
\frac{\phi(x)}{\mu} + \frac{\phi(y)}{\nu} &\leq \theta\phi\left(x\phi^{-1}\left(\frac{1}{\theta\mu}\right)\right) + \eta\phi\left(y\phi^{-1}\left(\frac{1}{\eta\nu}\right)\right) \\
&\leq (\theta+\eta)\phi\left(\frac{\theta\left(x\phi^{-1}\left(\frac{1}{\theta\mu}\right)\right) + \eta\left(y\phi^{-1}\left(\frac{1}{\eta\nu}\right)\right)}{\theta+\eta}\right) \\
&= \frac{\phi(ax+by)}{(\theta+\eta)^{-1}}
\end{aligned}$$

hold where according to (2.30), θ and η satisfy

$$a = \frac{\theta}{(\theta+\eta)(\mu\theta-1)}, \quad b = \frac{\eta}{(\theta+\eta)(\nu\eta-1)}.$$

In particular when $a = 1$, $b = \frac{1}{2}$, $\mu = 2$, $\nu = 3$ we get that $\theta = \eta = 1$, and

$$\frac{x(1+x)^{-1}}{2} + \frac{y(1+y)^{-1}}{3} \leq \frac{\left(\frac{x+\frac{1}{2}y}{2}\right)\left(1+\frac{x+\frac{1}{2}y}{2}\right)^{-1}}{\left(\frac{1}{2}\right)}.$$

The following is an example of N-quasiconvex function:

Example 2.15. Let $\varphi : [0,1) \to [1,\infty)$ be the convex function $\varphi(x) = \frac{1}{1-x}$ and let the 1-quasiconvex function $\phi : [0,1) \to [0,\infty)$ be $\phi(x) = \frac{x}{1-x}$. Then according to (2.13) because ϕ^{-1} satisfies inequality (2.15) we get that

$$\begin{aligned}
\frac{x(1-x)^{-1}}{\mu} + \frac{y(1-y)^{-1}}{\nu} &\geq \theta x(1+\mu\theta-x)^{-1} + \eta y(1+\nu\eta-y)^{-1} \\
&\geq \left(\theta x(1+\mu\theta)^{-1} + \eta y(1+\nu\eta)^{-1}\right)\left(1 - \frac{\theta x(1+\mu\theta)^{-1}}{\theta+\eta} - \frac{\eta y(1+\nu\eta)^{-1}}{\theta+\eta}\right)^{-1} \\
&+ \left(1 - \frac{\theta x(1+\mu\theta)^{-1}}{\theta+\eta} - \frac{\eta y(1+\nu\eta)^{-1}}{\theta+\eta}\right)^{-2} \frac{\theta\eta}{\theta+\eta}\left(y(1+\nu\eta)^{-1} - x(1+\mu\theta)^{-1}\right)^2,
\end{aligned}$$

where η and θ are computed by using the two equations with the given μ, ν, a, b:

$$a = \frac{\theta(1+\mu\theta)^{-1}}{\theta+\eta}, \quad b = \frac{\eta(1+\nu\eta)^{-1}}{\theta+\eta}.$$

In particular when $\mu = 2$, $\nu = 3$, $a = \frac{1}{6}$, $b = \frac{1}{8}$ we get that $\theta = \eta = 1$ and:

$$\begin{aligned}
\frac{x(1-x)^{-1}}{2} + \frac{y(1-y)^{-1}}{3} &\geq \frac{x\left(1-\frac{x}{3}\right)^{-1}}{3} + \frac{y\left(1-\frac{y}{4}\right)^{-1}}{4} \\
&\geq \left(\frac{x}{6}+\frac{y}{8}\right)\left(1-\frac{x}{6}-\frac{y}{8}\right)^{-1} + \frac{1}{2}\left(1-\frac{x}{6}-\frac{y}{8}\right)^{-2}\left(\frac{y}{4}-\frac{x}{3}\right)^2.
\end{aligned}$$

3. Remarks on Euler-Lagrange type inequalities for non-convex and for non-superquadratic functions

Inequalities obtained for a convex function and for a superquadratic function can be applied to get inequalities related to a function f which is twice differentiable on a finite interval $[a,b]$ because $\varphi(x) = f(x) - \frac{x^2}{2}\min_{a\le x\le b} f''(x)$ is convex.

Similarly if f is three times differentiable on $0 \le a < b < \infty$, and if $f(0) = f'(0) = 0$, then $\varphi(x) = f(x) - \frac{x^3}{6}\min_{a\le x\le b} f'''(x)$ is superquadratic on $[a,b]$ (see [4]).

From this consideration we get that:

Example 3.1. Let $f(x)$ be twice differentiable on $[a,b]$ then, as explained above, $\varphi(x) = f(x) - \frac{x^2}{2}\min_{a\le x\le b} f''(x)$ is convex. If the function φ satisfies all the conditions of Theorem 2.11 related to the function and to its inverse φ^{-1}, then

$$\frac{\varphi(x)}{\mu} + \frac{\varphi(y)}{\nu} \ge \frac{\varphi(ax+by)}{\overline{\lambda}}, \quad \overline{\lambda} = (\theta+\eta)^{-1}.$$

Therefore

$$\frac{f(x)}{\mu} + \frac{f(y)}{\nu} - \frac{f(ax+by)}{\overline{\lambda}} \ge \left(\frac{x^2}{\mu} + \frac{y^2}{\nu} - \frac{(ax+by)^2}{(\theta+\eta)^{-1}}\right)\min_{a\le x\le b} f''(x),$$

where

$$\theta\varphi^{-1}\left(\frac{1}{\mu\theta}\right) = a(\theta+\eta), \quad \eta\varphi^{-1}\left(\frac{1}{\nu\eta}\right) = b(\theta+\eta).$$

In particular, let $f(x) = -x^4$, $0 \le x \le 1$, then $\varphi(x) = 6x^2 - x^4$, $0 \le x \le \sqrt{3-\sqrt{8}} < 1$. It is easy to see that an inverse function is $\varphi^{-1}(x) = \sqrt{3-\sqrt{9-x}}$, $0 \le x \le 1$. As φ on $\left[0, \sqrt{3-\sqrt{8}}\right]$ and φ^{-1} on $[0,1]$ satisfy Theorem 2.11 we get that

$$\begin{aligned} \frac{\varphi(x)}{\mu} + \frac{\varphi(y)}{\nu} &= \frac{6x^2-x^4}{\mu} + \frac{6y^2-y^4}{\nu} \\ &\ge \frac{6(ax+by)^2-(ax+by)^4}{\overline{\lambda}} = \frac{\varphi(ax+by)}{\overline{\lambda}}, \quad \overline{\lambda} = (\theta+\eta)^{-1} \end{aligned} \tag{3.1}$$

where a, b, μ, ν, θ and η satisfy

$$\theta\varphi^{-1}\left(\frac{1}{\mu\theta}\right) = \theta\sqrt{3-\sqrt{9-\frac{1}{\mu\theta}}} = a(\theta+\eta), \tag{3.2}$$

$$\eta\varphi^{-1}\left(\frac{1}{\nu\eta}\right) = \eta\sqrt{3-\sqrt{9-\frac{1}{\nu\eta}}} = b(\theta+\eta).$$

For instance, take $a = 0.5\sqrt{3-\sqrt{8.5}}$, $b = 0.5\sqrt{3-\sqrt{8.75}}$, $\mu = 2$, $\nu = 4$, $\theta = \eta = 1$, and we get that

$$\begin{aligned} \frac{6x^2-x^4}{2} + \frac{6y^2-y^4}{4} \quad \ge \quad & 6\frac{\left(x0.5\sqrt{3-\sqrt{8.5}} + y0.5\sqrt{3-\sqrt{8.75}}\right)^2}{0.5} \\ & + \frac{\left(x0.5\sqrt{3-\sqrt{8.5}} + y0.5\sqrt{3-\sqrt{8.75}}\right)^4}{0.5} \end{aligned} \tag{3.3}$$

when $x, y \in \left[0, \sqrt{3-\sqrt{8}}\right]$.

We can rewrite (3.3) as:

$$\frac{-x^4}{2}+\frac{-y^4}{4}\geq\frac{\left(x0.5\sqrt{3-\sqrt{8.5}}+y0.5\sqrt{3-\sqrt{8.75}}\right)^4}{0.5}+\left(\frac{-6x^2}{2}+\frac{-6y^2}{4}+6\frac{\left(x0.5\sqrt{3-\sqrt{8.5}}+y0.5\sqrt{3-\sqrt{8.75}}\right)^2}{0.5}\right). \tag{3.4}$$

The last inequality emphasizes that we can obtain an Euler-Lagrange type inequality for a non-convex function with bounded second derivative. Here the non-convex function is $f(x)=-x^4$ on the interval $[0,1]$ where $\min_{0\leq x\leq 1} f''(x)=-12\leq\min_{0\leq x\leq\sqrt{3-\sqrt{8}}} f''(x)$ and therefore the function $\varphi(x)=-x^4+6x^2$ is convex on $0\leq x\leq\sqrt{3-\sqrt{8}}$, and φ^{-1} satisfies $\varphi^{-1}(A)\varphi^{-1}(B)\leq\varphi^{-1}(AB)$ when $0\leq A,B\leq 1$, by which we get (3.3) for this φ and (3.4) for the given f.

Similarly to Example 3.1, we get for a non-superquadratic function the following example:

Example 3.2. The function $f(x)=-x^6$, $x\geq 0$ is non-superquadratic. As $f'''(x)$ is bounded on $[0,1]$ and $\min_{0\leq x\leq 1} f'''(x)=-120$ then $\varphi(x)=-x^6+20x^3$ satisfies $\varphi(0)=\varphi'(0)=0$ and $\varphi'''(x)\geq 0$ and therefore $\varphi(x)$ is superquadratic and it satisfies Theorem 2.8 for $N=0$. Indeed, $\varphi^{-1}(x)=\sqrt[3]{10-\sqrt[2]{100-x}}$, $0\leq x\leq 1$ is an inverse function of $\varphi(x)$ on $0\leq x\leq\sqrt[3]{10-\sqrt[2]{99}}\leq 1$, and the inequality

$$\begin{aligned}&\frac{-x^6}{\mu}+\frac{-y^6}{\nu}+\frac{(ax+by)^6}{(\theta+\eta)^{-1}}+\theta\left(\left|x\varphi^{-1}\left(\frac{1}{\mu\theta}\right)-(ax+by)\right|\right)^6+\eta\left(\left|y\varphi^{-1}\left(\frac{1}{\nu\eta}\right)-(ax+by)\right|\right)^6\\&\geq 20\left(\frac{-x^3}{\mu}+\frac{-y^3}{\nu}+\frac{(ax+by)^3}{(\theta+\eta)^{-1}}+\theta\left(\left|x\varphi^{-1}\left(\frac{1}{\mu\theta}\right)-(ax+by)\right|\right)^3\right)+20\eta\left(\left|y\varphi^{-1}\left(\frac{1}{\nu\eta}\right)-(ax+by)\right|\right)^3.\end{aligned} \tag{3.5}$$

For instance when $\mu=2$, $\nu=4$, $a=\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}$, $b=\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}$ we get that $\theta=\eta=1$ and for $0\leq x$, y, $ax+by\leq\sqrt[3]{10-\sqrt[2]{100-x}}\leq 1$ and that

$$\begin{aligned}&\frac{-x^6}{2}+\frac{-y^6}{4}+\frac{\left(\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}x+\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}y\right)^6}{0.5}\\&+\theta\left(\left|\left(\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}x-\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}y\right)\right|\right)^6\\&+\eta\left(\left|\left(\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}x-\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}y\right)\right|\right)^6\\&\geq 20\left(\frac{-x^3}{2}+\frac{-y^3}{4}+\frac{\left(\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}x+\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}y\right)^3}{0.5}\right)\\&+20\theta\left(\left|\left(\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}x-\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}y\right)\right|\right)^3\\&+20\eta\left(\left|\left(\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.5}}x-\frac{1}{2}\sqrt[3]{10-\sqrt[2]{99.75}}y\right)\right|\right)^3.\end{aligned} \tag{3.6}$$

The last inequality emphasizes that we can obtain an Euler-Lagrange type inequality for a non-superquadratic function with bounded third derivative. Here, the non-superquadratic function is $f(x) = -x^6$ on the interval $[0, 1]$ where $\min_{0\le x\le 1} f'''(x) = -120 \le \min_{0\le x\le \sqrt{10-\sqrt{99}}} f'''(x)$ and therefore the function $\varphi(x) = -x^6 + 20x^4$ is superquadratic on $0 \le x \le \sqrt{10 - \sqrt{99}} < 1$, by which we get (3.5) and (3.6) for the given f.

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