Montes Taurus J. Pure Appl. Math. 5 (3), 23–36, 2023

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com
MTJPAM
ISSN: 2687-4814

Formulas for Fubini type numbers and polynomials of negative higher order

Neslihan Kilar [a]

[a] *Department of Computer Technologies, Bor Vocational School, Niğde Ömer Halisdemir University, Niğde TR-51700 Turkey*

Abstract

The present paper deals with the Fubini type numbers and polynomials with their generating functions and functional equations. By using these functions, some properties and applications of these polynomials are investigated. Many relations and computation formulas connected with the Stirling type numbers, the Apostol type polynomials and numbers of order $-r$, the Bernoulli polynomials of order $-r$, the Euler polynomials and numbers of order $-r$, the Fubini type numbers and polynomials of order $-r$ and combinatorial numbers are given. Applying the derivative operator to the generating functions of these polynomials, some formulas and combinatorial sums including these numbers and polynomials are also given. Moreover, applying the Riemann integral to some formulas, we derive several interesting finite combinatorial sums associated with the Bernstein basis functions, the Cauchy numbers and the Stirling type numbers.

Keywords: Cauchy numbers, Stirling numbers, Fubini type numbers and polynomials, Bernstein basis functions, combinatorial numbers, special numbers, generating function

2010 MSC: 05A15, 05A19, 11B68, 11B73

1. Introduction

The Fubini numbers, one of the families of special numbers, are used in various areas such as mathematics, combinatorial theory, engineering and mathematical physics. The Fubini numbers are also known as order Bell numbers and horse numbers. These numbers are used in many counting problems that have important applications in graph theory and number theory, especially in the number of weak orders on a set of k elements and certain plane trees (*cf.* [1, 2, 3]). In [9], Kilar and Simsek modified these numbers, they not only defined the Fubini type polynomials and numbers, but also obtained many interesting results related to them. Here, we use the generating functions and their functional equations methods to obtain many new identities, computation formulas and relations associated with the Fubini type numbers and polynomials. We give many results including the Cauchy numbers, the Stirling type numbers, the combinatorial numbers, the Bernstein basis functions, the Apostol type numbers and polynomials, the Euler polynomials and numbers, the Bernoulli polynomials and numbers, and combinatorial numbers.

Throughout this paper, we need the following notations and definitions. Let

$$\mathbb{N} = \{1, 2, 3, ...\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

†*Article ID:* MTJPAM-D-22-00003

Email address: neslihankilar@ohu.edu.tr and neslihankilar@gmail.com (Neslihan Kilar)

Received:13 February 2022, Accepted:1 April 2022, Published:27 May 2022

⋆*Corresponding Author:* Neslihan Kilar

and also $\mathbb{Z}$, $\mathbb{C}$ denotes the set of integers and the set of complex numbers, respectively. For $r \in \mathbb{N}$, $\theta \in \mathbb{C}$,

$$(\theta)_r = \binom{\theta}{r} r! = \theta(\theta-1)(\theta-2)\ldots(\theta-r+1)$$

with $(\theta)_0 = 1$ (*cf.* [2]-[30]).

The Apostol-Bernoulli polynomials of order $-r$ are defined by

$$\mathcal{H}_B(z,x,r;\lambda) = \left(\frac{\lambda e^z - 1}{z}\right)^r e^{zx} = \sum_{k=0}^{\infty} \mathcal{B}_k^{(-r)}(x;\lambda)\frac{z^k}{k!}, \tag{1.1}$$

where $r \in \mathbb{N}$ (*cf.* [24, 27, 29] and the references therein).

When $x = 0$ in (1.1), we get

$$\mathcal{B}_k^{(-r)}(\lambda) = \mathcal{B}_k^{(-r)}(0;\lambda)$$

which denotes the Apostol-Bernoulli numbers of order $-r$ (*cf.* [24, 27, 29]).

As a special case, if $\lambda = 1$ and $x = 0$ then we get

$$B_k^{(-r)}(x) = \mathcal{B}_k^{(-r)}(x;1)$$

and

$$B_k^{(-r)} = \mathcal{B}_k^{(-r)}(0;1)$$

which denotes the Bernoulli polynomials and numbers of order $-r$, respectively, (*cf.* [24, 27, 29]).

The Apostol-Euler polynomials of order $-r$ are defined by

$$\mathcal{H}_E(z,x,r;\lambda) = \left(\frac{\lambda e^z + 1}{2}\right)^r e^{zx} = \sum_{k=0}^{\infty} \mathcal{E}_k^{(-r)}(x;\lambda)\frac{z^k}{k!}, \tag{1.2}$$

where $r \in \mathbb{N}$ (*cf.* [24, 27, 29] and the references therein).

Setting $x = 0$ in (1.2), we have

$$\mathcal{E}_k^{(-r)}(\lambda) = \mathcal{E}_k^{(-r)}(0;\lambda)$$

which denotes the Apostol-Euler numbers of order $-r$ (*cf.* [24, 27, 29]).

As a special case, if $\lambda = 1$ and $x = 0$ then we get

$$E_k^{(-r)}(x) = \mathcal{E}_k^{(-r)}(x;1)$$

and

$$E_k^{(-r)} = \mathcal{E}_k^{(-r)}(0;1)$$

which denotes the Euler polynomials and numbers of order $-r$, respectively, (*cf.* [24, 27, 29]).

The λ-array polynomials are defined by

$$\mathcal{H}(z,x,d;\lambda) = \frac{(\lambda e^z - 1)^d}{d!} e^{zx} = \sum_{k=0}^{\infty} S_d^k(x;\lambda)\frac{z^k}{k!}, \tag{1.3}$$

where $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}_0$ (*cf.* [22]).

The λ-Stirling numbers of the second kind are defined by

$$\mathcal{H}_S(z,d;\lambda) = \frac{(\lambda e^z - 1)^d}{d!} = \sum_{k=0}^{\infty} S_2(k,d;\lambda)\frac{z^k}{k!}, \tag{1.4}$$

where $d \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [22, 29]).

From (1.4), one can easily deduce the following formula:

$$S_2(k,d;\lambda) = \frac{1}{d!}\sum_{v=0}^{d}(-1)^v\binom{d}{v}\lambda^{d-v}(d-v)^k \tag{1.5}$$

(*cf.* [22, 29]).

Substituting $x = 0$ into (1.3), one has the following result:

$$S_2(k,d;\lambda) = S_d^k(0;\lambda).$$

When $\lambda = 1$ in (1.4), we get the Stirling numbers of the second kind:

$$S_2(k,d) = S_2(k,d;1)$$

which are defined by

$$\mathcal{H}_{S_2}(z,d) = \frac{(e^z-1)^d}{d!} = \sum_{k=0}^{\infty} S_2(k,d)\frac{z^k}{k!} \tag{1.6}$$

and

$$x^k = \sum_{v=0}^{k} S_2(k,v)(x)_v \tag{1.7}$$

(*cf.* [2]-[30]).

The Stirling numbers of the first kind are defined by

$$\frac{(\ln(1+z))^d}{d!} = \sum_{k=0}^{\infty} S_1(k,d)\frac{z^k}{k!}$$

and

$$(x)_k = \sum_{v=0}^{k} S_1(k,v)x^v \tag{1.8}$$

(*cf.* [2]-[30]).

The Cauchy numbers are defined by

$$b_k(0) = \int_0^1 (x)_k\,dx \tag{1.9}$$

(*cf.* [21]; see also [27]). We note that, these numbers are also called Bernoulli numbers of the second kind.

The Bernstein basis functions are defined by

$$B_d^k(y) = \binom{k}{d}y^d(1-y)^{k-d}, \tag{1.10}$$

where $d = 0, 1, 2, ..., k$; $k \in \mathbb{N}_0$ and $y \in [0, 1]$ (*cf.* [18, 23]).

By using (1.10), one has the following the integral formula:

$$\int_0^1 B_d^k(y)dx = \frac{1}{k+1} \tag{1.11}$$

(*cf.* [4, Eq. (5.28), p. 254]; see also [18, 23]).

The numbers $y_1(k,d;\lambda)$ are defined by

$$\frac{(\lambda e^z+1)^d}{d!} = \sum_{k=0}^{\infty} y_1(k,d;\lambda)\frac{z^k}{k!}, \tag{1.12}$$

where $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}_0$ (*cf.* [24]).

By using (1.2) and (1.12), we get

$$E_k^{(-r)} = r!2^{-r} y_1(k, r; 1) \tag{1.13}$$

(*cf.* [24]). Since

$$r! y_1(k, r; 1) = B(k, r),$$

where $B(k, r)$ is given by

$$B(k, r) = \frac{d^k}{dt^k} \left(e^t + 1\right)^r |_{t=0}$$

(*cf.* [5]), we have

$$E_k^{(-r)} = 2^{-r} B(k, r) \tag{1.14}$$

(*cf.* [24, Eq. (29)]).

The Fubini type polynomials of order d are defined by

$$\mathcal{H}_a(z, x, d) = \frac{2^d}{(2 - e^z)^{2d}} e^{zx} = \sum_{k=0}^{\infty} a_k^{(d)}(x) \frac{z^k}{k!}, \tag{1.15}$$

where $|z| < \ln 2$ and $d \in \mathbb{N}_0$ (*cf.* [9]).

Setting $x = 0$ in (1.15), we get the Fubini type numbers of order d:

$$a_k^{(d)} = a_k^{(d)}(0)$$

(*cf.* [9]; see also [8], [10]-[12]).

By the aid of (1.15), one can see that

$$a_k^{(d)}(x) = \sum_{v=0}^{k} \binom{k}{v} a_v^{(d)} x^{k-v} \tag{1.16}$$

(*cf.* [9]; see also [8], [10]-[12]).

By using (1.4) and (1.15), we have

$$\sum_{v=0}^{k} \binom{k}{v} S_2\left(v, 2d; \frac{1}{2}\right) a_{k-v}^{(d)}(x) = \frac{x^k}{(2d)!2^d} \tag{1.17}$$

(*cf.* [8, Corollary 4.2, p. 28]).

The numbers $Y_k^{(-r)}(\lambda)$ are defined by

$$\left(\lambda^2 z + \lambda - 1\right)^r = 2^r \sum_{k=0}^{\infty} Y_k^{(-r)}(\lambda) \frac{z^k}{k!}, \tag{1.18}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) and $r \in \mathbb{N}$ (*cf.* [16]; see also [14, 15, 17]).

Note that the numbers $Y_k^{(-r)}(\lambda)$ and $Y_k^{(d)}(\lambda)$, $(d > 0)$, are so-called Simsek numbers of order $-r$ and Simsek numbers of order d, respectively. These numbers, studied by many researchers, are important from the point of view of applications in various fields such as probability theory, quasi-monomiality, character theory, analysis, Weyl group structure and combinatorics (*cf.* [6, 7], [13]-[17], [25]).

The numbers $\beta_k(d)$ are define by

$$\left(1 - \frac{z}{2}\right)^d = \sum_{k=0}^{\infty} \beta_k(d) \frac{z^k}{k!}, \tag{1.19}$$

where $d \in \mathbb{N}_0$ and $z \in \mathbb{C}$ with $|z| < 2$ (*cf.* [14]).

By using (1.19), we get

$$\beta_k(d) = (-1)^k 2^{-k} k! \binom{d}{k} \tag{1.20}$$

(*cf.* [14, Eq. (4.9)]).

We briefly summarize the results of this paper as follows:

In Section 2, many computation formulas and relations connected with the Fubini type polynomials and numbers, the Stirling type polynomials and numbers, and the numbers $\beta_n(d)$, are given.

In Section 3, using generating functions and their functional equations method, some formulas with the inclusion of the Fubini type polynomials and numbers, the combinatorial numbers, the Apostol type polynomials and numbers, the Euler polynomials and numbers, the Bernoulli polynomials are obtained.

In Section 4, by making use of differential equations, some derivative formulas for the Fubini type polynomials and numbers are presented.

In Section 5, by applying Riemann integral to some identities, some combinatorial sums involving the Fubini type polynomials and numbers, the Bernstein basis functions, the Cauchy numbers, and the λ-Stirling numbers, are given.

In Section 6 is the conclusions section.

2. Identities and formulas for Fubini type polynomials and numbers of negative higher order

In this section, by using generating functions with their functional equations of the Fubini polynomials and numbers, some properties of these numbers and polynomials are investigated. Moreover, some computation formulas and identities including the Fubini type polynomials and numbers of order $-r$, the λ-array polynomials, the numbers $\beta_k(d)$ and the Stirling type numbers, are given.

Using (1.15), we have the Fubini numbers and polynomials of order $-r$ by the following generating functions, respectively:

$$\mathcal{G}(z,r) = 2^{-r}\left(2-e^z\right)^{2r} = \sum_{k=0}^{\infty} a_k^{(-r)} \frac{z^k}{k!} \tag{2.1}$$

and

$$\mathcal{G}(z,x,r) = \mathcal{G}(z,r)e^{xz} = \sum_{k=0}^{\infty} a_k^{(-r)}(x) \frac{z^k}{k!}, \tag{2.2}$$

where $r \in \mathbb{N}$.

Combining (2.2) with (2.1), we get

$$\sum_{k=0}^{\infty} a_k^{(-r)}(x) \frac{z^k}{k!} = \sum_{k=0}^{\infty}\sum_{v=0}^{k} \binom{k}{v} x^{k-v} a_v^{(-r)} \frac{z^k}{k!}.$$

Comparing coefficient of $\frac{z^k}{k!}$ on both sides of the above equation, we have the following corollay:

Corollary 2.1 (*cf.* [11, Eq. (2.4)]). *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)}(x) = \sum_{v=0}^{k} \binom{k}{v} x^{k-v} a_v^{(-r)}. \tag{2.3}$$

Theorem 2.2. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = 2^{-r} \sum_{n=0}^{2r}\sum_{v=0}^{n} (-1)^{n+v} \binom{2r}{n}\binom{n}{v} (n-v)^k. \tag{2.4}$$

Proof. Using binomial theorem in the equation (2.1), we have

$$\sum_{n=0}^{2r}\binom{2r}{n}(-1)^n 2^{-r} n!\mathcal{H}_{S_2}(z,n)=\mathcal{G}(z,r).$$

From the above equation and (1.6), we get

$$\sum_{k=0}^{\infty} a_k^{(-r)}\frac{z^k}{k!}=\sum_{k=0}^{\infty}\sum_{n=0}^{2r}\binom{2r}{n}\frac{(-1)^n n!}{2^r}S_2(k,n)\frac{z^k}{k!}.$$

Comparing the coefficients of $\frac{z^k}{k!}$ on both sides of the above equation, we obtain

$$a_k^{(-r)}=2^{-r}\sum_{n=0}^{2r}(-1)^n\binom{2r}{n}n!S_2(k,n). \tag{2.5}$$

Combining (2.5) with (1.5) in which substituting $\lambda=1$, we get

$$a_k^{(-r)}=2^{-r}\sum_{n=0}^{2r}(-1)^n\binom{2r}{n}\sum_{v=0}^{n}(-1)^v\binom{n}{v}(n-v)^k.$$

Thus, the proof of theorem is completed. □

Using (2.4), the numbers $a_k^{(-r)}$ are computed as in Table 1:

$k \setminus r$	1	2	3	4	5	6
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
1	-1	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{5}{16}$	$-\frac{3}{16}$
2	0	2	3	3	$\frac{5}{2}$	$\frac{15}{8}$
3	2	2	$-\frac{9}{2}$	-11	$-\frac{115}{8}$	$-\frac{117}{8}$
4	6	-10	$-\frac{39}{2}$	3	$\frac{335}{8}$	$\frac{609}{8}$
5	14	-46	$\frac{81}{2}$	157	$\frac{875}{8}$	$-\frac{909}{8}$

Table 1. The numbers $a_k^{(-r)}$ for $k=0,1,2,\ldots,5$ and $r=1,2,\ldots,6$

Let $r\in\mathbb{N}$. Some special values of $a_k^{(-r)}$ are given as follows:

$$\begin{aligned}
a_0^{(-r)} &= 2^{-r},\\
a_1^{(-r)} &= -r2^{1-r},\\
a_2^{(-r)} &= \left(r^2-r\right)2^{2-k},\\
a_3^{(-r)} &= \left(-2r^3+6r^2-3r\right)2^{2-r}
\end{aligned}$$

and

$$a_4^{(-r)}=\left(4r^4-24r^3+36r^2-13r\right)2^{2-r}.$$

By using (2.4), for $k>1$, we derive the following result:

$$a_k^{(-r)}=\left(b_0r^k+b_1r^{k-1}+b_2r^{k-2}+\cdots+b_{k-2}r^2+b_{k-1}r\right)2^{2-r},$$

where $b_0,b_1,b_2,\ldots,b_{k-1}\in\mathbb{Z}$. Consequently, by using same method of Simsek [24, 26], we have the following question:

How can we compute the coefficients $b_0, b_1, b_2, \ldots, b_{k-1}$?

We see that these coefficients are compute, which is similar to method of Xu [30].

Since $n > k$, $S_2(k,n) = 0$, the equation (2.5) is reduced to the following result:

$$a_k^{(-r)} = 2^{-r} \sum_{n=0}^{k} (-1)^n (2r)_n S_2(k,n).$$

Combining the above equation with (1.8), we have

$$a_k^{(-r)} = 2^{-r} \sum_{n=0}^{k} (-1)^n S_2(k,n) \sum_{v=0}^{n} S_1(n,v)(2r)^v. \tag{2.6}$$

From the above equation, for $k = 0$ and $k = 1$, we get

$$\begin{aligned} a_0^{(-r)} &= 2^{-r}, \\ a_1^{(-r)} &= -r2^{1-r}. \end{aligned}$$

For $k > 1$, the equation (2.6) is reduced to the following result:

$$a_k^{(-r)} = 2^{-r} \sum_{n=1}^{k} (-1)^n S_2(k,n) \sum_{v=1}^{n} S_1(n,v)(2r)^v.$$

Interchanging the order of the above summations, we get

$$\left(b_0 r^k + b_1 r^{k-1} + \cdots + b_{k-1} r\right) 2^{2-r} = 2^{-r} \sum_{v=1}^{k} r^v \sum_{n=v}^{k} (-1)^n S_2(k,n) S_1(n,v) 2^v.$$

Therefore

$$b_{k-v} = \sum_{n=v}^{k} (-1)^n S_2(k,n) S_1(n,v) 2^{v-2}.$$

From the above equation, we have the following theorem:

Theorem 2.3. *Let $k \in \mathbb{N} \setminus \{1\}$. Then we have*

$$a_k^{(-r)} = \left(b_0 r^k + b_1 r^{k-1} + \cdots + b_{k-1} r\right) 2^{2-r},$$

where the coefficients b_{k-v} $(v = 1, 2, \ldots, k)$ are given by

$$b_{k-v} = \sum_{n=v}^{k} (-1)^n S_2(k,n) S_1(n,v) 2^{v-2}.$$

Using (2.3) and (2.4), the polynomials $a_k^{(-r)}(x)$ are computed as in Table 2:

$k \setminus r$	1	2	3	4
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{x-2}{2}$	$\frac{x-4}{4}$	$\frac{x-6}{8}$	$\frac{x-8}{16}$
2	$\frac{x(x-4)}{2}$	$\frac{x^2-8x+8}{4}$	$\frac{x^2-12x+24}{8}$	$3 - x + \frac{x^2}{16}$
3	$\frac{x^3-6x^2+4}{2}$	$\frac{x^3}{4} - 3x^2 + 6x + 2$	$\frac{x^3-18x^2+72x-36}{8}$	$\frac{x^3}{16} - \frac{3x^2}{2} + 9x - 11$

Table 2. The polynomials $a_k^{(-r)}(x)$ for $k = 0, 1, 2, 3$ and $r = 1, 2, 3, 4$

Let $k \in \mathbb{N}$. Some values of the polynomials $a_k^{(-r)}(x)$ are presented below:

$$\begin{aligned} a_0^{(-r)}(x) &= 2^{-r}, \\ a_1^{(-r)}(x) &= 2^{-r}(x-2r), \end{aligned}$$

and

$$a_2^{(-r)}(x) = 2^{-r}\left(x^2 - 4r(x+1) + 4r^2\right).$$

Combining (2.5) with (1.20), we obtain the following theorem:

Theorem 2.4. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = \sum_{n=0}^{2r} 2^{n-r}\beta_n(2r)S_2(k,n).$$

Theorem 2.5 (*cf.* [10, Eq. (11)])**.** *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)}(x) = (2r)!2^r S_{2r}^k\left(x;\frac{1}{2}\right). \tag{2.7}$$

Proof. Using (1.3) and (2.2), we get

$$\mathcal{G}(z,x,r) = (2r)!2^r\mathcal{H}\left(z,x,2r;\frac{1}{2}\right).$$

Thus,

$$\sum_{k=0}^{\infty} a_k^{(-r)}(x)\frac{z^k}{k!} = (2r)!2^r\sum_{k=0}^{\infty} S_{2r}^k\left(x;\frac{1}{2}\right)\frac{z^k}{k!}.$$

Comparing the coefficients of $\frac{z^k}{k!}$ on both sides of the above equation, we get the desired result. □

Putting $x = 0$ in (2.7), we have the following result:

Theorem 2.6. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = 2^r(2r)!S_2\left(k,2r;\frac{1}{2}\right). \tag{2.8}$$

Combining (2.8) with (1.5), we get the following corollary:

Corollary 2.7. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = \sum_{v=0}^{2r}(-1)^v\binom{2r}{v}2^{v-r}(2r-v)^k.$$

3. Relations among Apostol type numbers, combinatorial numbers and Fubini type numbers

In this section, by the aid of generating functions and their functional equation techniques, many identities and relations among the Apostol type polynomials and numbers of order $-r$, the Bernoulli polynomials of order $-r$, the Euler numbers and polynomials of order $-r$, the numbers $Y_k^{(-r)}(\lambda)$, the numbers $B(k,r)$, the numbers $y_1(k,r;\lambda)$, and the Fubini type numbers and polynomials of order $-r$ are obtained.

Theorem 3.1. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \geq r$. Then we have*

$$\mathcal{B}_{k-r}^{(-r)}(\lambda) = \frac{2^r}{(k)_r}\sum_{v=0}^{k}\frac{Y_v^{(-r)}(\lambda)S_2(k,v)}{\lambda^v}. \tag{3.1}$$

Proof. Replacing λz by $e^z - 1$ in (1.18), we have

$$2^{-r}\left(\lambda e^{z}-1\right)^{r}=\sum_{v=0}^{\infty}\lambda^{-v}Y_{v}^{(-r)}(\lambda)\frac{\left(e^{z}-1\right)^{v}}{v!}. \tag{3.2}$$

Combining (3.2) with (1.1) in which substituting $x = 0$ and (1.6), we get

$$\sum_{k=0}^{\infty}\mathcal{B}_{k}^{(-r)}(\lambda)\frac{z^{k+r}}{k!}=2^{r}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\lambda^{-v}Y_{v}^{(-r)}(\lambda)S_{2}(k,v)\frac{z^{k}}{k!}.$$

Therefore

$$\sum_{k=0}^{\infty}(k)_{r}\mathcal{B}_{k-r}^{(-r)}(\lambda)\frac{z^{k}}{k!}=2^{r}\sum_{k=0}^{\infty}\sum_{v=0}^{k}\lambda^{-v}Y_{v}^{(-r)}(\lambda)S_{2}(k,v)\frac{z^{k}}{k!}.$$

Comparing coefficient of $\frac{z^k}{k!}$ on both sides of the above equation, we have the desired result. □

Corollary 3.2. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \geq 2r$. Then we have*

$$a_{k}^{(-r)}=2^{r}(k)_{2r}\mathcal{B}_{k-2r}^{(-2r)}\left(\frac{1}{2}\right). \tag{3.3}$$

Proof. Substituting $\lambda = \frac{1}{2}$ and $x = 0$ into (1.1), we get

$$\frac{\left(e^{z}-2\right)^{2r}}{2^{2r}z^{2r}}=\sum_{k=0}^{\infty}\mathcal{B}_{k}^{(-2r)}\left(\frac{1}{2}\right)\frac{z^{k}}{k!}.$$

Using the above equation and (2.1), we obtain

$$\sum_{k=0}^{\infty}a_{k}^{(-r)}\frac{z^{k}}{k!}=2^{r}\sum_{k=0}^{\infty}(k)_{2r}\mathcal{B}_{k-2r}^{(-2r)}\left(\frac{1}{2}\right)\frac{z^{k}}{k!}.$$

Comparing coefficient of $\frac{z^k}{k!}$ on both sides of the above equation, we have the desired result. □

By (3.1) and (3.3), we obtain the following theorem:

Theorem 3.3. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_{k}^{(-r)}=2^{3r}\sum_{v=0}^{k}2^{v}Y_{v}^{(-2r)}\left(\frac{1}{2}\right)S_{2}(k,v).$$

Theorem 3.4. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_{k}^{(-r)}(\lambda)=\sum_{v=0}^{k}(-1)^{v+r}\lambda^{-v}Y_{v}^{(-r)}(-\lambda)S_{2}(k,v). \tag{3.4}$$

Proof. Replacing λ by $-\lambda$ in (3.2), we have

$$(-1)^{r}\left(\frac{\lambda e^{z}+1}{2}\right)^{r}=\sum_{v=0}^{\infty}(-1)^{v}\lambda^{-v}Y_{v}^{(-r)}(-\lambda)\frac{\left(e^{z}-1\right)^{v}}{v!}.$$

Using (1.2), (1.6) and the above equation, we obtain

$$(-1)^{r}\sum_{k=0}^{\infty}\mathcal{E}_{k}^{(-r)}(\lambda)\frac{z^{k}}{k!}=\sum_{k=0}^{\infty}\sum_{v=0}^{k}(-1)^{v}\lambda^{-v}Y_{v}^{(-r)}(-\lambda)S_{2}(k,v)\frac{z^{k}}{k!}.$$

Comparing coefficient of $\frac{z^k}{k!}$ on both sides of the above equation, we get the desired result. □

Corollary 3.5. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)}(x) = 2^{3k}\mathcal{E}_k^{(-2r)}\left(x; -\frac{1}{2}\right). \tag{3.5}$$

Proof. Substituting $\lambda = -\frac{1}{2}$ into (1.2), then combining the final equation with (2.1), we have

$$\sum_{k=0}^{\infty} a_k^{(-r)}(x)\frac{z^k}{k!} = 2^{3r}\sum_{k=0}^{\infty}\mathcal{E}_k^{(-2r)}\left(x; -\frac{1}{2}\right)\frac{z^k}{k!}.$$

Comparing coefficient of $\frac{z^k}{k!}$ on both sides of the above equation, we get the desired result. □

Setting $x = 0$ in (3.5), we obtain the Corollary 3.6:

Corollary 3.6. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = 2^{3r}\mathcal{E}_k^{(-2r)}\left(-\frac{1}{2}\right). \tag{3.6}$$

Remark 3.7. Using (3.4) and (3.6), we also obtain the Theorem 3.3.

Theorem 3.8. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \geq v$. Then we have*

$$a_k^{(-r)}(x) = 2^{-r}\sum_{v=0}^{2r}(-1)^v\binom{2k}{v}(k)_v B_{k-v}^{(-v)}(x).$$

Proof. From (1.1) and (2.2), we get

$$\mathcal{G}(z, x, r) = 2^{-r}\sum_{v=0}^{2r}(-1)^v\binom{2r}{v}z^v\mathcal{H}_B(z, x, v; 1).$$

Therefore

$$\sum_{k=0}^{\infty} a_k^{(-r)}(x)\frac{z^k}{k!} = 2^{-r}\sum_{k=0}^{\infty}\sum_{v=0}^{2r}(-1)^v\binom{2r}{v}(k)_v B_{k-v}^{(-v)}(x)\frac{z^k}{k!}.$$

Comparing the coefficients of $\frac{z^k}{k!}$ on both sides of the above equation, we get the desired result. □

Theorem 3.9. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)}(x) = \sum_{v=0}^{2r}(-1)^v\binom{2r}{v}2^{v-r}3^{2r-v}E_k^{(-v)}(x). \tag{3.7}$$

Proof. By using (1.2) and (2.2), we have

$$\mathcal{G}(z, x, r) = 2^{-r}\sum_{v=0}^{2r}(-1)^v\binom{2r}{v}2^v 3^{2r-v}\mathcal{H}_E(z, x, v; 1).$$

Therefore

$$\sum_{k=0}^{\infty} a_k^{(-r)}(x)\frac{z^k}{k!} = 2^{-r}\sum_{k=0}^{\infty}\sum_{v=0}^{2r}(-1)^v\binom{2r}{v}2^v 3^{2r-v}E_k^{(-v)}(x)\frac{z^k}{k!}.$$

Comparing the coefficients of $\frac{z^k}{k!}$ on both sides of the above equation, we get the desired result. □

Putting $x = 0$ in (3.7), we get the following result:

Corollary 3.10. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = \sum_{v=0}^{2r} (-1)^v \binom{2r}{v} 2^{v-r} 3^{2r-v} E_k^{(-v)}. \tag{3.8}$$

Combining (3.8) with (1.13), we obtain the Corollary 3.11:

Corollary 3.11. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = 2^{-r} \sum_{v=0}^{2r} (-1)^v \binom{2r}{v} 3^{2r-v} v! y_1(k, v; 1).$$

Combining (3.8) with (1.14), we have the following corollary:

Corollary 3.12. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$a_k^{(-r)} = 2^{-r} \sum_{v=0}^{2r} (-1)^v \binom{2r}{v} 3^{2r-v} B(k, v).$$

4. Derivative formulas for Fubini type polynomials: Approach to analysis of differential equations of generating functions

In this section, applying partial differential equations to the generating functions of Fubini type polynomials and numbers of order $-r$, some formulas and finite sums for the these polynomials and numbers are given.

Theorem 4.1. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$\frac{1}{2} \sum_{v=0}^{k} \binom{k}{v} \left(x a_v^{(-r)}(x) - a_{v+1}^{(-r)}(x) \right) = x a_k^{(-r)}(x) - r a_k^{(-r)}(x+1) - a_{k+1}^{(-r)}(x). \tag{4.1}$$

Proof. Applying the derivative operator $\frac{\partial}{\partial z}$ to equation (2.2), we obtain

$$-2r\mathcal{G}(z, x+1, r) + x\left(2 - e^z\right)\mathcal{G}(z, x, r) = \left(2 - e^z\right) \sum_{k=0}^{\infty} a_{k+1}^{(-r)}(x) \frac{z^k}{k!}.$$

Therefore

$$\sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{k}{v} \left(x a_v^{(-r)}(x) - a_{v+1}^{(-r)}(x) \right) \frac{z^k}{k!} = 2x \sum_{k=0}^{\infty} a_k^{(-r)}(x) \frac{z^k}{k!} - 2r \sum_{k=0}^{\infty} a_k^{(-r)}(x+1) \frac{z^k}{k!} - 2 \sum_{k=0}^{\infty} a_{k+1}^{(-r)}(x) \frac{z^k}{k!}.$$

Comparing the coefficients of $\frac{z^k}{k!}$ on both sides of the above equation, we arrive at the desired result. □

Putting $x = 0$ in (4.1), we get the following result:

Theorem 4.2. *Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{k} \binom{k}{v} a_{v+1}^{(-r)} = 2r a_k^{(-r)}(1) + 2a_{k+1}^{(-r)}.$$

Theorem 4.3. *Let $r, n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \geq n$. Then we have*

$$\frac{\partial^n}{\partial x^n} a_k^{(-r)}(x) = (k)_n \, a_{k-n}^{(-r)}(x). \tag{4.2}$$

Proof. Applying the derivative operator $\frac{\partial^n}{\partial x^n}$ to equation (2.2), we get

$$\sum_{k=0}^{\infty} \frac{\partial^n}{\partial x^n} a_k^{(-r)}(x) \frac{z^k}{k!} = z^n 2^{-r} (2 - e^z)^{2r} e^{zx}.$$

Therefore

$$\sum_{k=0}^{\infty} \frac{\partial^n}{\partial x^n} a_k^{(-r)}(x) \frac{z^k}{k!} = \sum_{k=0}^{\infty} (k)_n\, a_{k-n}^{(-r)}(x) \frac{z^k}{k!}.$$

Comparing the coefficients of $\frac{z^k}{k!}$ on both sides of the above equation, we get the desired result. □

When $n = 1$ in (4.2), we obtain the Corollary 4.4:

Corollary 4.4. *Let $k, r \in \mathbb{N}$. Then we have*

$$\frac{\partial}{\partial x} a_k^{(-r)}(x) = k a_{k-1}^{(-r)}(x).$$

5. Identities and combinatorial sums via the Riemann integral

In this section, applying Riemann integral to the some identities, several interesting combinatorial sums, identities involving the Cauchy numbers, the λ-Stirling numbers, the Fubini type polynomials and numbers of order d, and the Bernstein basis functions are given.

Theorem 5.1. *Let $d, k \in \mathbb{N}_0$. Then we have*

$$\frac{1}{k+1} = (2d)! 2^d \sum_{s=0}^{k} \binom{k}{s} S_2\left(s, 2d; \frac{1}{2}\right) \sum_{v=0}^{k-s} \binom{k-s}{v} \frac{a_v^{(d)}}{k-s+1-v}. \tag{5.1}$$

Proof. Combining (1.17) with (1.16), then applying the Riemann integral to the final equation, we have

$$\int_0^1 x^k dx = (2d)! 2^d \sum_{s=0}^{k} \binom{k}{s} S_2\left(s, 2d; \frac{1}{2}\right) \sum_{v=0}^{k-s} \binom{k-s}{v} a_v^{(d)} \int_0^1 x^{k-s-v} dx.$$

After some calculations, the proof of the theorem is completed. □

By combining (5.1) with (1.11), we have an integral formula for the Bernstein basis functions and the numbers $a_n^{(l)}$ as follows:

Corollary 5.2. *Let $j = 0, 1, ..., k$ and $k, d \in \mathbb{N}_0$. Then we have*

$$\int_0^1 B_j^k(x) dx = (2d)! 2^d \sum_{s=0}^{k} \binom{k}{s} S_2\left(s, 2d; \frac{1}{2}\right) \sum_{v=0}^{k-s} \frac{\binom{k-s}{v} a_v^{(d)}}{k-s+1-v}.$$

Theorem 5.3. *Let $s, d \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{k} S_2(k, v)\, b_v(0) = (2d)! 2^d \sum_{s=0}^{k} \binom{k}{s} S_2\left(s, 2d; \frac{1}{2}\right) \sum_{v=0}^{k-s} \frac{\binom{k-s}{v} a_v^{(d)}}{k-s+1-v}.$$

Proof. Combining (1.17) with (1.7), then applying the Riemann integral to the final equation, we have

$$\sum_{v=0}^{k} S_2(k,v) \int_0^1 (x)_v dx = (2d)!2^d \sum_{s=0}^{k} \binom{k}{s} S_2\left(s, 2d; \frac{1}{2}\right) \int_0^1 a_{k-s}^{(d)}(x)\,dx. \tag{5.2}$$

From (1.9), (1.16) and (5.2), we obtain

$$\sum_{v=0}^{k} S_2(k,v)\, b_v(0) = (2d)!2^d \sum_{s=0}^{k} \binom{k}{s} S_2\left(s, 2d; \frac{1}{2}\right) \sum_{v=0}^{k-s} \frac{\binom{k-s}{v} a_v^{(d)}}{k-s+1-v}.$$

Thus, the proof of the theorem is completed. □

6. Conclusion

The Fubini type polynomials and numbers were studied by the aid of generating functions and their functional equations in this paper. Next, some properties of these numbers and polynomials were examined, and various identities, relations, computation and explicit formulas involving the Euler polynomials and numbers, the Bernoulli polynomials, the Apostol type polynomials and numbers, the Stirling type polynomials and numbers, combinatorial numbers, and these numbers and polynomials were given. Additionally, by applying Riemann integral and derivative formulas to these polynomials and numbers, some combinatorial and finite sums, and formulas including some special numbers were also given. Thus, the results of this paper can be used many areas in almost all branches of mathematics, number theory, combinatorial theory, engineering and mathematical physics.

Acknowledgments

This paper is dedicated to our colleagues and all humanity who died or was affected due to the coronavirus pandemic.

The author would like to thank to all referees for their valuable comments.

Author Contributions: This paper has only one author.

Conflicts of Interest: The author declare no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] A. Cayley, *On the analytical forms called trees*, Second part, Philosophical Magazine, Series IV **18 (121)**, 374–378, 1859.

[2] L. Comtet, *Advanced combinatorics: The art of finite and infinite expansions*, D. Reidel Publications, Dordrecht and Boston, 1974.

[3] J.-M. De Koninck, *Those fascinating numbers*, American Mathematical Society, 2009.

[4] R. Goldman, *Pyramid algorithms: A dynamic programming approach to curves and surfaces for geometric modeling*, Morgan Kaufmann Publishers, R. Academic Press, San Diego, 2002.

[5] R. Golombek, *Aufgabe 1088*, Elem. Math. **49**, 126–127, 1994.

[6] S. Khan, T. Nahid and M. Riyasat, *Partial derivative formulas and identities involving 2-variable Simsek polynomials*, Bol. Soc. Mat. Mex. **26**, 1–13, 2020.

[7] S. Khan, T. Nahid and M. Riyasat, *Properties and graphical representations of the 2-variable form of the Simsek polynomials*, Vietnam J. Math. **50**, 95–109, 2022.

[8] N. Kilar, *Fubini type numbers and their generating functions*, Akdeniz University, MSc Thesis in Mathematics, Antalya, 2017.

[9] N. Kilar and Y. Simsek, *A new family of Fubini numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials*, J. Korean Math. Soc. **54 (5)**, 1605–1621, 2017.

[10] N. Kilar and Y. Simsek, *Some relationships between Fubini type polynomials and other special numbers and polynomials*, AIP Conf. Proc. **2116**, 2019; Article ID: 100017.

[11] N. Kilar and Y. Simsek, *Identities and relations for Fubini type numbers and polynomials via generating functions and p-adic integral approach*, Publ. Inst. Math. (Beograd) (N.S.) **106 (120)**, 113–123, 2019.

[12] N. Kilar and Y. Simsek, *Formulae to Fubini type numbers emerge from application of p-adic integrals*, GU J Sci, Part A **8 (4)**, 402–410, 2021.

[13] I. Kucukoglu, *Computational and implementational analysis of generating functions for higher order combinatorial numbers and polynomials attached to Dirichlet characters*, Math. Methods Appl. Sci. **45 (9)**, 5043–5066, 2022.

[14] I. Kucukoglu and Y. Simsek, *New formulas and numbers arising from analyzing combinatorial numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **3 (3)**, 238–259, 2021.

[15] I. Kucukoglu and Y. Simsek, *Matrix representations for a certain class of combinatorial numbers associated with Bernstein basis functions and cyclic derangements and their probabilistic and asymptotic analyses*, Appl. Anal. Discrete Math. **15**, 45–68, 2021.

[16] I. Kucukoglu, B. Simsek and Y. Simsek, *Generating functions for new families of combinatorial numbers and polynomials: Approach to Poisson-Charlier polynomials and probability distribution function*, Axioms **8**, 2019; Article ID: 112.

[17] I. Kucukoglu, B. Simsek and Y. Simsek, *An approach to negative hypergeometric distribution by generating function for special numbers and polynomials*, Turk. J. Math. **43**, 2337–2353, 2019.

[18] G. G. Lorentz, *Bernstein polynomials*, Chelsea Pub. Comp., New York, 1986.

[19] T. Nahid and M. Ali, *Several characterizations of Bessel functions and their applications*, Georgian Math. J. **29 (1)**, 83–93, 2022.

[20] T. Nahid and C. S. Ryoo, *2-Variable Fubini-degenerate Apostol-type polynomials*, Asian-Eur. J. Math. 2021; DOI:10.1142/S179355712250 0929.

[21] S. Roman, *The umbral calculus*, Dover Publications, New York, 2005.

[22] Y. Simsek, *Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Theory Appl. **2013**, 2013; Article ID: 87.

[23] Y. Simsek, *Analysis of the Bernstein basis functions: An approach to combinatorial sums involving binomial coefficients and Catalan numbers*, Math. Methods Appl. Sci. **38 (14)**, 3007–3021, 2015.

[24] Y. Simsek, *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math. **12**, 1–35, 2018.

[25] Y. Simsek, *Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals*, Turk. J. Math. **42**, 557–577, 2018.

[26] Y. Simsek, *Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers*, J. Math. Anal. Appl. **477**, 1328–1352, 2019.

[27] Y. Simsek, *Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.

[28] Y. Simsek, *Interpolation functions for new classes special numbers and polynomials via applications of p-adic integrals and derivative operator*, Montes Taurus J. Pure Appl. Math. **3 (1)**, 38–61, 2021.

[29] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci. **5 (3)**, 390–444, 2011.

[30] A. Xu, *On an open problem of Simsek concerning the computation of a family of special numbers*, Appl. Anal. Discrete Math. **13**, 61–72, 2019.