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Generalized quantum Airy differential operator in a complex domain

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Abstract

In this work, we express a generalization of Airy functions in virtue of the q-calculus, in a complex domain. Based on this generalization, we formulate the Airy equation and study its behavior in view of the geometric function theory. This formula will be considered in some classes of analytic functions. In this investigation, we act the suggested q-Airy operator on the subclass of normalized analytic functions. Our method is given by the theory of subordination and superordination. Some examples will be illustrated in the sequel. Moreover, an application is formulated for finding the upper solution of a complex wave diffusive equation using the suggested q-operator.

Keywords: Fractional calculus, quantum calculus, analytic function, univalent function, subordination, open unit disk

2010 MSC: 30C45, 30C55

1. Introduction

Fundamental series of special functions and polynomials and their results are documented to have wide performances, in mathematical analysis. These functions are appreciated additionally in all fields of sciences such as computer science, optical studies and chemical processing. The theory of geometric function is ironic with altered kinds of these special functions formulating convolution operators acting on different types of classes of analytic functions in the open unit disk $\bigcirc := \{\xi \in \mathbb{C} : |\xi| < 1\}$. For examples: Carlson-Shaffer operator [2], hypergeometric linear operator [8, 16] and Fox-Write linear operator [7]. Most of these linear operators are generalized by assuming the quantum calculus. As a consequence, it appears new generations of classes of analytic functions called the quantum analytic functions (q-analytic functions), for example the subclass of q-starlike univalent functions and q-convex analytic functions. These quantum classes have improved many facts in the geometric function theory, such as the coefficient bounds, distortion inequalities, and differential subordination and superordination results.

The Jackson calculus (q-calculus) [12] is an actual critical part of the investigation in the interior field of traditional mathematical analysis. It acts potentially on the generalization of the different types of differential, integral and convolution operators. It is a complete notion of study in Mathematics, which has its antique origins, as well as a transformed possibility in the present times. It is significant to note that the extended antiquity of the Jackson q-calculus ages back to the effort of Bernoulli, Abel and Euler (see [4], [9]-[11], [19]).

Airy functions [1] are special functions dominated by the hypergeometric function of a complex variable fulfilling the Airy equation

$$\varphi''(\xi) - \xi\,\varphi(\xi) = 0, \quad \xi \in \mathbb{C},$$

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which can be expressed by the formula

$$\left(\frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right) - \frac{\xi^3}{\left(\frac{\xi\varphi'(\xi)}{\varphi(\xi)}\right)} = 0, \quad \xi \in \mathbb{C}.$$

More interesting formula can be presented as follows:

$$\left(1 + \frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right) - \frac{\xi^3}{\left(\frac{\xi\varphi'(\xi)}{\varphi(\xi)}\right)} = 1, \quad \xi \in \mathbb{C}.$$

The above formula can be studied geometrically, because it involves the convexity: $\left(1 + \frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right)$ and starlikeness: $\left(\frac{\xi\varphi'(\xi)}{\varphi(\xi)}\right)$ expressions, when φ is a normalized analytic function in the open unit disk satisfying $\varphi(0) = \varphi'(0) - 1 = 0$.

These differential equations show very essential character in the practical sciences such as optics studies, industries, economy, Quantum mechanics, and astrophysics. The most benefit of Airy functions in mathematical researches is to progress the areas of singular functions and statistical studies [13]. The prescription of the Airy function of a complex variable is expressed as tails:

$$\begin{aligned} A\imath(\xi) &= \frac{1}{2\pi i}\int_{\bigcirc} \exp\left(\frac{\zeta^3}{3} - \xi\zeta\right) d\zeta \\ &= \sum_{n=0}^{\infty}\left(\frac{3^{n/3}\sin\left(\pi\,\frac{2(n+1)}{3}\right)\Gamma\left(\frac{n+1}{3}\right)}{\pi 3^{2/3}\Gamma(n+1)}\right)\xi^n \\ &= \frac{1}{3^{2/3}\Gamma(2/3)} - \frac{\xi}{3^{1/3}\Gamma(1/3)} + \cdots + O(\xi^7). \end{aligned}$$

Using the Euler's reflection formula

$$\Gamma(1-\chi)\Gamma(\chi) = \frac{\pi}{\sin\pi\chi},$$

we have

$$A\imath(\xi) = \sum_{n=0}^{\infty}\left(\frac{3^{n/3}\Gamma\left(\frac{n+1}{3}\right)}{3^{2/3}\Gamma(n+1)\Gamma\left(1-\frac{2(n+1)}{3}\right)\Gamma\left(\frac{2(n+1)}{3}\right)}\right)\xi^n. \tag{1.1}$$

Let Δ denoted the class of normalized functions in $\bigcirc$, as follows:

$$\varphi(\xi) = \xi + \sum_{n=2}^{\infty}\varphi_n\xi^n, \quad \xi \in \bigcirc. \tag{1.2}$$

Two functions $\varphi, \phi \in \Delta$ are convoluted if they satisfy

$$\varphi(\xi) * \phi(\xi) = \xi + \sum_{n=2}^{\infty}\varphi_n\phi_n\xi^n, \quad \xi \in \bigcirc,$$

where $\phi(\xi) = \xi + \sum_{n=2}^{\infty}\phi_n\xi^n$. And they are subordinated $\phi \prec \varphi$ if there occurs an analytic function w with $|w(\xi)| \le |\xi| < 1$ such that $\phi = (\varphi(w))$ (see [14]). Note that the normalized Airy operator is formulated in [5, 6]. Modify the function

$A\imath(\xi)$ by multiplying Eq. (1.1) by $(3^{2/3})\Gamma(2/3)\xi \approx 0.8004379779825515\xi$, we get [5]

$$\begin{aligned}
\mathbb{A}\imath(\xi) &:= (3^{2/3})\Gamma(2/3)\xi A\imath(\xi)\\
&= (3^{2/3})\Gamma(2/3)\sum_{n=0}^{\infty}\left(\frac{3^{n/3}\Gamma\left(\frac{n+1}{3}\right)}{3^{2/3}\Gamma(n+1)\Gamma\left(1-\frac{2(n+1)}{3}\right)\Gamma\left(\frac{2(n+1)}{3}\right)}\right)\xi^{n+1}\\
&= (3^{2/3})\Gamma(2/3)\sum_{n=1}^{\infty}\left(\frac{3^{n/3-1}\Gamma\left(\frac{n}{3}\right)}{\Gamma(n)\Gamma\left(1-\frac{2n}{3}\right)\Gamma\left(\frac{2n}{3}\right)}\right)\xi^{n}\\
&= \xi+\sum_{n=2}^{\infty}\left(\frac{3^{(n-1)/3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{n}{3}\right)}{\Gamma(n)\Gamma\left(1-\frac{2n}{3}\right)\Gamma\left(\frac{2n}{3}\right)}\right)\xi^{n}.
\end{aligned}$$

It is clear that $\Lambda \in \Delta$. Proceeding, define an operator $\mathbb{A}\imath : \Delta \to \Delta$ using the convolution product, as follows:

$$(\mathbb{A}\imath * \varphi)(\xi) = \xi+\sum_{n=2}^{\infty}\left(\frac{3^{(n-1)/3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{n}{3}\right)}{\Gamma(n)\Gamma\left(1-\frac{2n}{3}\right)\Gamma\left(\frac{2n}{3}\right)}\right)\varphi_n\xi^{n}. \tag{1.3}$$

Our aim is to generalize (1.3) utilizing the quantum calculus. Then study its behavior geometrically.

2. Quantum concept

The formal q-shifted factorials is given for an integer $\xi \in \mathbb{C}$ (*cf.* [12])

$$(\xi;q)_\ell = \prod_{\imath=0}^{\ell-1}(1-q^\imath\xi), \quad \ell \in \mathbb{N},\ (\xi;q)_0 = 1. \tag{2.1}$$

The q-shifted formula is obtained by using (2.1) and the well known gamma function

$$(q^\xi;q)_\ell = \frac{\Gamma_q(\xi+\ell)(1-q)^\ell}{\Gamma_q(\xi)}, \quad \Gamma_q(\xi) = \frac{(q;q)_\infty(1-q)^{1-\xi}}{(q^\xi;q)_\infty}, \tag{2.2}$$

where

$$\Gamma_q(\xi+1) = \frac{\Gamma_q(\xi)(1-q^\xi)}{1-q}, \quad q \in (0,1)$$

and

$$(\xi;q)_\infty = \prod_{\imath=0}^{\infty}(1-q^\imath\xi). \tag{2.3}$$

The q-derivative is given as follows:

$$\partial_q\, h(\xi) = \frac{h(\xi)-h(q\xi)}{\xi(1-q)}, \tag{2.4}$$

where

$$\partial_q\,(\xi^\nu) = \left(\frac{1-q^\nu}{1-q}\right)\xi^{\nu-1}.$$

Furthermore, the concept of the q-binomial formula obtains equality

$$(x-y)_b = x^b \left(-\frac{y}{x}; q\right)_b. \tag{2.5}$$

The creators of [19] considered generalized q-Mittag-Leffler function

$$\mathcal{E}^{\varsigma}_{\nu,\mu}(\xi; q) = \sum_{n=0}^{\infty} \frac{(q^{\varsigma}; q)_n}{(q; q)_n} \frac{\xi^n}{\Gamma_q(\nu n + \mu)}. \tag{2.6}$$

Consequently, we have

$$\begin{aligned} [\mathbb{A}\iota * \varphi]_q(\xi) &= \xi + \sum_{n=2}^{\infty} \left(\frac{3^{(n-1)/3}\Gamma_q\left(\frac{2}{3}\right)\Gamma_q\left(\frac{n}{3}\right)}{\Gamma_q(n)\Gamma_q\left(1-\frac{2n}{3}\right)\Gamma_q\left(\frac{2n}{3}\right)} \right) \varphi_n \xi^n \\ &:= \xi + \sum_{n=2}^{\infty} [A_n]_q \varphi_n \xi^n, \end{aligned} \tag{2.7}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Extra investigation can be located in [4], [9]-[11]. We aim to study q-operator (2.7) geometrically using the theory of subordination and superordination [14]. Therefore, next section deals with the preliminaries that will be used.

3. Arguments

The outcomes of recent study are indicated in view of differential subordination scheme via the resulting initiations:

Argument 3.1 (*cf.* [14]). Let $f(\xi)$ and $g(\xi)$ be two convex univalent analytic functions defined in $\bigcirc$, so that $f(0) = g(0)$. Furthermore, the subordination holds for a constant $t \neq 0$, $\Re(t) \geq 0$,

$$f(\xi) + t^{-1} f'(\xi) \prec g(\xi)$$

implies

$$f(\xi) \prec g(\xi).$$

Argument 3.2 (*cf.* [14]). Assume that

$$\mathbb{L}[a, n] = \left\{ h : h(\xi) = a + a_n \xi^n + a_{n+1}\xi^{n+1} + \cdots \right\},$$

indicates the general class of holomorphic functions, where $a \in \mathbb{C}$ and n is a positive integer. If $t \in \mathbb{R}$ then

$$\Re \{h(\xi) + t\,\xi\, h'(\xi)\} > 0 \Rightarrow \Re\,(h(\xi)) > 0.$$

Moreover, if $t > 0$ and $h \in \mathbb{L}[1, n]$, then there are specific constants $c_1 > 0$ and $c_2 > 0$ formalizing the inequality

$$h(\xi) + t\xi h'(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{c_1}$$

and

$$h(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{c_2}.$$

Argument 3.3 (*cf.* [18]). Assume that $\hbar, p \in \mathbb{L}[a,n]$, where p is univalent and convex in $\bigcirc$ and for $\Bbbk_1, \Bbbk_2 \in \mathbb{C}, \Bbbk_2 \neq 0$, then the inequality

$$\Bbbk_1 \hbar(\xi) + \Bbbk_2\, \xi \hbar'(\xi) \prec \Bbbk_1 p(\xi) + \Bbbk_2 \xi\, p'(\xi)$$

yields

$$\hbar(\xi) \prec p(\xi), \quad \xi \in \bigcirc.$$

Argument 3.4 (*cf.* [15]). Assume that $g, p \in \mathbb{L}[a,n]$, such that p is univalent convex in $\bigcirc$ with $g(\xi) + \Bbbk\, \xi g'(\xi)$ is univalent then the inequality

$$p(\xi) + \Bbbk\, \xi\, p'(\xi) \prec g(\xi) + \Bbbk \xi g'(\xi)$$

gives

$$p(\xi) \prec g(\xi), \quad \xi \in \bigcirc.$$

Argument 3.5 (*cf.* [3]). Suppose that $\hbar, \hslash, \eth \in \mathbb{L}[a,n]$ and $\eth$ is univalent convex in $\bigcirc$ having the inequalities

$$\hbar \prec \eth, \quad \hslash \prec \eth$$

then

$$\Bbbk \hbar + (1-\Bbbk)\hslash \prec \eth, \quad \Bbbk \in [0,1].$$

4. Outcomes

We start with the following definition:

Definition 4.1. Assume that $\varphi \in \Delta$ in the class $[\Sigma_\sigma(p)]_q$ if it achieves the subordination

$$(1-\sigma)\ \xi^{-1}[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q \prec p(\xi) = \frac{a\xi+1}{b\xi+1}, \tag{4.1}$$

$$\Big(\xi \in \bigcirc, \nu, \sigma \in [0,1], |a| = |b| = 1\Big),$$

where p is univalent convex in $\bigcirc$.

For insistent,

$$p(\xi) = \frac{a\xi+1}{b\xi+1} = \mathbb{P}_{a,b}(\xi),$$

which is convex univalent extremely function in $\bigcirc$ of the set

$$\mathcal{P} := \left\{ p \in \bigcirc : p(\xi) = 1 + \sum_{n=1}^{\infty} p_i \xi^n,\ \Re(p(\xi)) > 0 \right\}.$$

Suppose the functional $\mathbb{O} : \bigcirc \to \bigcirc$, where

$$\mathbb{O}(\xi) := (1-\sigma)\ \xi^{-1}[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma[\mathbb{A}\imath * \varphi(\xi)]'_q. \tag{4.2}$$

Consequently, by Definition 4.1, we get the following inequality

$$\mathbb{O}(\xi) \prec \mathbb{P}_{a,b}(\xi) = \frac{a\xi+1}{b\xi+1}, \quad \xi \in \bigcirc.$$

Theorem 4.2. *Suppose that $\varphi \in [\Sigma_\sigma(p)]_q$. If*

$$\begin{aligned}\Re\{\mathbb{O}(\xi)\} &= \Re\left\{1+\sum_{n=1}^{\infty}[A_{n+1}]_q\,(1+\sigma n)\,\varphi_{n+1}\xi^n\right\}\\ &:= \Re\left\{1+\sum_{n=1}^{\infty}\Theta_n\right\}>0\end{aligned}$$

then

$$|\varphi_{n+1}| \le \frac{\left(\int_0^{2\pi}|e^{-i\vartheta n}|\,d\mathfrak{T}(\theta)\right)}{2[A_{n+1}]_q(1+\sigma n)},$$

such that $d\mathfrak{T}$ is an estimate of probability. Moreover, if

$$\Re\left(e^{i\vartheta}\mathbb{O}(\xi)\right)>0, \quad \xi \in \bigcirc,\ \vartheta \in \mathbb{R}$$

then $\varphi \in \left[\Sigma_\sigma\left(\frac{a\xi+1}{b\xi+1}\right)\right]_q$ that is

$$\mathbb{O}(\xi) \approx \frac{a\xi+1}{b\xi+1}, \quad \xi \in \bigcirc.$$

Proof. In view of the condition of the theorem, we obtain

$$\Re(\mathbb{O}(\xi)) = \Re\left(1+\sum_{n=1}^{\infty}\Theta_n\xi^n\right)>0.$$

Therefore, the Carathéodory positivist Lemma displays

$$|\Theta_n| \le 2\int_0^{2\pi}|e^{-in\vartheta}|\,d\mathfrak{T}(\theta),$$

where $d\mathfrak{T}$ is an estimate of probability. As a result, we are able to achieve

$$|\varphi_{n+1}| \le \frac{\left(\int_0^{2\pi}|e^{-in\vartheta}|\,d\mathfrak{T}(\vartheta)\right)}{2[A_{n+1}]_q(1+\sigma n)}.$$

Moreover, if

$$\Re\left(e^{i\vartheta}\mathbb{O}(\xi)\right)>0, \quad \xi \in \bigcirc, \quad \vartheta \in \mathbb{R}$$

then in view of [17, Theorem 1.6, p. 22], we obtain

$$\mathbb{O}(\xi) \approx \frac{a\xi+1}{b\xi+1}, \quad \xi \in \bigcirc.$$

Hence, $\varphi \in \left[\Sigma_\sigma\left(\frac{a\xi+1}{b\xi+1}\right)\right]_q$. □

The necessary and sufficient expresses for the sandwich result of the functional $\mathbb{O}(xi)$ are indicated in the following outcomes.

Theorem 4.3. *Assume that*

$$\sigma\xi[\mathbb{A}\imath * \varphi(\xi)]_q'' + [\mathbb{A}\imath * \varphi(\xi)]_q' \prec p_2(\xi) + \xi p_2'(\xi), \tag{4.3}$$

where $p_2(0) = 1$ and convex in $\bigcirc$. Additionally, assume that $\mathbb{O}(\xi)$ univalent in $\bigcirc$ such that $\Psi \in \mathbb{H}[p_1(0), 1] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of all injective holomorphic functions f satisfying $\lim_{\xi \in \partial\bigcirc} f \neq \infty$ and

$$p_1(\xi) + \xi\, p_1'(\xi) \prec \sigma\xi[\mathbb{A}\imath * \varphi(\xi)]_q'' + [\mathbb{A}\imath * \varphi(\xi)]_q'. \tag{4.4}$$

Then

$$p_1(\xi) \prec \mathbb{O}(\xi) \prec p_2(\xi)$$

and $p_1(\xi)$ is the best sub-dominant and $p_2(\xi)$ is the best dominant.

Proof. Since,

$$\mathbb{O}(\xi) = \left(\frac{1-\sigma}{\xi}\right) [\mathbb{A}\imath * \varphi(\xi)]_q + \sigma[\mathbb{A}\imath * \varphi(\xi)]_q'$$

then, a computation yields

$$\begin{aligned}\mathbb{O}(\xi) + \xi\mathbb{O}'(\xi) &= \sigma[\mathbb{A}\imath * \varphi(\xi)]_q' + \frac{(\xi(\sigma\xi[\mathbb{A}\imath * \varphi(\xi)]_q'' - (\sigma-1)[\mathbb{A}\imath * \varphi(\xi)]_q') + (\sigma-1)[\mathbb{A}\imath * \varphi(\xi)]_q)}{\xi} \\ &+ \frac{((1-\sigma)[\mathbb{A}\imath * \varphi(\xi)]_q)}{\xi} \\ &= \sigma\chi[\mathbb{A}\imath * \varphi(\xi)]_q'' + [\mathbb{A}\imath * \varphi(\xi)]_q'.\end{aligned}$$

Consequently, we obtain

$$p_1(\xi) + \xi p_1'(\xi) \prec \mathbb{O}(\xi) + \xi\mathbb{O}'(\xi) \prec p_2(\xi) + \xi p_2'(\xi).$$

Hence, Arguments 3.3 and 3.4 provide the request result. □

Theorem 4.4. *Suppose that v is a univalent convex function in $\bigcirc$ with $v(0) = 0$ and*

$$[\mathbb{A}\imath * \varphi(\xi)]_q \prec v(\xi), \quad \xi[\mathbb{A}\imath * \varphi(\xi)_q]' \prec v(\xi).$$

Then

$$\Bbbk\ [\mathbb{A}\imath * \varphi(\xi)]_q + (1-\Bbbk)\ \xi[\mathbb{A}\imath * \varphi(\xi)]_q' \prec v(\xi), \quad \Bbbk \in [0,1].$$

Proof. In view of Eq. (1.3), clearly we have

$$\begin{aligned}\Bbbk\ [\mathbb{A}\imath * \varphi(\xi)]_q + (1-\Bbbk)\ \xi[\mathbb{A}\imath * \varphi(\xi)]_q &= \Bbbk\left(\xi + \sum_{n=2}^{\infty}\left(\frac{3^{(n-1)/3}\Gamma_q\left(\frac{2}{3}\right)\Gamma_q\left(\frac{n}{3}\right)}{\Gamma_q(n)\Gamma_q\left(1-\frac{2n}{3}\right)\Gamma_q\left(\frac{2n}{3}\right)}\right)\varphi_n\xi^n\right) \\ &+ (1-\Bbbk)\left(\xi + \sum_{n=2}^{\infty}\left(\frac{3^{(n-1)/3}\Gamma_q\left(\frac{2}{3}\right)\Gamma_q\left(\frac{n}{3}\right)}{\Gamma_q(n)\Gamma_q\left(1-\frac{2n}{3}\right)\Gamma_q\left(\frac{2n}{3}\right)}\right)n\varphi_n\xi^n\right) \\ &= \xi + \sum_{n=2}^{\infty}\left(\frac{3^{(n-1)/3}\Gamma_q\left(\frac{2}{3}\right)\Gamma_q\left(\frac{n}{3}\right)}{\Gamma_q(n)\Gamma_q\left(1-\frac{2n}{3}\right)\Gamma_q\left(\frac{2n}{3}\right)}\right)(\Bbbk + (1-\Bbbk)n)\,\varphi_n\xi^n \in \Delta.\end{aligned}$$

A straightforward application of Argument 3.5, we receive the outcome. □

Theorem 4.5. *Let $\sigma_2 \le \sigma_1 < 0$ and $\varphi \in \Delta$. Then*

$$[\Sigma_{\sigma_2}(p)]_q \subset [\Sigma_{\sigma_1}(p)]_q.$$

Proof. Let $\varphi \in [\Sigma_{\sigma_2}(p)]_q$. Formulate the next functional

$$\phi(\xi) = \frac{[\mathbb{A}\imath * \varphi(\xi)]_q}{\xi}, \quad \phi(0) = 1.$$

A calculation entails that

$$(1-\sigma_2)\,\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma_2[\mathbb{A}\imath * \varphi(\xi)]'_q = \phi(\xi) + \sigma_2(\xi\,\phi'(\xi)). \tag{4.5}$$

Which yields

$$\phi(\xi) + \sigma_2\,(\xi\,\phi'(\xi)) \prec \frac{a\xi+1}{b\xi+1}.$$

Employing Argument 3.1, where $\sigma_2 > 0$ gets

$$\phi(\xi) \prec \frac{a\xi+1}{b\xi+1}. \tag{4.6}$$

Since $0 < \sigma_1/\sigma_2 < 1$ and $\frac{a\xi+1}{b\xi+1}$ is convex univalent in $\mathbb{O}$, we get

$$\begin{aligned}
(1-\sigma_2)\,\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma_1[\mathbb{A}\imath * \varphi(\xi)]'_q &= (1-\sigma_1)\phi(\xi) + \sigma_1\,[\mathbb{A}\imath * \varphi(\xi)]'_q \\
&= (1-\sigma_1)\phi(\xi) + \sigma_1\,(\xi\phi'(\xi) + \phi(\xi))\,, \\
&= (1-\sigma_1)\phi(\xi) + \sigma_1\,(\xi\phi'(\xi) + \phi(\xi)) + \left(\frac{\sigma_1}{\sigma_2}\phi(\xi) - \frac{\sigma_1}{\sigma_2}\phi(\xi)\right) \\
&= \frac{\sigma_1}{\sigma_2}(1-\sigma_2)\phi(\xi) + \sigma_2\,(\xi\,\phi'(\xi) + \ \phi(\xi)) + \left(1 - \frac{\sigma_1}{\sigma_2}\right)\phi(\xi) \\
&= \frac{\sigma_1}{\sigma_2}\left[\frac{(1-\sigma_2)}{\xi}[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma_2[\mathbb{A}\imath * \varphi(\xi)]'_q\right] + \left(1 - \frac{\sigma_1}{\sigma_2}\right)\phi(\xi) \\
&\prec \frac{a\xi+1}{b\xi+1} = p(\xi).
\end{aligned}$$

Consequently, we arrive at $\varphi \in [\Sigma_{\sigma_1}(p)]_q$. □

Theorem 4.6. *Suppose that*

$$\mathbb{O}(\xi) = (1-\sigma_2)\,\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q.$$

Then

$$\left(\frac{[\mathbb{A}\imath * \varphi(\xi)]'_q}{\chi}\right)\hbar_1 + [\mathbb{A}\imath * \varphi(\xi)]_q[\hbar_1 + 3\hbar_2] + \hbar_2\xi\,[\mathbb{A}\imath * \varphi(\xi)]''_q \prec \left(\frac{1+\xi}{1-\xi}\right)^{c_1}$$
$$\Rightarrow \mathbb{O}(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{c_2},$$

where $c_1 > 0, c_2 > 0, \hbar_1 = 1-\sigma,\ \hbar_2 = \sigma > 0$.

Proof. A calculation implies that

$$\begin{aligned}
\mathbb{O}(\xi) + \xi\mathbb{O}'(\xi) &= \frac{(1-\sigma)}{\xi}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q + \xi\left(\frac{(1-\sigma)}{\xi}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q\right)' \\
&= \left(\frac{[\mathbb{A}\imath * \varphi(\xi)]'_q}{\xi}\right)\hbar_1 + [\mathbb{A}\imath * \varphi(\xi)]_q[\hbar_1 + 3\hbar_2] + \hbar_2\xi[\mathbb{A}\imath * \varphi(\xi)]''_q \\
&\prec \left(\frac{1+\xi}{1-\xi}\right)^{c_1}.
\end{aligned}$$

According to Argument 3.2 with $t = 1$, we get

$$\mathbb{O}(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{c_2}.$$

□

5. Implementation

In light of the recommended operator q-operator $[\mathbb{A}\imath * \varphi(\xi)]_q$, which is given in the class $\left[\Sigma_\sigma\left(\frac{1+\xi}{1-\xi}\right)\right]_q$, we present the fractional 2D-wave diffusive equation using the notion of fractional calculus. The complex diffusive wave equation's upper bound is investigated, where the equation estimated the level of the liquid.
The model simply is formulated by

$$\left(\frac{1-\sigma}{\xi}\right)[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \frac{a\xi+1}{b\xi+1}. \tag{5.1}$$

The solution of (5.1) is reported by the following result.

Theorem 5.1. *Assume the class* $\left[\Sigma_\sigma\left(\frac{1+\xi}{1-\xi}\right)\right]_q$, $\sigma \in (0,1]$. *Then the outcome of the differential equation*

$$(1-\sigma)\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \frac{\xi+1}{1-\xi}, \quad \xi \in \bigcirc$$

is formulated by

$$[\mathbb{A}\imath * \varphi(\xi)]_q \approx \xi\left(\frac{2\xi\ {}_2F_1\left(1, 1+\frac{1}{\sigma}, 2+\frac{1}{\sigma}, \xi\right)}{\sigma+1} + 1\right), \tag{5.2}$$

where ${}_2F_1\,(a,b,c;\xi)$ *indicates the hypergeometric function.*

Proof. Suppose that $\varphi \in \left[\Sigma_\sigma\left(\frac{1+\xi}{1-\xi}\right)\right]_q$. Then it achieves the differential equation

$$\left(\frac{1-\sigma}{\xi}\right)[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \frac{\omega(\xi)+1}{1-\omega(\xi)},$$

where $\omega(0) = 0$ and $|\omega| \le |\xi| < 1$. This leads to

$$[\mathbb{A}\imath * \varphi(\xi)]_q = \xi^{(\sigma-1)/\sigma}\int_0^\xi -z^{1/(\sigma-1)}\left(\frac{\omega(z)+1}{\sigma(\omega(z)-1)}\right)dz.$$

In order to determined the upper solution, we consume $\omega(\xi) = \xi$. So, we get

$$(1-\sigma)\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma[\mathbb{A}\imath * \varphi(\xi)]'_q = \frac{\xi+1}{1-\xi}.$$

Rearrange the above equation, as follows:

$$[\mathbb{A}\imath * \varphi(\xi)]'_q + \frac{1-\sigma}{\sigma\xi}\,[\mathbb{A}\imath * \varphi(\xi)]_q = \left(\frac{1}{\sigma}\right)\left(\frac{1+\xi}{1-\xi}\right).$$

Multiplying the above equation by the functional

$$\tau(\xi) = \exp\left(\int \frac{1-\sigma}{\sigma\xi}d\xi\right),$$

then, we obtain

$$\xi^{1/\sigma-1}[\mathbb{A}\imath * \varphi(\xi)]'_q - \frac{[\mathbb{A}\imath * \varphi(\xi)]_q\left((1-\sigma)\xi^{1/\sigma-2}\right)}{\sigma} = \left(\frac{\xi^{1/\sigma-1}}{\sigma}\right)\left(\frac{1+\xi}{1-\xi}\right).$$

Hence, it follows the solution (5.2). □

Similarly, we obtain the next result for functions in the class $\left[\Sigma_\sigma\left(\frac{1+\xi}{1-\xi}\right)^\sigma\right]_q$, $\sigma \in (0,1]$.

Theorem 5.2. *Consider the class of analytic functions* $\left[\Sigma_\sigma\left(\frac{1+\xi}{1-\xi}\right)^\sigma\right]_q$, $\sigma \in (0,1]$. *Then the solution of the fractional quantum differential equation*

$$(1-\sigma)\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \left(\frac{\xi+1}{1-\xi}\right)^\sigma, \quad \xi \in \bigcirc$$

is given by the formula

$$[\mathbb{A}\imath * \varphi(\xi)]_q \approx \frac{\xi^{(\sigma-1)/\sigma)}\left((\sigma+1)\xi^{1/\sigma}\left(\frac{\xi+1}{1-\xi}\right)^\sigma F_1\left(\frac{1}{\sigma};\sigma,-\sigma;1+\frac{1}{\sigma};\xi,-\xi\right)\right)}{\sigma^2\,F_1(\xi)+\sigma^2F_2(\xi)+(\sigma+1)F_3(\xi)}, \tag{5.3}$$

where

$$\begin{aligned} F_1(\xi) &:= \xi F_1\left(1+\frac{1}{\sigma};\sigma,1-\sigma;2+\frac{1}{\sigma};\xi,-\xi\right)\\ F_2(\xi) &:= \xi F_1\left(1+\frac{1}{\sigma};\sigma+1,-\sigma;2+\frac{1}{\sigma};\xi,-\xi\right)\\ F_3(\xi) &:= F_1\left(\frac{1}{\sigma};\sigma,-\sigma;1+\frac{1}{\sigma};\xi,-\xi\right). \end{aligned}$$

Proof. Suppose that $\varphi \in \left[\Sigma_\sigma\left(\frac{1+\xi}{1-\xi}\right)^\sigma\right]_q$. Then, we have

$$(1-\sigma)\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \left(\frac{\omega(\xi)+1}{1-\omega(\xi)}\right)^\sigma.$$

As a consequence, we get

$$[\mathbb{A}\imath * \varphi(\xi)]_q = \xi^{(\sigma-1)/\sigma}\int_0^\xi -z^{1/(\sigma-1)}\left(\frac{\omega(z)+1}{\sigma(\omega(z)-1)}\right)^\sigma dz.$$

The upper solution is given, when $\omega(\xi)=\xi$. Therefore, we have

$$(1-\sigma)\xi^{-1}\,[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma[\mathbb{A}\imath * \varphi(\xi)]'_q = \left(\frac{\xi+1}{1-\xi}\right)^\sigma.$$

Rearrange the above equation as follows:

$$[\mathbb{A}\imath * \varphi(\xi)]'_q + \frac{1-\sigma}{\sigma\xi}\,[\mathbb{A}\imath * \varphi(\xi)]_q = \left(\frac{1}{\sigma}\right)\left(\frac{1+\xi}{1-\xi}\right)^\sigma.$$

Multiplying the above equation by the functional

$$\tau(\xi) = \exp\left(\int \frac{1-\sigma}{\sigma\xi}d\xi\right),$$

then, we obtain

$$\xi^{1/\sigma-1}[\mathbb{A}\imath * \varphi(\xi)]'_q - \frac{[\mathbb{A}\imath * \varphi(\xi)]_q\left((1-\sigma)\xi^{1/\sigma-2}\right)}{\sigma} = \left(\frac{\xi^{1/\sigma-1}}{\sigma}\right)\left(\frac{1+\xi}{1-\xi}\right)^\sigma.$$

Thus, it follows the solution (5.3). □

In general, we have the following result:

Theorem 5.3. *Consider the class of analytic functions* $\left[\Sigma_{\sigma}\left(\frac{1+\xi}{1-\xi}\right)^{\upsilon}\right]_q$, $\sigma \in (0,1]$, $\upsilon > 0$. *Then the solution of the fractional quantum differential equation*

$$\left(\frac{1-\sigma}{\xi}\right)[\mathbb{A}\imath * \varphi(\xi)]_q + \sigma\,[\mathbb{A}\imath * \varphi(\xi)]'_q = \left(\frac{\xi+1}{1-\xi}\right)^{\upsilon}, \quad \xi \in \bigcirc$$

is given by the formula

$$[\mathbb{A}\imath * \varphi(\xi)]_q \approx \frac{\xi^{(\sigma-1)/\sigma}\left((\sigma+1)\xi^{1/\sigma}\left(\frac{\xi+1}{1-\xi}\right)^{\upsilon} F_1\left(\frac{1}{\sigma};\upsilon,-\upsilon;1+\frac{1}{\sigma};\xi,-\xi\right)\right)}{\sigma\,\upsilon\,G_1(\xi)+\sigma\,\upsilon\,G_2(\xi)+(\sigma+1)G_3(\xi)}, \tag{5.4}$$

where

$$G_1(\xi) := \xi F_1\left(1+\frac{1}{\sigma};\upsilon,1-\upsilon;2+\frac{1}{\sigma};\xi,-\xi\right)$$
$$G_2(\xi) := \xi F_1\left(1+\frac{1}{\sigma};\upsilon+1,-\upsilon;2+\frac{1}{\sigma};\xi,-\xi\right)$$
$$G_3(\xi) := F_1\left(\frac{1}{\sigma};\upsilon,-\upsilon;1+\frac{1}{\sigma};\xi,-\xi\right).$$

6. Conclusion

The recent analysis introduced a generalization of the quantum Airy operator in a complex domain utilizing the quantum calculus. We communicated it in a linear convolution operator performing on a normalized function. The suggested operator is used to describe a class of holomorphic functions, and the sufficient conditions for the sandwich theorem are investigated (Theorem 4.3).

For next works, one can present a new subclass of analytic functions called Airy class denoting by Λ, fulfilling the structure

$$\left(1+\frac{\xi\varphi''(\xi)}{\varphi'(\xi)}\right) - \frac{\xi^3}{\left(\frac{\xi\varphi'(\xi)}{\varphi(\xi)}\right)} \prec p(\xi), \quad \xi \in \mathbb{C},$$

where $p(\xi) \in \mathcal{P}$ with $p(0) = 1$ and $\varphi \in \Delta$. Furthermore, one can suggest any types of analytic functions in the open unit disk, such as the meromorphic functions, multi-valent functions and harmonic functions.

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References

[1] G. B. Airy, *On the intensity of light in the neighbourhood of a caustic*, Trans. Cambridge Philos. Soc. **6**, 379–402, 1838.

[2] B. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15 (4)**, 737–745, 1984.

[3] G. Dong and M. Liu, *On certain subclass of Bazilevic functions*, J. Inequal. Pure Appl. Math. **8 (1)**, 2007; Article ID: 12.

[4] R. W. Ibrahim and D. Baleanu, *On quantum hybrid fractional conformable differential and integral operators in a complex domain*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **115**, 2021; Article ID: 31.

[5] R. W. Ibrahim and D. Baleanu, *On a new linear operator formulated by Airy functions in the open unit disk*, Adv. Difference Equ. **2021**, 2021; Article ID: 366.

[6] R. W. Ibrahim and D. Baleanu, *Generalized quantum integro-differential fractional operator with application of 2D-shallow water equation in a complex domain*, Axioms **10 (4)**, 2021; Article ID: 342.

[7] R. W. Ibrahim and M. Darus, *New classes of analytic functions involving generalized Noor integral operator*, J. Inequal. Appl. **2008**, 2008; Article ID: 390435.

[8] R. W. Ibrahim and M. Darus, *On analytic functions associated with the Dziok-Srivastava linear operator and Srivastava-Owa fractional integral operator*, Arab. J. Sci. Eng. **36 (3)**, 441–450, 2011.

[9] R. W. Ibrahim, R. M. Elobaid and S. J. Obaiys, *A class of quantum Briot-Bouquet differential equations with complex coefficients*, Mathematics **8 (5)**, 2020; Article ID: 794.

[10] R. W. Ibrahim, R. M. Elobaid and S. J. Obaiys, *On subclasses of analytic functions based on a quantum symmetric conformable differential operator with application*, Adv. Difference Equ. **2020**, 2020; Article ID: 325.

[11] R. W. Ibrahim, S. B. Hadid and S. Momani, *Generalized Briot-Bouquet differential equation by a quantum difference operator in a complex domain*, Int. J. Dyn. Control **8 (3)**, 762–771, 2020.

[12] F. Jackson, *q-form of Taylor's theorem*, Messenger Math. **38**, 62–64, 1909.

[13] B. Jinho, K. Liechty and G. Schehr, *On the joint distribution of the maximum and its position of the* $Airy_2$ *process minus a parabola*, J. Math. Phys. **53 (8)**, 2012; Article ID: 083303.

[14] S. S. Miller and P. T. Mocanu, *Differential subordinations: Theory and applications*, CRC Press, Boca Raton, FL, USA, 2000.

[15] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, Theory Appl. **48 (10)**, 815–826, 2003.

[16] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. **39 (5)**, 1057–1077, 1987.

[17] S. Ruscheweyh, *Convolutions in geometric function theory*, Les Presses De L'Universite De Montreal, Montreal, 1982.

[18] T. Shanmugam, S. Sivasubramanian and H. M. Srivastava, *Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations*, Integral Transforms Spec. Funct. **17 (12)**, 889–899, 2006.

[19] S. K. Sharma and R. Jain, *On some properties of generalized q-Mittag Leffler function*, Math. AEterna **4 (6)**, 613–619, 2014.