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The q-analogue of a specific property of second order linear recurrences

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Abstract

A translated recurrent sequence of rank two is related to the Fibonacci sequence, this property is generalized in this paper using a q-analogue of Fibonacci sequence suggested by J. Cigler. We give some specialization to the generalized Fibonacci and Lucas sequences.

Keywords: Fibonacci polynomials, Lucas polynomials, q-analogues

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1. Introduction

The Fibonacci and Lucas numbers, denoted respectively by (F_n) and (L_n), are defined by

$$\begin{cases} F_0 = 0,\ F_1 = 1, \\ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \end{cases} \quad \text{and} \quad \begin{cases} L_0 = 2,\ L_1 = 1, \\ L_n = L_{n-1} + L_{n-2} \quad (n \geq 2). \end{cases}$$

These sequences have an explicit form via the binomial coefficients,

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \quad (n \geq 0), \tag{1.1}$$

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} \quad (n \geq 1). \tag{1.2}$$

It is well known, see [17], that any sequence (G_n) satisfying $G_{n+2} = G_{n+1} + G_n$ verify, for any $k \geq 1$

$$G_{n+k} = F_{k-1} G_n + F_k G_{n+1}. \tag{1.3}$$

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Many identities satisfying by Fibonacci and Lucas numbers are related to Equality (1.3). In order to extend it to Carlitz's q-Fibonacci polynomials and Cigler's q-Fibonacci polynomials, we use the common q-Fibonacci polynomials proposed by Cigler in [14]. Also, we use the q-Lucas polynomials suggested by the authors [7] in order to give some identities related to (1.3). We can find other papers on q-analogue on Fibonacci sequence and on bi-periodic Fibonacci sequence in [11, 13, 15, 16]. The Carlitz' q-Fibonacci polynomials [10] and the Cigler's q-Fibonacci polynomials [12] are defined respectively by $\mathbf{F}_0(z) = 0$ and for $n \geq 0$,

$$\mathbf{F}_{n+1}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q z^k,$$
$$\mathbf{F}_{n+1}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q z^k$$

with the q-notations

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

In [14], Cigler proposed as q-Fibonacci and q-Lucas polynomials the following unified approaches

$$\mathbf{F}_{n+1}(z,m) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q z^k, \tag{1.4}$$

$$\mathbf{Luc}_n(z,m) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(1+m)\binom{k}{2}} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q z^k. \tag{1.5}$$

He also found that the q-Fibonacci polynomials satisfy the following relations

$$\mathbf{F}_{n+2}(z,m) = \mathbf{F}_{n+1}(z,m) + q^n z \mathbf{F}_n\left(q^{m-1}z, m\right), \tag{1.6}$$

$$\mathbf{F}_{n+2}(z,m) = \mathbf{F}_{n+1}(qz,m) + qz \mathbf{F}_n\left(q^{m+1}z, m\right). \tag{1.7}$$

For $m = 1$ we recover the Carlitz' q-Fibonacci polynomials which satisfy the recursions

$$\mathbf{F}_{n+2}(z,1) = \mathbf{F}_{n+1}(z,1) + q^n z \mathbf{F}_n(z,1),$$
$$\mathbf{F}_{n+2}(z,1) = \mathbf{F}_{n+1}(qz,1) + qz \mathbf{F}_n\left(q^2 z, m\right).$$

These two recursions can be expressed by the relation

$$\begin{pmatrix} 0 & 1 \\ q^{n-1}z & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q^{n-2}z & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ z & 1 \end{pmatrix} = \begin{pmatrix} z\mathbf{F}_{n-1}(qz,1) & \mathbf{F}_n(z,1) \\ z\mathbf{F}_n(qz,1) & \mathbf{F}_{n+1}(z,1) \end{pmatrix}, \tag{1.8}$$

which permits to consider the polynomials $\mathbf{Luc}_n(z,1)$ as the trace of the last product.

For $m = 0$ we recover the first Cigler's approach to q-Fibonacci polynomials and q-Lucas polynomials, this approach is generalized in [8] where we introduce a q-analogue for bi^{s}nomial coefficients.

In [7], we suggested two alternatives to the q-Lucas polynomials $\mathbf{Luc}_n(z,m)$, the q-Lucas polynomials of the first kind denoted $\mathbf{L}_n(z,m)$ and the q-Lucas polynomials of the second kind denoted $\mathbb{L}_n(z,m)$, defined respectively by

$$\mathbf{L}_n(z,m) := \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + \frac{[k]_q}{[n-k]_q}\right) z^k, \tag{1.9}$$

$$\mathbb{L}_n(z,m) := \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^{n-2k} \frac{[k]_q}{[n-k]_q}\right) z^k, \tag{1.10}$$

and we showed that the q-Lucas polynomials of the first kind $\mathbf{L}_n(z,m)$ satisfy the same recursion given by relation (1.6)

$$\mathbf{L}_{n+2}(z,m)=\mathbf{L}_{n+1}(z,m)+q^n z\mathbf{L}_n\left(q^{m-1}z,m\right),$$

and the q-Lucas polynomials of the second kind $\mathbb{L}_n(z,m)$ satisfy the same recursion given by relation (1.7)

$$\mathbb{L}_{n+2}(z,m)=\mathbb{L}_{n+1}(qz,m)+qz\mathbb{L}_n\left(q^{m+1}z,m\right).$$

As we deal with recurrent sequences, we can extend it to the negative integers. For example it's established for the q-Fibonacci polynomials [14], that

$$\mathbf{F}_{-n}(z,m)=(-1)^{n-1}q^{-m\binom{n}{2}}\frac{\mathbf{F}_n\left(q^{-mn}z,m\right)}{\left(q^{-mn}z\right)^n}, \tag{1.11}$$

and for the q-Lucas polynomials of the both kinds, see [6], the authors proposed

$$\begin{aligned}\mathbf{L}_{-n}(z,m) &= (-1)^n q^{-m\binom{n}{2}}\frac{\mathbb{L}_n\left(q^{-mn}z,m\right)}{\left(q^{-mn}z\right)^n},\\ \mathbb{L}_{-n}(z,m) &= (-1)^n q^{-m\binom{n}{2}}\frac{\mathbf{L}_n\left(q^{-mn}z,m\right)}{\left(q^{-mn}z\right)^n},\end{aligned}$$

thus the q-Lucas polynomials suggested by Cigler satisfy $\mathbf{Luc}_n(z,m)=\left(\mathbf{L}_n(z,m)+\mathbb{L}_n(z,m)\right)/2$.

Observing that the q-Fibonacci polynomials satisfy the two recursions given by relations (1.6) and (1.7), we expand the equality (1.3) for the sequences satisfying (1.6), and for the sequences satisfying (1.7).
We need to introduce some notations.

Notation 1.1. We consider the following sets according to relations (1.6) and (1.7):

$\Omega_m=\left\{U=(U_n(z))_{n\in\mathbb{Z}}: U_{n+2}(z)=U_{n+1}(z)+q^n zU_n\left(q^{m-1}z\right) \text{ with } U_0(z), U_1(z)\in\mathbb{R}[z]\right\}$,

$\Theta_m=\left\{U=(U_n(z))_{n\in\mathbb{Z}}: U_{n+2}(z)=U_{n+1}(qz)+qzU_n\left(q^{m+1}z\right) \text{ with } U_0(z), U_1(z)\in\mathbb{R}[z]\right\}$,

and the operators $E, z, Q_m, \widehat{Q}$ such that: $(EU)_n(z)=U_{n+1}(z), (zU)_n(z)=zU_n(z), (Q_mU)_n(z)=U_n(q^m z)$, $\left(\widehat{Q}U\right)_n(z)=q^nU_n(z)$.

Remark 1.2. By the applications $\Psi:\Omega_m\longrightarrow(\mathbb{R}[z])^2$ and $\Psi:\Theta_m\longrightarrow(\mathbb{R}[z])^2$ such that $\Psi\left((U_n(z))_{n\in\mathbb{Z}}\right)=\binom{U_0(z)}{U_1(z)}$, each of the sets Ω_m,Θ_m endowing with operations of addition of sequences and multiplication with real number is a real vectorial space isomorphic to $(\mathbb{R}[z])^2$.

Proposition 1.3. *The real vectorial space Ω_m is preserved by the operations zQ_{m-1}, EQ_{-1}, and the real vectorial space Θ_m is preserved by E and $\widehat{Q}zQ_{m-1}$.*

Proof. Let $U\in\Omega_m$, and let $a=zQ_{m-1}U$ and $b=EQ_{-1}U$. For $n\in\mathbb{Z}$, we have $a_{n+1}(z)+q^nza_n\left(q^{m-1}z\right)=z\left(U_{n+1}\left(q^{m-1}z\right)+q^{n+m-1}zU_n\left(q^{2m-2}z\right)\right)=zU_{n+2}\left(q^{m-1}z\right)=a_{n+2}(z)$ and $b_{n+1}(z)+q^nzb_n\left(q^{m-1}z\right)=U_{n+2}(z/q)+q^nzU_n\left(q^{m-2}z\right)=U_{n+3}(z/q)=b_{n+2}(z)$.

Let $U\in\Theta_m$, $c=EU$ and $d=\widehat{Q}zQ_{m-1}U$. For $n\in\mathbb{Z}$, we have $c_{n+1}(qz)+qzc_n\left(q^{m+1}z\right)=U_{n+2}(qz)+qzU_{n+1}(z)=U_{n+3}\left(q^{m+1}z\right)=c_{n+2}(z)$ and

$d_{n+1}(qz)+qzd_n\left(q^{m+1}z\right)=z\left(q^{n+2}U_{n+1}(q^mz)+q^{n+m+2}zU_n\left(q^{2m}z\right)\right)=q^{n+2}zU_{n+2}\left(q^{m-1}z\right)=d_{n+2}(z/q)$. □

Proposition 1.4. *The isomorphism $\Psi:\Omega_m\longrightarrow(\mathbb{R}[z])^2$ satisfy*

$$\Psi^{-1}\left(z^i\Psi(U)\right)=z^iQ_{m-1}^{-i}U \text{ for every } U\in\Omega_m \text{ such that } \Psi(U)\in\mathbb{R}^2, \tag{1.12}$$

where $z\binom{p(z)}{t(z)}=\binom{zp(z)}{zt(z)}$.

The isomorphism $\Psi:\Theta_m\longrightarrow(\mathbb{R}[z])^2$ satisfy

$$\begin{aligned}\Psi^{-1}\left((qz)^i\Psi(U)\right) &= z^i\widehat{Q}^iQ_{m-1}^{-i}U \text{ for every } U\in\Theta_m \text{ such that } \Psi(U)\in\{0\}\times\mathbb{R}, &(1.13)\\ \Psi^{-1}\left(z^i\Psi(U)\right) &= z^i\widehat{Q}^iQ_{m-1}^{-i}U \text{ for every } U\in\Theta_m \text{ such that } \Psi(U)\in\mathbb{R}\times\{0\}, &(1.14)\end{aligned}$$

where $z\binom{p(z)}{t(z)}=\binom{zp(z)}{zt(z)}$.

Proof. For $U \in \Omega_m$ such that $\Psi(U) \in \mathbb{R}^2$, we have $\Psi\left(z^i Q_{m-1}^{-i} U\right) = \Psi\left(z^i U\right) = \binom{z^i U_0(z)}{z^i U_1(z)} = z^i \Psi(U)$.

Then $z^i Q_{m-1}^{-i} U = \Psi^{-1}\left(z^i \Psi(U)\right)$, because $\Psi^{-1}\left(z^i \Psi(U)\right) \in \Omega_m$ and according to Proposition 1.3 $z^i Q_{m-1}^{-i} U \in \Omega_m$.

For $U \in \Theta_m$ such that $\Psi(U) \in \{0\} \times \mathbb{R}$ we have $\Psi\left(z^i \widehat{Q}^i Q_{m-1}^{-i} U\right) = \Psi\left(z^i \widehat{Q}^i U\right) = \binom{0}{(qz)^i U_1(z)} = (qz)^i \Psi(U)$.

Then $\Psi^{-1}\left((qz)^i \Psi(U)\right) = z^i \widehat{Q}^i Q_{m-1}^{-i} U$, because $\Psi^{-1}\left((qz)^i \Psi(U)\right) \in \Theta_m$ and according to Proposition 1.3 $z^i \widehat{Q}^i Q_{m-1}^{-i} U \in \Theta_m$.

For $U \in \Theta_m$ such that $\Psi(U) \in \mathbb{R} \times \{0\}$ we have $\Psi\left(z^i \widehat{Q}^i Q_{m-1}^{-i} U\right) = \Psi\left(z^i \widehat{Q}^i U\right) = \binom{z^i U_1}{0} = z^i \Psi(U)$.

Then $\Psi^{-1}\left(z^i \Psi(U)\right) = z^i \widehat{Q}^i Q_{m-1}^{-i} U$, because $\Psi^{-1}\left(z^i \Psi(U)\right) \in \Theta_m$ and according to Proposition 1.3 $z^i \widehat{Q}^i Q_{m-1}^{-i} U \in \Theta_m$. □

Remark 1.5. As $\Psi\left((z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}}\right) = \binom{1}{0}$, $\Psi\left((\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right) = \binom{0}{1}$ and $\Psi\left(\left(q^{n-1} z\mathbf{F}_{n-1}\left(q^{m-1} z, m\right)\right)_{n\in\mathbb{Z}}\right) = \binom{1}{0}$, the pair of sequences $\left((z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}}, (\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)$ will be considered as canonical basis of real vectorial space $\left\{U \in \Omega_m \text{ such that } \Psi(U) \in \mathbb{R}^2\right\}$ and the pair of sequences $\left(\left(q^{n-1} z\mathbf{F}_{n-1}\left(q^{m-1} z, m\right)\right)_{n\in\mathbb{Z}}, (\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)$ will be considered as canonical basis of real vectorial space $\left\{U \in \Theta_m \text{ such that } \Psi(U) \in \mathbb{R}^2\right\}$.

Proposition 1.6. *Let $G \in \Omega_m \cup \Theta_m$, for a negative integer n we have*

$$G_n(z) = p(z) + q(1/z),$$

where $p(z)$ and $q(z)$ are two polynomials.

Proof. For $G \in \Omega_m$ with $G_0(z) = \sum_{i=0}^{s} \alpha_i z^i$ and $G_1(z) = \sum_{i=0}^{r} \beta_i z^i$, we have

$$\Psi(G) = \sum_{i=0}^{s} \alpha_i z^i \begin{pmatrix}1\\0\end{pmatrix} + \sum_{i=0}^{r} \beta_i z^i \begin{pmatrix}0\\1\end{pmatrix} = \Psi^{-1}\left(z^i \Psi(U)\right) = z^i Q_m^{-i} U.$$

And according to Remark 1.5 and Proposition 1.4, we have

$$(G(z))_{n\in\mathbb{Z}} = \sum_{i=0}^{s} \alpha_i z^i Q_m^{-i} (z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}} + \sum_{i=0}^{r} \beta_i z^i Q_m^{-i} (\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}.$$

With the expression (1.11) we get the result.

For $G \in \Theta_m$, we use the same method. □

2. Main results

We need to introduce the two following correspondences.

For $p(z) = \sum_{i=c}^{d} \alpha_i z^i$ with $c, d \in \mathbb{Z}$, we consider the two operators $\widetilde{p}$ and $\widehat{p}$ defined by

$$\widetilde{p}(z) = \sum_{i=c}^{d} a_i z^i Q_{m-1}^{-i} \text{ and } \widehat{p}(z) = \sum_{i=c}^{d} a_i z^i \widehat{Q}^i.$$

Now, we give the generalization of the equality (1.3).

Theorem 2.1. *For $G \in \Omega_m$ and $G \in \Theta_m$ we have respectively*

$$E^k Q_{-k} (G_n(z))_{n\in\mathbb{Z}} = \widetilde{G}_k\left(z/q^k\right)(z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}} + \widetilde{G}_{k+1}\left(z/q^k\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}, \quad (2.1)$$

$$E^k (G_n(z))_{n\in\mathbb{Z}} = \widehat{\widetilde{G}}_k(z)\left(q^{n-1} z\mathbf{F}_{n-1}\left(q^{m-1} z, m\right)\right)_{n\in\mathbb{Z}} + \widehat{\widetilde{G}}_{k+1}\left(\frac{z}{q}\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}. \quad (2.2)$$

Proof. Let $G \in \Omega_m$ and $k \in \mathbb{Z}$, with the Proposition 1.6, we put $G_k(z) = \sum_{i=r}^{s} \alpha_i z^i$ and $G_{k+1}(z) = \sum_{i=l}^{t} \beta_i z^i$, where $r, s, l, t \in \mathbb{Z}$.

The two sequences $E^k Q_{-k}(G_n(z))_{n\in\mathbb{Z}}$ and $\widetilde{G}_k\left(z/q^k\right)(z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}} + \widetilde{G}_{k+1}\left(z/q^k\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}$ belong to Ω_m According to Proposition 1.3. Then it suffices to show that $\Psi\left(E^k Q_{-k}((G_n(z))_{n\in\mathbb{Z}})\right) = \Psi\, \widetilde{G}_k\left(z/q^k\right)(z\mathbf{F}_{n-1}(z))_{n\in\mathbb{Z}} + \widetilde{G}_{k+1}\left(z/q^k\right)(\mathbf{F}_n(z))_{n\in\mathbb{Z}}\right)$.

We have $\Psi\left(E^k Q^{-k}((G_n(z))_{n\in\mathbb{Z}})\right) = \binom{G_k(z/q^k)}{G_{k+1}(z/q^k)} = \sum_{i=r}^{s} \alpha_i \left(z/q^k\right)^i \binom{1}{0} + \sum_{i=l}^{t} \beta_i \left(z/q^k\right)^i \binom{0}{1}$, we also have

$$\begin{aligned}
&\Psi\left(\widetilde{G}_k\left(z/q^k\right)(z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}} + \widetilde{G}_{k+1}\left(z/q^k\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)\\
= \;&\Psi\left(\sum_{i=r}^{s} \alpha_i \left(z/q^k\right)^i Q_m^{-i}(z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}} + \sum_{i=l}^{t} \beta_i \left(z/q^k\right)^i Q_m^{-i}(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)\\
= \;&\sum_{i=r}^{s} \frac{\alpha_i}{q^{ik}} \Psi\left(z^i Q_m^{-i}(z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}}\right) + \sum_{i=l}^{t} \frac{\beta_i}{q^{ik}} \Psi\left(z^i Q_m^{-i}(\mathbf{F}(z, m))_{n\in\mathbb{Z}}\right).
\end{aligned}$$

According to (1.12) we get

$$\begin{aligned}
&\Psi\left(\widetilde{G}_k\left(z/q^k\right)(z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}} + \widetilde{G}_{k+1}\left(z/q^k\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)\\
= \;&\sum_{i=r}^{s} \alpha_i \left(z/q^k\right)^i \Psi((z\mathbf{F}_{n-1}(q^m z, m))_{n\in\mathbb{Z}}) + \sum_{i=l}^{t} \beta_i \left(z/q^k\right)^i \Psi((\mathbf{F}_n(z, m))_{n\in\mathbb{Z}})\\
= \;&\sum_{i=r}^{s} \alpha_i \left(z/q^k\right)^i \begin{pmatrix}1\\0\end{pmatrix} + \sum_{i=l}^{t} \beta_i \left(z/q^k\right)^i \begin{pmatrix}0\\1\end{pmatrix}.
\end{aligned}$$

Let $G \in \Theta_m$, and $k \in \mathbb{Z}$, with the Proposition 1.6, we put $G_k(z) = \sum_{i=r}^{s} \alpha_i z^i$ and $G_{k+1}(z) = \sum_{i=0}^{t} \beta_i z^i$ where $r, s, l, t \in \mathbb{Z}$.

The two sequences $E^k(G_n(z))_{n\in\mathbb{Z}}$ and $\widehat{\widetilde{G}}_k(z)\left(q^{n-1} z\mathbf{F}_{n-1}(z)\right)_{n\in\mathbb{Z}} + \widehat{\widetilde{G}}_{k+1}\left(\frac{z}{q}\right)(\mathbf{F}_n(z))_{n\in\mathbb{Z}}$ belong to Θ_m according to Proposition 1.3, then it suffices to show that

$$\Psi\left(E^k(G_n(z))_{n\in\mathbb{Z}}\right) = \Psi\left(\widehat{\widetilde{G}}_k(z)\left(q^{n-1} z\mathbf{F}_{n-1}(z)\right)_{n\in\mathbb{Z}} + \widehat{\widetilde{G}}_{k+1}\left(\frac{z}{q}\right)(\mathbf{F}_n(z))_{n\in\mathbb{Z}}\right).$$

We have $\Psi\left(E^k((G_n(z))_{n\in\mathbb{Z}})\right) = \binom{G_k(z)}{G_{k+1}(z)} = \sum_{i=r}^{s} \alpha_i z^i \binom{1}{0} + \sum_{i=0}^{t} \beta_i z^i \binom{0}{1}$, we also have

$$\begin{aligned}
&\Psi\left(\widehat{\widetilde{G}}_k(z)\left(q^{n-1} z\mathbf{F}_{n-1}(z, m)\right)_{n\in\mathbb{Z}} + \widehat{\widetilde{G}}_{k+1}\left(\frac{z}{q}\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)\\
= \;&\Psi\left(\sum_{i=r}^{s} \alpha_i z^i \widehat{Q}^i Q_m^{-i}\left(q^{n-1} z\mathbf{F}_{n-1}(z, m)\right)_{n\in\mathbb{Z}} + \sum_{i=l}^{t} \beta_i \left(\frac{z}{q}\right)^i \widehat{Q}^i Q_m^{-i}(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)\\
= \;&\sum_{i=r}^{s} \alpha_i \Psi\left(z^i \widehat{Q}^i \left(q^{n-1} z\mathbf{F}_{n-1}(z, m)\right)_{n\in\mathbb{Z}}\right) + \sum_{i=l}^{t} \beta_i \Psi\left(\left(\frac{z}{q}\right)^i \widehat{Q}^i (\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right).
\end{aligned}$$

According to (1.13) and (1.14) we get

$$\begin{aligned}
&\Psi\left(\widehat{\widetilde{G}}_k(z)\left(q^{n-1} z\mathbf{F}_{n-1}(z, m)\right)_{n\in\mathbb{Z}} + \widehat{\widetilde{G}}_{k+1}\left(\frac{z}{q}\right)(\mathbf{F}_n(z, m))_{n\in\mathbb{Z}}\right)\\
= \;&\sum_{i=r}^{s} \alpha_i z^i \Psi\left(\left(q^{n-1} z\mathbf{F}_{n-1}(z, m)\right)_{n\in\mathbb{Z}}\right) + \sum_{i=l}^{t} \beta_i z^i \Psi((\mathbf{F}_n(z, m))_{n\in\mathbb{Z}})\\
= \;&\sum_{i=r}^{s} \alpha_i z^i \begin{pmatrix}1\\0\end{pmatrix} + \sum_{i=l}^{t} \beta_i z^i \begin{pmatrix}0\\1\end{pmatrix},
\end{aligned}$$

□

In particular, if k is fixed we get the following results.

Corollary 2.2. *For $G \in \Omega_m$ and $G \in \Theta_m$ we have respectively, for $n \geq 0$;*

$$G_{k+n}\left(z/q^k\right) = G_k\left(z/q^k\right) * z\mathbf{F}_{n-1}\left(q^m z, m\right) + G_{k+1}\left(z/q^k\right) * \mathbf{F}_n\left(z, m\right), \tag{2.3}$$

$$G_{k+n}\left(z\right) = G_k\left(z\right) \Delta q^{n-1} z\mathbf{F}_{n-1}\left(q^{m-1} z, m\right) + G_{k+1}\left(\frac{z}{q}\right) \Delta \mathbf{F}_n\left(z, m\right), \tag{2.4}$$

where $\left(\sum_{i=c}^d \alpha_i z^i\right) * U_n(z) = \sum_{i=c}^d \alpha_i z^i U_n\left(q^{(m-1)i} z\right)$ *and* $\left(\sum_{i=c}^d \alpha_i z^i\right) \Delta U_n(z) = \sum_{i=c}^d \alpha_i z^i q^{ni} U_n\left(q^{(m-1)i} z\right)$ *with* $c, d \in \mathbb{Z}$.

Corollary 2.3. *The situation of negative indices: for $G \in \Omega_m$ and $G \in \Theta_m$ we have respectively, for $n \geq 0$;*

$$G_{k-n}\left(z/q^k\right) = (-1)^n q^{m\binom{n+1}{2}} \left(G_k\left(z/q^k\right) * \frac{\mathbf{F}_{n+1}\left(q^{-mn} z, m\right)}{z^n} - G_{k+1}\left(z/q^k\right) * \frac{\mathbf{F}_n\left(q^{-mn} z, m\right)}{z^n}\right),$$

$$G_{k-n}\left(z\right) = (-1)^n q^{m\binom{n+1}{2}} \left(G_k\left(z\right) \Delta \frac{\mathbf{F}_{n+1}\left(q^{-mn-1} z, m\right)}{z^n} - G_{k+1}\left(\frac{z}{q}\right) \Delta \frac{\mathbf{F}_n\left(q^{-mn} z, m\right)}{z^n}\right).$$

Proof. It suffices to use (1.11). □

With the Carlitz approach we get the particular case $m = 1$ of Corollary 2.2 by an other way:

For $m = 1$, and $p(z) = \sum_{i=c}^d \alpha_i z^i$, we have $p(z) * U_n(z) = p(z) U_n(z)$ and $p(z) \Delta U_n(z) = p(q^n z) U_n(z)$ $(n \geq 0)$, thus relation (2.3) becomes, for $G \in \Omega_1$ and $n \geq 0$, as follows

$$G_{k+n}\left(z/q^k\right) = G_k\left(z/q^k\right) z\mathbf{F}_{n-1}\left(qz, 1\right) + G_{k+1}\left(z/q^k\right) \mathbf{F}_n\left(z, 1\right), \tag{2.5}$$

and relation (2.4) becomes, for $G \in \Theta_1$ and $n \geq 0$, as follows

$$G_{k+n}\left(z\right) = G_k\left(q^n z\right)\left(q^{n-1} z\mathbf{F}_{n-1}\left(z, 1\right)\right) + G_{k+1}\left(q^{n-1} z\right)\left(\mathbf{F}_n\left(z, 1\right)\right). \tag{2.6}$$

The two identities (2.5) and (2.6) can be obtained by (1.8):

Let $G \in \Omega_1$, since $E^k Q^{-k} G \in \Omega_1$ then for $C(z) = \begin{pmatrix} 0 & 1 \\ z & 1 \end{pmatrix}$ we have

$$\begin{aligned}
\begin{pmatrix} G_{k+n}\left(z/q^k\right) \\ G_{k+n+1}\left(z/q^k\right) \end{pmatrix} &= C\left(q^{n-1} z\right) C\left(q^{n-2} z\right) \cdots C(z) \begin{pmatrix} G_k\left(z/q^k\right) \\ G_{k+1}\left(z/q^k\right) \end{pmatrix} \\
&= \begin{pmatrix} z\mathbf{F}_{n-1}\left(qz, 1\right) & \mathbf{F}_n\left(z, 1\right) \\ z\mathbf{F}_n\left(qz, 1\right) & \mathbf{F}_{n+1}\left(z, 1\right) \end{pmatrix} \begin{pmatrix} G_k\left(z/q^k\right) \\ G_{k+1}\left(z/q^k\right) \end{pmatrix}.
\end{aligned}$$

Then

$$G_{k+n}\left(z/q^k\right) = z\mathbf{F}_{n-1}\left(qz, 1\right) G_k\left(z/q^k\right) + \mathbf{F}_n\left(z, 1\right) G_{k+1}\left(z/q^k\right).$$

For $G_n(z) \in \Theta_1$, the sequence $g_n(z) := G_{n+k}\left(q^{1-n} z\right)$ satisfy the recursion

$$g_{n+2}(z) = g_{n+1}(z) + q^{-n} z g_n(z).$$

According to (2.5) with $k = 0$, we get

$$g_n(z) = g_0(z) z\mathbf{Fib}_{n-1}\left(q^{-1} z, 1\right) + g_1(z) \mathbf{Fib}_n\left(z, 1\right),$$

where $\mathbf{Fib}_{n+1}(z, 1) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{-k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_{1/q} z^k = \mathbf{F}_{n+1}\left(q^{-n} z, 1\right)$.

Then

$$g_n(z) = g_0(z) z\mathbf{F}_{n-1}\left(q^{1-n} z, 1\right) + g_1(z) \mathbf{F}_n\left(q^{1-n} z, 1\right),$$

therefore

$$g_n\left(q^{n-1} z\right) = g_0\left(q^{n-1} z\right) q^{n-1} z\mathbf{F}_{n-1}\left(z, 1\right) + g_1\left(q^{n-1} z\right) \mathbf{F}_n\left(z, 1\right),$$

meaning

$$G_{n+k}(z) = G_k\left(q^n z\right) q^{n-1} z\mathbf{F}_{n-1}\left(z, 1\right) + G_{k+1}\left(q^{n-1} z\right) \mathbf{F}_n\left(z, 1\right).$$

3. Some specializations

If we replace the sequence G by the sequence of q-Fibonacci polynomials $(\mathbf{F}_n(z,m))_{n\in\mathbb{Z}}$ in Corollary 2.2, we obtain two q-analogues of the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$ satisfied by the Fibonacci numbers, see for instance [17],

$$\mathbf{F}_{2n+1}(z/q^n, m) = \mathbf{F}_n(z/q^n, m) * (z\mathbf{F}_n(q^m z, m)) + \mathbf{F}_{n+1}(z/q^n, m) * (\mathbf{F}_{n+1}(z,m)), \quad (3.1)$$

$$\mathbf{F}_{2n+1}(z,m) = \mathbf{F}_n(z,m)\,\Delta\left(q^n z\mathbf{F}_n\left(q^{m-1}z, m\right)\right) + \mathbf{F}_{n+1}(z/q, m)\,\Delta(\mathbf{F}_{n+1}(z,m)). \quad (3.2)$$

Also, we draw two q-analogues of the identity $F_{2n} = F_n F_{n-1} + F_{n+1}F_n$, see [17],

$$\mathbf{F}_{2n}(z/q^n, m) = \mathbf{F}_n(z/q^n, m) * (z\mathbf{F}_{n-1}(q^m z, m)) + \mathbf{F}_{n+1}(z/q^n, m) * (\mathbf{F}_n(z,m)), \quad (3.3)$$

$$\mathbf{F}_{2n}(z,m) = \mathbf{F}_n(z,m)\,\Delta\left(q^{n-1}z\mathbf{F}_{n-1}\left(q^{m-1}z, m\right)\right) + \mathbf{F}_{n+1}(z/q, m)\,\Delta(\mathbf{F}_n(z,m)). \quad (3.4)$$

For generalizing the equalities; $L_{2n+1} = L_nF_n + L_{n+1}F_{n+1}$ and $L_{2n} = L_nF_{n-1} + L_{n+1}F_n$ satisfied by the Fibonacci and Lucas numbers [17], we replace the sequence G by the sequence of q–Lucas polynomials of the first kind $\mathbf{L}_n(z,m)$ in identity (2.3), we obtain

$$\mathbf{L}_{2n+1}(z/q^n, m) = \mathbf{L}_n(z/q^n, m) * z\mathbf{F}_n(q^m z, m) + \mathbf{L}_{n+1}(z/q^n, m) * \mathbf{F}_{n+1}(z,m), \quad (3.5)$$

$$\mathbf{L}_{2n}(z/q^n, m) = \mathbf{L}_n(z/q^n, m) * z\mathbf{F}_{n-1}(q^m z, m) + \mathbf{L}_{n+1}(z/q^n, m) * \mathbf{F}_n(z,m) \quad (3.6)$$

as a first q-deformation.

Also, we draw a second q-analogue to replace the sequence $G_n(z)$ by the q-Lucas polynomials of the second king $\mathbb{L}_n(z)$ in identity (2.4), we obtain

$$\mathbb{L}_{2n+1}(z,m) = \mathbb{L}_n(z,m)\,\Delta q^n z\mathbf{F}_n\left(q^{m-1}z, m\right) + \mathbb{L}_{n+1}(z/q, m)\,\Delta\mathbf{F}_{n+1}(z,m), \quad (3.7)$$

$$\mathbb{L}_{2n}(z,m) = \mathbb{L}_n(z,m)\,\Delta q^{n-1}z\mathbf{F}_{n-1}\left(q^{m-1}z, m\right) + \mathbb{L}_{n+1}(z/q, m)\,\Delta(\mathbf{F}_n(z,m)). \quad (3.8)$$

For Carlitz's approach the q-Fibonacci polynomials and the q-Lucas polynomials become as follows

$$\begin{aligned}
\mathbf{F}_{2n+1}(z/q^n, 1) &= \mathbf{F}_n(z/q^n, 1)\, z\mathbf{F}_n(qz, 1) + \mathbf{F}_{n+1}(z/q^n, 1)\,\mathbf{F}_{n+1}(z,1), \\
\mathbf{F}_{2n+1}(z, 1) &= \mathbf{F}_n(q^n z, 1)\, q^n z\mathbf{F}_n(z, 1) + \mathbf{F}_{n+1}(q^n z, 1)\,\mathbf{F}_{n+1}(z,1),
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_{2n}(z/q^n, 1) &= \mathbf{F}_n(z/q^n, 1)\, z\mathbf{F}_{n-1}(qz, 1) + \mathbf{F}_{n+1}(z/q^n, 1)\,\mathbf{F}_n(z,1), \\
\mathbf{F}_{2n}(z, 1) &= \mathbf{F}_n\left(q^{n-1}z, 1\right) q^{n-1}z\mathbf{F}_{n-1}(z, 1) + \mathbf{F}_{n+1}\left(q^{n-1}z, 1\right)\mathbf{F}_n(z,1),
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{2n+1}(z/q^n, 1) &= \mathbf{L}_n(z/q^n, 1)\, z\mathbf{F}_n(qz, 1) + \mathbf{L}_{n+1}(z/q^n, 1)\,\mathbf{F}_{n+1}(z,1), \\
\mathbf{L}_{2n}(z/q^n, 1) &= \mathbf{L}_n(z/q^n, 1)\, z\mathbf{F}_{n-1}(qz, 1) + \mathbf{L}_{n+1}(z/q^n, 1)\,\mathbf{F}_n(z,1),
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}_{2n+1}(z, 1) &= \mathbb{L}_n(q^n z, 1)\, q^n z\mathbf{F}_n(z, 1) + \mathbb{L}_{n+1}(q^n z, 1)\,\mathbf{F}_{n+1}(z,1), \\
\mathbb{L}_{2n}(z, 1) &= \mathbb{L}_n\left(q^{n-1}z, 1\right) q^{n-1}z\mathbf{F}_{n-1}(z, 1) + \mathbb{L}_{n+1}\left(q^{n-1}z, 1\right)\mathbf{F}_n(z,1).
\end{aligned}$$

4. The bivariate case

In [14], Cigler consider as q-Fibonacci sequence of bivariate polynomials the following

$$\mathbf{F}_{n+1}(x,y,m) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k}y^k,$$

and he found that the q-Fibonacci sequence of bivariate polynomials satisfy

$$\mathbf{F}_{n+2}(x,y,m) = x\mathbf{F}_{n+1}(x,y,m) + q^n y\mathbf{F}_n\left(x, q^{m-1}y, m\right), \quad (4.1)$$

$$\mathbf{F}_{n+2}(x,y,m) = x\mathbf{F}_{n+1}(x,qy,m) + qy\mathbf{F}_n\left(x, q^{m+1}y, m\right). \quad (4.2)$$

As we establish Theorem 2.1, we show that relations (2.1) and (2.2) can be extended to the sequences of bivariate polynomials via Corollaries 2.2 and 2.3.

For a bivariate polynomial $p(x,y) = \sum a_{i,j}x^i y^j$, we denote by Q_x, Q_y, $\widetilde{p}$, $\widehat{p}$ the operators

$$\begin{aligned} Q_y(p(x,y)) &= p(x,qy), \\ \widetilde{p}(x,y) &= \sum a_{i,j}x^i y^j Q_y^{-j}, \\ \widehat{p}(x,y) &= \sum a_{i,j}x^i \widehat{Q}^j y^j. \end{aligned}$$

Corollary 4.1. *For $(G_n(x,y))_{n\in\mathbb{Z}}$ satisfying* (4.1) *and* (4.2) *respectively, we have*

$$\begin{aligned} E^k Q_y^{-k}(G_n(x,y))_{n\in\mathbb{Z}} &= \widetilde{G}_k\left(x, y/q^k\right)(y\mathbf{F}_{n-1}(x, q^m y, m))_{n\in\mathbb{Z}} + \widetilde{G}_{k+1}\left(x, y/q^k\right)(\mathbf{F}_n(x,y,m))_{n\in\mathbb{Z}}, \\ E^k(G_n(x,y))_{n\in\mathbb{Z}} &= \widehat{\widetilde{G}}_k(x,y)\left(q^{n-1}y\mathbf{F}_{n-1}\left(x, q^{m-1}y, m\right)\right)_{n\in\mathbb{Z}} + \widehat{\widetilde{G}}_{k+1}\left(x, \frac{y}{q}\right)(\mathbf{F}_n(x,y,m))_{n\in\mathbb{Z}}. \end{aligned}$$

Now, we give an other formulation of bivariate polynomials according to relations

$$\begin{aligned} \mathbf{F}_n(x,y,m) &= x^{n-1}\mathbf{F}_n\left(y/x^2, m\right), \\ \mathbf{F}_n(1,z,m) &= \mathbf{F}_n(z,m). \end{aligned}$$

We define a correspondence between the sequences of polynomials and the sequences of bivariate polynomials as follows

$$(U_n(z))_{n\in\mathbb{Z}} \longmapsto (\varphi(U)_n(x,y))_{n\in\mathbb{Z}n} \text{ such that } \varphi(U)_n(x,y) = x^{n-1}U_n\left(y/x^2\right).$$

Remark 4.2. 1) For $(U_n(z))_{n\in\mathbb{Z}} \in \Omega_m$ the sequences of bivariate polynomials $(\varphi(U)_n(x,y))_{n\in\mathbb{Z}n}$ satisfy

$$\varphi(U)_{n+2}(x,y) = x\varphi(U)_{n+1}(x,y) + q^n y\varphi(U)_n(x,y/q). \quad (4.3)$$

2) For $(U_n(z))_{n\in\mathbb{Z}} \in \Theta_m$ the sequences of bivariate polynomials $\varphi(U)_n(x,y)$ satisfy

$$\varphi(U)_{n+2}(x,y/q) = x\varphi(U)_{n+1}(x,y) + y\varphi(U)_n(x,y). \quad (4.4)$$

With this correspondence, we find a q-Lucas sequence of bivariate polynomials of the first and the second kind denoted respectively by $\mathbf{L}_n(x,y,m)$ and $\mathbb{L}_n(x,y,m)$

$$\begin{aligned} \mathbf{L}_n(x,y,m) &= \sum_{k=0}^{\lfloor n/2\rfloor} q^{\binom{k}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + \frac{[k]_q}{[n-k]_q}\right) x^{n-1-2k}y^k, \\ \mathbb{L}_n(x,y,m) &= \sum_{k=0}^{\lfloor n/2\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^{n-2k}\frac{[k]_q}{[n-k]_q}\right) x^{n-1-2k}y^k. \end{aligned}$$

The sequence $(\mathbf{F}_n(2X,1,m))$ seems to be a q-analogue of Chebyshev polynomials of the second kind, and the two sequences $(\mathbf{L}_n(2X,1,m))$ and $(\mathbb{L}_n(2X,1,m))$ can be considered as two q-analogues of Chebyshev polynomials of the first kind (see for instance [4]). Using Corollary 4.1 or the correspondence $U_n \longmapsto \varphi(U)_n$ the identities given as application of relations (2.1) and (2.2) are related to Chebyshev polynomials by the application $(x,y) \longrightarrow (2X,1)$.

For the Pell and Pell-Lucas polynomials (see for instance [5]), we use the application $(x,y) \longrightarrow (2,X)$.

5. Conclusion

The generalization given in the present paper for the linear recurrence of order two can be extended in two ways. Firstly, it is known that the Tribonacci sequence can not be deduced as sums of elements lying on transversals of a Pascal triangle, as we can deduce it in [9]. For doing, we use the triangle of Delannoy for wich the sum of elements lying on the principal diagonal gives the Tribonacci sequence, we refer to [2, 3]. As a second extension, for a given positive integer r, we think that the approach presented here works fo the r-Fibonacci sequence and its companion family of sequences: the (r, s)-Lucas sequence, for $s = 1, 2, ..., r$, as developed in [1].

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