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Bounds on the covering radius of some classes code, simplex code and MacDonald code in R

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Abstract

In this paper, study of the codes in $R = \mathbb{Z}_2\mathbb{R}$, where $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$, $u^2 = 0$. Its related parameter of codes over $R = \mathbb{Z}_2\mathbb{R}$, with different distance are discussed. The block repetition codes over R are defined and the covering radius for block repetition codes, simplex code and macdonald code of type α and type β in R are obtained.

Keywords: Code, finite ring, additive codes, parameter, different distance, block repetition code, simplex code, Macdonald code

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1. Introduction

In the last five decade, codes over finite commutative rings have been studied. The objective of codes over finite rings is that they can be associated with codes in finite fields through the Gray maps are studied. Recently, coding theory over finite commutative non-chain rings is a hot research topic and there has been substantial interest in the class of additive codes. Delsarte contributes to the algebraic theory of association scheme where the main idea is to characterize the subgroups of the underlying abelian group in a given association scheme (*cf.* [16, 17]).

In coding theory, the covering radius is an one of the important geometric parameter of codes. It not only indicates the maximum error correcting capability of codes, but also relates to some practical problems such as the data compression and transmission. Studying of the covering radius of codes has attracted many coding scientists for almost 50 years. In [14], the covering radius of linear codes over binary finite fields was studied.

In [1], [5]-[7], have been extensively studied for the additive codes over $\mathbb{Z}_2\mathbb{Z}_4$. The huge results were made available on the simplex codes over finite fields and finite rings are [4, 11, 12, 20, 21]. In [8]-[10], the authors, in particular, gave lower and upper bounds on the covering radius of codes over the ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ with assigned to different distance and they explained the covering radius of discrete repetition codes, simplex codes and Macdonald codes (type α and β over R). The above results motivate us to work in this correspondence.

1.1. Preliminaries

In $\mathbb{Z}_2$ and $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$, $u^2 = 0$ be the rings of integers modulo 2 and let $\mathbb{Z}_2^n$ and $\mathbb{R}^n$, where n is the space of n-tuples over these ring. A finite ring $R = \mathbb{Z}_2\mathbb{R} = \{00, 01, 0u, 0u_1, 10, 11, 1u, 1u_1\}$, where $\mathbb{R} = \{0, 1, u, u_1\}, u^2 = 0, u_1 = 1 + u$ with integer modulo is 2.

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Definition 1.1. A non-empty set C is a subset of the ring R is called a *code*, and a *linear code* C of R is an R-submodule of R^n.

In this section, some preliminary results are given [5, 7]. A non-empty set C is a R-additive code if it is a subgroup of $\mathbb{Z}_2^\gamma \times \mathbb{R}^\delta$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^\lambda \times \mathbb{R}^\mu$ for some λ and μ and type of C is a $2^\lambda \mathbb{R}^\mu$ as a group. It pursue, the codewords of C is $2^{\lambda+2\mu}$ and $2^{\lambda+\mu}$ be the number of order for two codewords in C.

The Gray map: $\mu : \mathbb{R} \to \mathbb{Z}_2^2$ is

$$\mu(0) = (00),$$

$$\mu(1) = (01),$$

$$\mu(u) = (11)$$

and

$$\mu(1+u) = (10).$$

In general, the Gray map $\rho : \mathbb{Z}_2^\gamma \times \mathbb{R}^\delta \to \mathbb{Z}_2^n$, is defined by

$$\rho(v, w) = (v, \mu(w_1), \cdots, \mu(w_\delta)),\ \forall v \in \mathbb{Z}_2^\gamma \text{ and } (w_1, \cdots, w_\delta) \in \mathbb{R}^\delta,\ \text{with } n = \gamma + 2\delta.$$

Therefore, the binary image of a R-additive code under the general Gray map is called a *R-linear code* of length $n = \gamma + 2\delta$.

The Hamming weight of C is $wt_H(C) = \{wt(c) | c \in C \text{ and } c \neq 0\}$.

In [3], the Bachoc weight of x is defined as

$$w_B(x_i) = \begin{cases} 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i = 1 \text{ and} \\ 2 & \text{if } x_i = u, (1+u). \end{cases}$$

Hence the Bachoc weight is defined by $wt_B(x) = wt_H(v) + wt_B(w)$, where $x = (v, w) \in \mathbb{Z}_2^\gamma \times \mathbb{R}^\delta$, $v = (v_1, \cdots, v_\gamma) \in \mathbb{Z}_2^\gamma$ and $w = (w_1, \cdots, w_\delta) \in \mathbb{R}^\delta$.

Example 1.2. Let $x = (11\ 1u\ 1{+}1u) \in \mathbb{R}$, then $wt_B(x) = wt_B(11\ 1u\ 11{+}u) = wt_H(1\ 1\ 1){+}wt_B(1\ u\ 1{+}u) = (1\ 1\ 1){+}(1\ 2\ 2) = (1+1\ 1+2\ 1+2) = (2\ 3\ 3)$.

The Chinese Euclidean weight of x is given

$$wt_{CE}(x_i) = \begin{cases} 0 & \text{if } x_i = 0, \\ 2 & \text{if } x_i = 1, (1+u) \text{ and} \\ 4 & \text{if } x_i = u \end{cases}$$

(*cf.* [22]).

Define, $w_{CE}(x) = wt_H(u) + wt_{CE}(w)$, where $x = (u, w) \in \mathbb{Z}_2^\gamma \times \mathbb{R}^\delta$ and $u = (u_1, \cdots, u_\gamma) \in \mathbb{Z}_2^\gamma$ and $w = (w_1, \cdots, w_\delta) \in \mathbb{R}^\delta$.

If $c_1, c_2 \in C$, be any two distinct codewords of D distance is defined as $d_D(C) = \{d_D(c_1, c_2) | c_1 - c_2 \neq 0 \text{ and } c_1, c_2 \in C\}$. The minimum D weight of C is $d_D(C) = \min\{d_D(c_1, c_2) | c_1 - c_2 \neq 0 \text{ and } c_1, c_2 \in C\}$. In C is a *linear code C*, thus $d_D(C) = \min\{w_D(c) | c \neq 0 \in C\}$. Therefore, $d_D(c_1, c_2) = w_D(c_1 - c_2)$. Let $C \subseteq R^n$ is a linear code, where n is a length of code, the number of codewords N and the minimum D distance d_D is called a (n, N, d_D) code in R, where $D = \{\text{Bachoc(B), Chinese Euclidean(CE)}\}$.

2. The covering radius of codes and block repetition codes over R

Let C be a covering radius of a code with the smallest number r such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^{\gamma} \times \mathbb{R}^{\delta} = R$ and thus the *covering radius* of a code C over R with respect to the different distance, such as (Bachoc, Chinese Euclidean) is given $r_d(C) = \max_{v \in R}\{\min_{c \in C} d(v, c)\}$.

In $F_q = \{0, 1, \beta_2, \cdots, \beta_{q-1}\}$ is a finite field. Let $C = \{\bar{\beta} = (\beta\beta \cdots \beta) | \beta \in F_q\}$ be a q-ary *repetition code* C over F_q and its related parameter of the repetition code C is an $[n, 1, n]$. Therefore, $r(C) = \lceil \frac{n(q-1)}{q} \rceil$ this true for binary repetition code. In [8]-[10], the authors studied for different classes of repetition codes over $\mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$ and their covering radius has been obtained. Now, generalize those results for codes over $R = \mathbb{Z}_2\mathbb{R}, u^2 = 0$.

Consider the repetition codes over R. Let a fixed $1 \le i \le 7$ and for all $1 \le j \ne i \le 7, n_j = 0$, then the code $C^n = C^{n_i}$ is it denoted by C_i. Thus, the seven basic repetition codes are the following table,

Generator Matrix	Code	Parameters
$G_1 = [\overbrace{01 \cdots 01}^{n_{1(3)}}] = G_3$	$C_{1(3)} = \{c_0, c_1, c_2, c_3, c_4\}$	$(n_{1(3)}, 4, d_B = n, d_{CE} = 2n)$
$G_2 = [\overbrace{0v \cdots 0v}^{n_2}]$	$C_2 = \{c_0, c_2\}$	$(n_2, 2, d_B = 2n, d_{CE} = 2n)$
$G_4 = [\overbrace{10 \cdots 10}^{n_4}]$	$C_4 = \{c_0, c_1\}$	$(n_4, 2, d_B = n, d_{CE} = n)$
$G_5 = [\overbrace{11 \cdots 11}^{n_{5(7)}}] = G_7$	$C_{5(7)} = \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$	$(n_{5(7)}, 1, d_B = n, d_{CE} = n)$
$G_6 = [\overbrace{1u \cdots 1u}^{n_6}]$	$C_6 = \{c_0, c_2, c_4, c_6\}$	$(n_6, 4, d_B = n, d_{CE} = n)$

where $c_0 = (00 \cdots 00)$, $c_1 = (01 \cdots 01)$, $c_2 = (0u \cdots 0u)$, $c_3 = (0u_1 \cdots 0u_1)$, $c_4 = (10 \cdots 10)$, $c_5 = (11 \cdots 11)$, $c_6 = (1u \cdots 1u)$, $c_7 = (1u_1 \cdots 1u_1)$ and $u_1 = 1 + u$.

Theorem 2.1. *Prove that,*

(1) $\frac{5n}{8} \le r_B(C_1) = r_B(C_3) \le 2n$,

(2) $\frac{n}{2} \le r_B(C_2) \le 2n$,

(3) $\frac{n}{4} \le r_B(C_4) \le \frac{5n}{2}$,

(4) $\frac{7n}{8} \le r_B(C_5) = r_B(C_7) \le \frac{7n}{4}$ *and*

(5) $\frac{3n}{4} \le r_B(C_6) \le \frac{7n}{4}$, *here $r_B(C_j)$ be a covering radius of code $C_{j, 1 \le j \le 7}$ with Bachoc weight.*

Proof. If $c \in C_{j, 1 \le j \le 7}$ be a codeword of code C_j in R. Let $t_i(c)_{, 0 \le i \le 7}$ is the number of occurrences of symbol i in the codeword c.

Let $x \in R^n$ by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum_{j=0}^{7} t_j = n$, then

$d_B(x, \overline{00}) = n - t_0 + t_2 + t_3 + t_5 + 2t_6 + 2t_7$, $d_B(x, \overline{01}) = n - t_1 + t_0 + t_3 + 2t_4 + t_6 + 2t_7$,
$d_B(x, \overline{0u}) = n - t_2 + t_0 + t_1 + 2t_4 + 2t_5 + t_7$, $d_B(x, \overline{0u_1}) = n - t_3 + t_1 + t_2 + t_4 + 2t_5 + 2t_6$,
$d_B(x, \overline{10}) = n - t_4 + t_1 + 2t_2 + 2t_3 + t_6 + t_7$, $d_B(x, \overline{11}) = n - t_5 + 2t_0 + t_2 + 2t_3 + t_4 + t_7$,
$d_B(x, \overline{1u}) = n - t_6 + 2t_0 + 2t_1 + t_3 + t_4 + t_5$, $d_B(x, \overline{1u_1}) = n - t_7 + t_0 + 2t_1 + 2t_2 + t_5 + t_6$.

In code, $C_1 = C_3 \in R$, therefore, $d_B(x, C_1) = d_B(x, C_3) = \min\{d_B(x, \overline{00}), d_B(x, \overline{01}), d_B(x, \overline{0u}), d_B(x, \overline{0u_1})\} \le \frac{4n - n + 5n}{4} \le 2n$, then $r_B(C_1) = r_B(C_3) \le 2n$.

If $x = (\overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{01 \cdots 01}^{\frac{n}{4}} \overbrace{0u \cdots 0u}^{\frac{n}{4}} \overbrace{0u_1 \cdots 0u_1}^{\frac{n}{4}}) \in R^n$, then $d_B(x, \overline{00}) = d_B(x, \overline{01}) = d_B(x, \overline{0u}) = d_B(x, \overline{0u_1}) = \frac{n}{8} + 2(\frac{n}{8}) + 2(\frac{n}{8}) = \frac{5n}{8}$. Thus $r_B(C_1) = r_B(C_3) \ge \frac{5n}{8}$ and so $\frac{5n}{8} \le r_B(C_1) = r_B(C_3) \le 2n$.

In code, $C_2 \in R$, $d_B(x, C_2) = \min\{d_B(x, \overline{00}), d_B(x, \overline{0u})\} \leq \frac{2n-n+3n}{2} \leq 2n$. Then $r_B(C_2) \leq 2n$.

If $x = (\overbrace{00\cdots 00}^{\frac{n}{2}}\overbrace{0u\cdots 0u}^{\frac{n}{2}}) \in R^n$, then $d_B(x, \overline{00}) = d_B(x, \overline{0u}) = 2(\frac{n}{4}) = \frac{n}{2}$. Thus $r_B(C_2) \geq \frac{n}{2}$ and so $\frac{n}{2} \leq r_B(C_2) \leq 2n$. The remaining part of proof is follows from part 1 and 2 for code C_4, C_5, C_6. □

Theorem 2.2. *In Chinese Euclidean weight of code of $C_{j, 1\leq j\leq 7}$, to find*

(1) $n \leq r_{CE}(C_1) = r_{CE}(C_3) \leq \frac{11n}{4}$,

(2) $n \leq r_{CE}(C_2) \leq \frac{5n}{2}$,

(3) $\frac{n}{4} \leq r_{CE}(C_4) \leq 4n$,

(4) $\frac{5n}{4} \leq r_{CE}(C_5) = r_{CE}(C_7) \leq \frac{5n}{2}$ *and*

(5) $\frac{5n}{4} \leq r_{CE}(C_6) \leq \frac{5n}{2}$.

Proof. In code C_i, $_{i=1 \text{ to } 7}$ with Chinese Euclidean weight is apply to Theorem 2.1. □

Block repetition code in R

The block repetition code C^n over R is a R-additive code.

Let $G = [\overbrace{g_1}^{n_1}\ \overbrace{g_2}^{n_2}\ \overbrace{g_3}^{n_3}\ \overbrace{g_4}^{n_4}\ \overbrace{g_5}^{n_5}\ \overbrace{g_6}^{n_6}\ \overbrace{g_7}^{n_7}]$ be a generator matrix with the parameters of $C^n : [n = \sum_{j=1}^{7} n_j, 8, d_B = \min\{n_4+n_5+n_6+n_7, n_1+2n_2+2n_3+n_5+2n_6+2n_7\}, d_{CE} = \min\{n_4+n_5+n_6+n_7, 4(n_4+n_5+n_6+n_7)\}]$, where $g_1 = 0101\cdots 01$, $g_2 = 0u0u\cdots 0u$, $g_3 = 0u_10u_1\cdots 0u_1$, $g_4 = 1010\cdots 10$, $g_5 = 1111\cdots 11$, $g_6 = 1u1u\cdots 1u$, $g_7 = 1u_11u_1\cdots 1u_1$.

Theorem 2.3. *Let C^n be the block repetition code in R with length is n. Then*

(1) $\frac{5(n_1+n_3)+4n_2+2n_4+7(n_5+n_7)+6n_6}{8} \leq r_B(C^n) \leq \frac{18(n_1+n_3+n_6)+17n_2+15(n_4+n_5+n_6+n_7)}{8}$ *and*

(2) $\frac{4(n_1+n_2+n_3)+n_4+5(n_5+n_6+n_7)}{4} \leq r_{CE}(C^n) \leq \frac{6(n_1+n_2+n_3)+8n_5+5(n_5+n_6+n_7)}{2}$.

Proof. Using [14], Theorem 2.1 and Theorem 2.2, thus

- $\frac{5(n_1+n_3)+4n_2+2n_4+7(n_5+n_7)+6n_6}{8} \leq r_B(C^n)$ and
- $\frac{4(n_1+n_2+n_3)+n_4+5(n_5+n_6+n_7)}{4} \leq r_{CE}(C^n)$.

Let $x = x_1x_2x_3x_4x_5x_6x_7 \in R^n$ where x_1 is $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and similarly others x_2, x_3, x_4, x_5, x_6, x_7 are (b_i), (c_i), (d_i), (e_i), (f_i), (g_i),$_{i=0,1,2,3,4,5,6,7}$ respectively such that $n_1 = \sum_{j=0}^{7} a_j$, $n_2 = \sum_{j=0}^{7} b_j$, $n_3 = \sum_{j=0}^{7} c_j$, $n_4 = \sum_{j=0}^{7} d_j$, $n_5 = \sum_{j=0}^{7} e_j$, $n_6 = \sum_{j=0}^{7} f_j$, $n_7 = \sum_{j=0}^{7} g_j$. Then, $r_B(C^n) \leq \frac{18(n_1+n_3+n_6)+17n_2+15(n_4+n_5+n_6+n_7)}{8}$ and $r_{CE}(C^n) \leq \frac{6(n_1+n_2+n_3)+8n_5+5(n_5+n_6+n_7)}{2}$. □

3. Simplex codes of type α and type β in R

In this part, to consider the construction of simplex codes of type α and type β over R.

Let $G^{\alpha}_{2,k}(S^{\alpha}_{2,k}) = \left[\begin{array}{c|c} 00\cdots 0 & 11\cdots 1 \\ \hline G^{\alpha}_{2,k-1} & G^{\alpha}_{2,k-1} \end{array}\right]$, for $k \geq 2$, be the generator matrix of $S^{\alpha}_{2,k}$ of the binary simplex code of type α and $G^{\alpha}_{2,1} = [0, 1]$.

The simplex codes $S^{\alpha}_{4,k}$ of type α over $\mathbb{R}$ were defined in [4]. Therefore, the generator matrix

$$G^{\alpha}_{\mathbb{R},k}(S^{\alpha}_{\mathbb{R},k}) = \left[\begin{array}{c|c|c|c} 00\cdots 0 & 11\cdots 1 & uu\cdots u & u_1u_1\cdots u_1 \\ \hline G^{\alpha}_{\mathbb{R},k-1} & G^{\alpha}_{\mathbb{R},k-1} & G^{\alpha}_{\mathbb{R},k-1} & G^{\alpha}_{\mathbb{R},k-1} \end{array}\right], \text{ for } k \geq 2,$$

where $G^{\alpha}_{\mathbb{R},1} = [0\ 1\ u\ u_1]$ and $u_1 = 1 + u$.

The generator matrix of S^{α}_{k}, the simplex code of type α over R is defined as the concatenation of 2^{2k} copies of the generator matrix of $S^{\alpha}_{2,k}$ and 2^k copies of the generator matrix of $S^{\alpha}_{\mathbb{R},k}$ given by

$$\Theta^{\alpha}_{k} = \left[\begin{array}{c|c|c|c|c|c|c|c} G^{\alpha}_{2,k} & G^{\alpha}_{2,k} & \cdots & G^{\alpha}_{2,k} & G^{\alpha}_{\mathbb{R},k} & G^{\alpha}_{\mathbb{R},k} & \cdots & G^{\alpha}_{\mathbb{R},k} \end{array}\right], \ k \geq 1. \quad (3.1)$$

The standard form of Θ^{α}_{k} of the generator matrix of S^{α}_{k} is given by

$$\Theta^{\alpha}_{k} = \left[\begin{array}{c|c|c|c} 00\ 00\cdots 00 & 01\ 01\cdots 01 & \cdots & 1u_1 1u_1\cdots 1u_1 \\ \hline \Theta^{\alpha}_{k-1} & \Theta^{\alpha}_{k-1} & \cdots & \Theta^{\alpha}_{k-1} \end{array}\right], \text{ for } k \geq 2,$$

where

$$\Theta^{\alpha}_{1} = [00\ 01\ 0u\ 0u_1\ 10\ 11\ 1u\ 1u_1] \text{ and } u_1 = 1 + u.$$

The parameter of the simplex code of type α over R : $[l = 2^{3k+1}, M = 2^{k_0}\mathbb{R}^{k_1}]$, here l is a length and M is the number of codewords.

Let $k = 1$ with $k_0 = 0$ and $k_1 = 1$, that all of the codewords of the simplex code $S^{\alpha}_{1} = \{c_0, c_1, c_2, c_3\}$ are generated by Θ^{α}_{1}, where $c_0 = (00\ 00\ 00\ 00\ 00\ 00\ 00\ 00)$, $c_1 = (00\ 01\ 0u\ 0u_1\ 10\ 11\ 1u\ 1u_1)$, $c_2 = (00\ 0u\ 00\ 0u\ 00\ 0u\ 00\ 0u)$, $c_3 = (00\ 0u_1\ 0u\ 01\ 10\ 1u_1\ 1u\ 11)$ and $u_1 = 1 + u$.

In type β simplex code S^{β}_{k} : $\left[l = 2^k(2^{k-2} + 1)(2^k - 1), M = 2^{k_0}\mathbb{R}^{k_1}, \text{ for some } k_0 \text{ and } k_1\right]$, be the parameter of the simplex code of type β over R.

The generator matrix of S^{β}_{k} is the concatenation of 2^k copies of the generator matrix of $S^{\beta}_{2,k}$ and 2^{k-1} copies of the generator matrix of $S^{\beta}_{\mathbb{R},k}$ given by

$$\Theta^{\beta}_{k} = \left[\begin{array}{c|c|c|c|c|c|c|c} G^{\beta}_{2,k} & G^{\beta}_{2,k} & \cdots & G^{\beta}_{2,k} & G^{\beta}_{\mathbb{R},k} & G^{\beta}_{\mathbb{R},k} & \cdots & G^{\beta}_{\mathbb{R},k} \end{array}\right], \text{ for } k \geq 2, \quad (3.2)$$

where $G^{\beta}_{2,k} = \left[\begin{array}{c|c} 11\cdots 1 & 00\cdots 0 \\ \hline G^{\alpha}_{2,k-1} & G^{\beta}_{2,k-1} \end{array}\right]$, is a generator matrix of the binary simplex code of type β and for $k \geq 3$, with $G^{\beta}_{2,2} = \left[\begin{array}{c|c} 11 & 0 \\ \hline 01 & 1 \end{array}\right]$.

Therefore, $G^{\beta}_{\mathbb{R},k} = \left[\begin{array}{c|c|c} 11\cdots 1 & 00\cdots 0 & uu\cdots u \\ \hline G^{\alpha}_{\mathbb{R},k-1} & G^{\beta}_{\mathbb{R},k-1} & G^{\beta}_{\mathbb{R},k-1} \end{array}\right]$, is the generator matrix of the simplex code over $\mathbb{R}$ and for $k \geq 3$, with

$$G^{\beta}_{\mathbb{R},2} = \left[\begin{array}{c|c|c} 1111 & 0 & u \\ \hline 01uu_1 & 1 & 1 \end{array}\right].$$

Obtain, the following

Theorem 3.1. *Prove that,*

(1) $r_B(S^{\alpha}_{k}) \leq \frac{2^k(3.2^{2k-1}+2^{2k+2}-1)}{3}$ *and*

(2) $r_{CE}(S^{\alpha}_{k}) \leq 2^k(3.2^{2k-1} + 2^{2k+1} - 3)$, *here $r_d(S^{\alpha}_{k})$ be a covering radius of type α-simplex codes in R with different distance.*

Proof. In R-Simplex codes of type α have a Bachoc weight equal to 2^{2k} or $2^{2k} + 1$. From the matrix (3.1), [14] and Theorem 2.3 with different distance, then

$$\begin{aligned} r_B(S^{\alpha}_{k}) &\leq r_B(2^{2k}S^{\alpha}_{2,k}) + r_B(2^kS^{\alpha}_{\mathbb{R},\ k}) \\ &\leq 2^{2k}r_B(S^{\alpha}_{2,\ k}) + 2^kr_B(S^{\alpha}_{\mathbb{R},\ k}) \\ &\leq 2^{2k}r_H(S^{\alpha}_{2,\ k}) + 2^kr_B(S^{\alpha}_{\mathbb{R},\ k}) \\ &\leq 2^{2k}\left(2^{k-1}\right) + 2^k\left[\left(4.2^{2(k-1)} + 4.2^{2(k-2)} + \cdots + 4.2^{2.1}\right) + r_B(S^{\alpha}_{\mathbb{R},\ 1})\right] \\ r_B(S^{\alpha}_{k}) &\leq \frac{2^k(3.2^{2k-1} + 2^{2k+2} - 1)}{3}. \end{aligned}$$

The remaining part of proof is unification from part 1 but different distance. □

Theorem 3.2. *Find the following*

(1) $r_B(S_k^{\beta}) \leq \frac{2^{3k}+3(2^{2k-1}+2^{3(k-1)}-3\cdot 2^{2k-3}-2^{k-1})-20\cdot 2^{k-1}}{3}$ *and*

(2) $r_{CE}(S_k^{\beta}) \leq 2^{3k-1} - 8\cdot 2^{k-1}$,

where $r_{B,(CE)}(S_k^{\beta})$ *are the covering radius of type* α*-simplex codes in* R *with respective distance.*

Proof. From (3.2), [14] and Theorem 2.3 with different distance, so

$$\begin{aligned} r_B(S_k^{\beta}) &\leq r_B(2^k S_{2,\,k}^{\beta}) + r_L(2^{k-1} S_{\mathbb{R},\,k}^{\beta}) \\ &\leq 2^k r_B(S_{2,\,k}^{\beta}) + 2^{k-1} r_B(S_{\mathbb{R},\,k}^{\beta}) \\ &\leq 2^k r_B(S_{2,\,k}^{\beta}) + 2^{k-1} r_B(S_{\mathbb{R},\,k}^{\beta}) \\ &\leq 2^{k-1}\left(2^k-1\right) + 2^{k-1}\left(\frac{2^{2k+1}+3\cdot 4^{k-1}-9\cdot 2^{k-2}-20}{3}\right) \\ r_B(S_k^{\beta}) &\leq \frac{2^{3k}+3(2^{2k-1}+2^{3(k-1)}-3\cdot 2^{2k-3}-2^{k-1})-20\cdot 2^{k-1}}{3}. \end{aligned}$$

The Proof of second part uses, first part and Chinese Euclidean distance. □

4. MacDonald codes of type α and type β in R

Let $M_{k,t}(q)$ be a q-ary MacDonald code over the finite field $\mathbb{F}_q$ is a unique parameter $\left[\frac{q^k-q^t}{q-1}, k, q^{k-1}-q^{t-1}\right]$ linear code in which every non-zero codeword has weight either q^{k-1} or $q^{k-1}-q^{t-1}$ [18]. The the covering radius of MacDonald codes over a finite field and given many exact values for smaller dimension are studied in [19]. Defined the MacDonald codes over a ring using the generator matrices of the simplex codes ref. [15].

Let $G_{k,t}^{\alpha}$ be the matrix obtained from G_k^{α} by deleting columns corresponding to the columns of G_t^{α}, $2 \leq t \leq k-1$. That is,

$$G_{k,t}^{\alpha} = \left[\begin{array}{cc} G_k^{\alpha} & \backslash \frac{\mathbf{0}}{G_t^{\alpha}} \end{array}\right] \tag{4.1}$$

and let $G_{k,t}^{\beta}$ be the matrix obtained from G_k^{β} by deleting columns corresponding to the columns of G_t^{β}. That is,

$$G_{k,t}^{\beta} = \left[\begin{array}{cc} G_k^{\beta} & \backslash \frac{\mathbf{0}}{G_t^{\beta}} \end{array}\right], \tag{4.2}$$

where $[A\backslash B]$ denotes the matrix obtained from the matrix A by deleting the columns of the matrix B and $\mathbf{0}$ is a $(k-t)\times 2^{2t}\left((k-t)\times 2^{t-1}(2^t-1)\right)$. The parameters in MacDonald codes of α-type and β-type is $\left[4^k-4^t, k\right]$ and $\left[(2^{k-1}-2^{t-1})(2^k+2^t-1), k\right]$ code over R.

Construct the type α and type β MacDonald codes over $\mathbb{Z}_2\mathbb{R}$:

The type α and type β by using the generator matrix of the $\mathbb{Z}_2\mathbb{R}$-simplex codes of type α and type β. If $1 \leq t \leq k-1$, let $\Theta_{k,t}^{\alpha}$ (resp., $\Theta_{k,t}^{\beta}$) be the matrix of MacDonald codes $M_{k,t}^{\alpha}$ (resp.,$M_{k,t}^{\beta}$) with parameters $\left[2^{3k+1}-2^{k+t}(2^k-2^t)\right]$ $\left(\text{resp., } \left[2^{2k-1}(2^{2k-1}+1)(2^k-1)-2^{k+t-1}(2^{2t-3}+1)(2^t-1)\right]\right)$ obtained from Θ_k^{α} (resp., Θ_k^{β}) by deleting columns corresponding to the columns of the matrix Θ_t^{α} and $0_{2^{2t}\times(k-t)}$ (resp., Θ_t^{β} and $0_{(2^t-1)\times(k-t)}$). That is, for $k \geq 1$,

$$\Theta_{k,t}^{\alpha} = \left[m_{k,t}^{\alpha}|\cdots|m_{k,t}^{\alpha}|G_{k,t}^{\alpha}|\cdots|G_{k,t}^{\alpha}\right], \tag{4.3}$$

where $m_{k,t}^{\alpha}$(resp., $G_{k,t}^{\alpha}$) repeat 2^{2k} (resp., 2^k) times in $\Theta_{k,t}^{\alpha}$ for $k \geq 3$,

$$\Theta_{k,t}^{\beta} = \left[m_{k,t}^{\beta}|\cdots|m_{k,t}^{\beta}|G_{k,t}^{\beta}|\cdots|G_{k,t}^{\beta}\right], \tag{4.4}$$

where $m_{k,t}^{\beta}$ (resp., $G_{k,t}^{\beta}$) repeat 2^{2k} (resp., 2^{k-1}) times in $\Theta_{k,t}^{\beta}$. In Lee distance and Euclidean distance ref. to [13] and obtain the following

Theorem 4.1. *For $t \leq r \leq k$,*

$$(1)\quad r_L\left(M^{\alpha}_{k,t}\right) \leq \left[2^{3k+1} - 2^{k+r}\left(2^r + 2^k\right)\right] + \left[2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_L\left(M^{\alpha,4}_{k,t}\right)\right],$$

$$(2)\quad r_E\left(M^{\alpha}_{k,t}\right) \leq \left[\frac{2^{3(k+1)} - 2^{k+r}\left(3\cdot 2^r + 5\cdot 2^k\right)}{3}\right] + \left[2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_E\left(M^{\alpha,4}_{k,t}\right)\right],$$

$$(3)\quad r_B\left(M^{\alpha}_{k,t}\right) \leq \left[\frac{7\cdot 2^{3k} - 2^{k+r}\left(4\cdot 2^r + 3\cdot 2^k\right)}{3}\right] + \left[2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_B\left(M^{\alpha,4}_{k,t}\right)\right],$$

$$(4)\quad r_{CE}\left(M^{\alpha}_{k,t}\right) \leq \left[3\cdot 2^{3k} - 2^{k+r}\left(2^k + 2\cdot 2^r\right)\right] + \left[2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_{CE}\left(M^{\alpha,4}_{k,t}\right)\right].$$

Proof. Use, the matrix(4.3), [14] and Theorem 2.3, thus

$$\begin{aligned}
r_L(M^{\alpha}_{k,t}) &\leq r_L(2^{2\cdot k} M^{\alpha,2}_{k,t}) + r_L\left(2^k M^{\alpha,2}_{k,t}\right),\\
&\leq 2^{2\cdot k} r_L\left(M^{\alpha,2}_{k,t}\right) + 2^k r_L\left(M^{\alpha,4}_{k,t}\right),\\
&\leq 2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_L\left(M^{\alpha,4}_{k,t}\right),\\
&\leq 2^{2\cdot k}\left(2^k - 2^r\right) + 2^k(2^{2\cdot k} - 2^{2\cdot r}) + 2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_L\left(M^{\alpha,4}_{k,t}\right),\\
r_L(M^{\alpha}_{k,t}) &\leq \left[2^{3k+1} - 2^{k+r}(2^r + 2^k)\right] + \left[2^{2\cdot k} r_H\left(M^{\alpha,2}_{k,t}\right) + 2^k r_L\left(M^{\alpha,4}_{k,t}\right)\right].
\end{aligned}$$

The remaining part of proof is follows in part 1. □

Theorem 4.2. *For $t \leq r \leq k$,*

$$(1)\quad r_L\left(M^{\beta}_{k,t}\right) \leq \left[2^{3k-1} - 2^{k+r-1}\left(2^{k+1} + 2^r - 1\right)\right] + \left[2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_L\left(M^{\beta,4}_{k,t}\right)\right],$$

$$(2)\quad r_E\left(M^{\beta}_{k,t}\right) \leq \left[6\cdot 2^{3k} - 2^{k+r}\left(2^k + 5\cdot 2^r + 6\right) - 6\cdot 2^{2k}\right] + \left[2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_E\left(M^{\beta,4}_{k,t}\right)\right],$$

$$(3)\quad r_B\left(M^{\beta}_{k,t}\right) \leq \frac{6\left(2^{3k} - 2^{2k+r}\right) + 4\left(2^{3k} - 2^{k+2r}\right) + 3\left(2^{3k-2} - 2^{k+2(r-1)}\right) + 9\left(2^{k+r-1} - 2^{2k-1}\right)}{6} + \left[2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_B\left(M^{\beta,4}_{k,t}\right)\right],$$

$$(4)\quad r_{CE}\left(M^{\beta}_{k,t}\right) \leq \left[2^{3k+1} - 2^{k+r}\left(2^k + 2^r + 1\right) - 2^{2k}\right] + \left[2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_{CE}\left(M^{\beta,4}_{k,t}\right)\right].$$

Proof. Use, the matrix (4.4), [14] and Theorem 2.3, so

$$\begin{aligned}
r_L\left(M^{\beta}_{k,t}\right) &\leq r_L\left(2^{2\cdot k} M^{\beta,2}_{k,t}\right) + r_L\left(2^k M^{\beta,2}_{k,t}\right),\\
&\leq 2^{2\cdot k} r_L\left(M^{\beta,2}_{k,t}\right) + 2^k r_L\left(M^{\beta,4}_{k,t}\right),\\
&\leq 2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_L\left(M^{\beta,4}_{k,t}\right),\\
&\leq 2^{2\cdot k}\left(2^k - 2^r\right) + 2^k\left[\left(2^{k-1}\left(2^k - 1\right) - 2^{r-1}\left(2^r - 1\right)\right)\right] + 2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_L\left(M^{\beta,4}_{k,t}\right),\\
r_L\left(M^{\beta}_{k,t}\right) &\leq \left[2^{3k-1} - 2^{k+r-1}\left(2^{k+1} + 2^r - 1\right)\right] + \left[2^{2\cdot k} r_H\left(M^{\beta,2}_{k,t}\right) + 2^k r_L\left(M^{\beta,4}_{k,t}\right)\right].
\end{aligned}$$

The remaining part of proof is pursue in part 1. □

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