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On some properties of nonuniform multi-wavelet Bessel sequences on spectrum

Owais Ahmad [a], Ishtaq Ahmad [b], Mobin Ahmad [c], Neyaz A. Sheikh [d]

[a] *Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar - 190 006, Jammu and Kashmir, India*
[b] *Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar - 190 006, Jammu and Kashmir, India*
[c] *Department of Mathematics, Faculty of Science, Jazan University, Jazan - 45142, Saudi Arabia*
[d] *Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar - 190 006, Jammu and Kashmir, India*

Abstract

In this paper, we study some properties of Nonuniform multiwavelet Bessel sequences in Sobolev spaces on spectrum.

Keywords: Fourier transform, Bessel sequence, Sobolev space, spectral pair

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1. Introduction

Frames are redundant systems of vectors for a Hilbert space, which can yield many different and stable representations for a given vector. The notion of frames in a Hilbert space was initially given by Duffin and Schaeffer [15] in the domain of non-harmonic Fourier series. In the ares image l processing, it has become very beneficial in analyzing the completeness and stability of linear discrete signal/image representations. Until the ground-breaking work on wavelet frames by Daubechies et al. [13], this concept has not generated much interest. Due to the redundancy of frames, the frame has become an important tool in machine learning, quantum field theory, sampling analysis.

During last two decades, there is a tremendous influx of work on the development of wavelet theory on local fields. An initial work based on wavelet sets via Fourier transform approach was established by R. L. Benedetto and J. J. Benedetto [10]. Recently a lot of work was carried on the develpment of wavelet frames, Gabor frames, wave packet systems by various researchers [1]-[4], [6]-[9], [21]-[24].

Motivated and inspired by the above work, we in this study nonuniform multiwavelet Bessel sequences on local fields.

The paper is structured as follows. In section 2, we recall some on local fields, definition of Sobolov spaces and also discuss some auxiliarry results about Bessel sequences. In Section 3, we provide the complete characterization of multiwavelet Bessel wavelet sequences in Sobolev spaces over local fields.

†*Article ID:* MTJPAM-D-21-00036
Email addresses: siawoahmad@gmail.com (Owais Ahmad), ishtiyaqahmadun@gmail.com (Ishtaq Ahmad), msyed@jazanu.edu.sa (Mobin Ahmad), neyaznit@yahoo.co.in (Neyaz A. Sheikh)
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⋆*Corresponding Author:* Owais Ahmad

2. Fourier transforms on local fields

The Fourier transform of $\varphi \in L^1(\mathbb{K})$ is denoted by $\hat{\varphi}(\xi)$ and defined by

$$\mathcal{F}\{\varphi(x)\} = \hat{\varphi}(\xi) = \int_{\mathbb{K}} \varphi(x)\overline{\chi_\xi(x)}\, dx.$$

It is noted that

$$\hat{\varphi}(\xi) = \int_K \varphi(x)\overline{\chi_\xi(x)}dx = \int_{\mathbb{K}} \varphi(x)\chi(-\xi x)\, dx.$$

The Fourier transform on $\mathbb{K}$ enjoys the following properties:

- The map $\varphi \to \hat{\varphi}$ is a bounded linear transformation of $L^1(\mathbb{K})$ into $L^\infty(\mathbb{K})$, and $\left\|\hat{\varphi}\right\|_\infty \le \left\|\varphi\right\|_1$.
- If $\varphi \in L^1(\mathbb{K})$, then $\hat{\varphi}$ is uniformly continuous.
- If $\varphi \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\left\|\hat{\varphi}\right\|_2 = \left\|\varphi\right\|_2$.

The Fourier transform of a function $\varphi \in L^2(\mathbb{K})$ is defined by

$$\hat{\varphi}(\xi) = \lim_{k\to\infty} \hat{\varphi}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} \varphi(x)\overline{\chi_\xi(x)}\, dx,$$

where $\varphi_k = \varphi\, \mathbf{\Phi}_{-k}$ and $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $\varphi \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of φ as

$$\hat{\varphi}(u(n)) = \int_{\mathfrak{D}} \varphi(x)\overline{\chi_{u(n)}(x)}\, dx.$$

From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series $\sum_{n\in\mathbb{N}_0} \hat{\varphi}(u(n))\chi_{u(n)}(x)$ of φ converges to φ in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\left\|\varphi\right\|_2^2 = \int_{\mathfrak{D}} \left|\varphi(x)\right|^2 dx = \sum_{n\in\mathbb{N}_0} \left|\hat{\varphi}(u(n))\right|^2.$$

For $s \in \mathbb{K}$, the Sobolev space $\mathbb{H}^s(\mathbb{K})$ consists of all distributions φ such that

$$||\varphi||^2_{\mathbb{H}^s(\mathbb{K})} = \int_{\mathbb{K}} |\widehat{\varphi}(\zeta)|^2 \left(1 + ||\zeta||_2^2\right)^s d\zeta < \infty,$$

where $||.||_2$ denotes the Euclidean norm on $\mathbb{K}$. It is noted that, $H^s(\mathbb{K})$ is a separable Hilbert space under the definition of the inner product

$$\langle \phi, \psi\rangle_{\mathbb{H}^s(\mathbb{K})} = \int_{\mathbb{K}} \widehat{\phi}(\zeta)\overline{\widehat{\psi}(\zeta)} \left(1 + ||\zeta||_2^2\right) d\zeta, \quad \phi,\ \psi \in \mathbb{H}^s(\mathbb{K}).$$

Obviously, $\mathbb{H}(\mathbb{K}) = L^2(\mathbb{K})$ and $\mathbb{H}^{s_1} \subseteq \mathbb{H}^{s_2}(\mathbb{K})$ iff $s_1 \ge s_2$. Furthermore, for every $\psi \in \mathbb{H}^{-s}(\mathbb{K})$,

$$\langle \phi, \psi\rangle = \int_{\mathbb{K}} \widehat{\phi}(\zeta)\overline{\widehat{g}(\zeta)} d\zeta, \quad \phi \in \mathbb{H}^s(\mathbb{K})$$

gives a continuous functional on $\mathbb{H}^s\,(\mathbb{K})$.

Given an integer $N \ge 1$ and an odd integer r with $1 \le r \le qN - 1$, r and N are relatively prime, we consider the translation set Λ which is not necessarily a group as

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathcal{Z} = \left\{\frac{u(r)k}{N} + u(n) : n \in \mathbb{N}_0,\ k = 0, 1\right\}.$$

For $\phi, \psi : \mathbb{K} \to \mathbb{C}$, we define

$$[\phi, \psi]_t(\xi) = \sum_{\lambda\in\Lambda} \phi(\xi+\lambda)\overline{\psi(\xi+\lambda)}\left(1+\|\cdot+\lambda\|_2^2\right)^t, \quad t \in \mathbb{K}.$$

By $\Gamma_\mathfrak{p}$ a full set of $q\mathbb{N}_0/\mathbb{N}_0$, i.e a set of representatives of distinct cosets of $q\mathbb{N}_0/\mathbb{N}_0$. We write

$$f_{j,\lambda}(\xi) = (qN)^{\frac{j}{2}} f\left(\left(\mathfrak{p}^{-1}N\right)^{-j}\xi - \lambda\right)$$

and

$$f^s_{j,\lambda}(\xi) = (qN)^{-js} f_{j,\lambda}(\xi) = (qN)^{j(\frac{1}{2}-s)} f\left(\left(\mathfrak{p}^{-1}N\right)^{-j}\xi - \lambda\right)$$

for a distribution f, $j \in \mathbb{Z}$, $\lambda \in \Lambda$ and $s \in \mathbb{K}$.
Given $r \in \mathbb{N}$, let $\varphi = (\varphi_1, \varphi_2, ..., \varphi_r)^T \in (\mathbb{H}^s(\mathbb{K}))^r$ be an $\mathfrak{p}$-refinable function vector satisfying the refinement equation, i.e., there exists an $r \times r$ order matrix $\widehat{a}$, called refinement mask symbol such that

$$\widehat{\varphi}(\mathfrak{p}^{-1}N\xi) = \widehat{a}^l(\xi)\widehat{\varphi}(\xi) \;\; a.e\; \xi \in \mathbb{K}. \tag{2.1}$$

Given $N \in \mathbb{N}$, wavelet function vectors $\psi_\ell = (\psi^\ell_1, \psi^\ell_2, ..., \psi^\ell_r)^T$ with $1 \le \ell \le qN-1$ are defined by

$$\widehat{\psi}(\mathfrak{p}^{-1}N\xi) = \widehat{b}^l(\xi)\widehat{\varphi}(\xi) \;\; 1 \le \ell \le qN-1, \tag{2.2}$$

where $\widehat{b}^l(\xi) = (\widehat{b}^\ell_{n,m}(\xi))^r_{n,m}$ with $1 \le \ell \le qN-1$ being a sequence of $r \times r$ order matrices of $\mathbb{N}_0$-periodic measurable functions on $\mathbb{K}$ called wavelet masks symbol. Define a multi-wavelet system

$$\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_\ell) = \{\varphi_{n;0,\lambda} : n = 1,2,...,r; \lambda \in \Lambda\} \cup \{\psi^{\ell,s}_{n;j,\lambda} : n = 1,2,...,r; j \in \mathbb{N}_0, \; \lambda \in \Lambda, \;\; 1 \le \ell \le qN-1\}. \tag{2.3}$$

$\mathcal{W}^s(\varphi; \psi_1, \psi_2, ...\psi_{qN-1})$ is called a nonuniform multi-wavelet Bessel sequence (NUMWBS) in $\mathbb{H}^s(\mathbb{K})$ if there exists $B > 0$ such that

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda} |\langle f, \varphi_{n;0,\lambda}\rangle_{\mathbb{H}^s\mathbb{K}}|^2 + \sum_{n=1}^{r}\sum_{l=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda} |\langle f, \psi^{\ell,s}_{n;j,\lambda}\rangle_{\mathbb{H}^s(\mathbb{K})}|^2 \le B\|f\|^2_{\mathbb{H}^s(\mathbb{K})}, \;\; \forall \;\; f \in \mathbb{H}^s(\mathbb{K}),$$

where B is called a bessel bound; it is called a nonuniform multi-wavelet frame (NUMWF) in $\mathbb{H}^s(\mathbb{K})$ if there exist $0 < A \le B < \infty$ such that

$$A\|f\| \le \sum_{n=1}^{r}\sum_{\lambda\in\Lambda} |\langle f, \varphi_{n;0,\lambda}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda} |\langle f, \psi^{\ell,s}_{n;j,\lambda}\rangle|^2 \le B\|f\|^2, \; \textit{for all} \;\; f \in \mathbb{H}^s(\mathbb{K}),$$

where A and B are called frame bounds.

3. Properties of nonuniform multi wavelet Bessel sequences in sobolev spaces

In this section, we provide some necessary lemmas which are used for later. By a standard argument, we have

Lemma 3.1. *Let $s \in \mathbb{K}$, define $\mathcal{P}$ by*

$$\widehat{\mathcal{P}f}(\xi) = (1+\|.\|_2^2)^{s/2}\widehat{f}(\xi)$$

for $f \in \mathbb{H}^s(\mathbb{K})$ or $L^2(\mathbb{K})$. Then $\mathcal{P}$ is a unitary operator both from $\mathbb{H}^s(\mathbb{K})$ onto $L^2(\mathbb{K})$ and $L^2(\mathbb{K})$ onto $\mathbb{H}^{-s}(\mathbb{K})$.

Lemma 3.2. *Let $s \in \mathbb{K}$ and $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_{qN-1})$ is a NUMWBS in $\mathbb{H}^s(\mathbb{K})$ with Bessel bound B if and only if*

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda} |\langle f, \varphi_{n;0,\lambda}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda} |\langle f, \psi^{\ell,s}_{n;j,\lambda}\rangle|^2 \le B\|f\|^2 \, \textit{for} \;\; f \in \mathbb{H}^{-s}(\mathbb{K}). \tag{3.1}$$

Proof. By Lemma 3.1, we know that $\mathcal{W}^s(\varphi;\psi_1,\psi_2,...,\psi_{qN-1})$ is a MWBS in $\mathbb{H}^s(\mathbb{K})$ with Bessel bound B if and only if

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda}|\langle f,\mathcal{P}\varphi_{n;0,\lambda}\rangle|^2+\sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda}|\langle f,\mathcal{P}\psi^{\ell,s}_{n;j,k}\rangle|^2\leq B\|f\|^2 \text{ for } f\in L^2(\mathbb{K}). \tag{3.2}$$

Since $\mathcal{P}$ is a unitary operator, we have

$$\langle f,\mathcal{P}\varphi_{n;0,k}\rangle=\langle\mathcal{P}f,\varphi_{n;0,k}\rangle \text{ and } \langle f,\mathcal{P}\psi^{\ell,s}_{n;j,k}\rangle=\langle\mathcal{P}f,\psi^{\ell,s}_{n;j,k}\rangle$$

and

$$\|f\|^2_{L^2(\mathbb{K})}=\|\mathcal{P}f\|_{\mathbb{H}^{-s}(\mathbb{K})}.$$

It follows that (3.2) is equivalent to

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda}|\langle f,\mathcal{P}\varphi^n_{0,\lambda}\rangle|^2+\sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda}|\langle f,\mathcal{P}\psi^{\ell,s}_{n;j,\lambda}\rangle|^2\leq B\|\mathcal{P}f\|^2 \text{ for } f\in L^2(\mathbb{K}). \tag{3.3}$$

This leads to the Lemma since $\mathcal{P}$ is a unitary operator from $L^2(\mathbb{K})$ to $\mathbb{H}^{-s}(\mathbb{K})$ by Lemma 3.1. □

Lemma 3.3. *Let $0\neq s\in\mathbb{K}$ and $\varphi=(\varphi_1,\varphi_2,...,\varphi_r)^T\in(H^s(\mathbb{K}))^r$. If $[\widehat{\varphi}_n,\widehat{\varphi}_n]_t\in L^\infty(\mathbb{K})$ for some $t>s$ with $n=1,2,...,r$, then*

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda}|\langle g,\varphi_{n;0,\lambda}\rangle|^2\leq\sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\|g\|^2_{H^{-s}(\mathbb{K})} \tag{3.4}$$

for $g\in H^{-s}(\mathbb{K})$.

Proof. Since for any $n\in\{1,2,...,r\}$ $\varphi_n\in H^s(\mathbb{K})$ and $g\in H^{-s}(\mathbb{K})$, we have $\widehat{g\overline{\varphi_n}}\in L^2(\mathbb{K})$. Applying the Plancheral theorem and the Parseval identity, by a simple computation we have

$$\begin{aligned}\sum_{\lambda\in\Lambda}|\langle g,\varphi_n(\xi-\lambda)\rangle|^2 &= \sum_{\lambda\in\Lambda}\left|\int_{\mathbb{K}}\widehat{g}(\xi)\overline{\widehat{\varphi}(\xi)}\chi_\lambda(\xi)d\xi\right|^2\\ &= \sum_{\lambda\in\Lambda}\left|\sum_{\lambda'\in\Lambda}\int_{\mathfrak{D}}\widehat{g}(\xi+\lambda')\overline{\widehat{\varphi}(\xi+\lambda')}\chi_\lambda(\xi)d\xi\right|^2\\ &= \int_{\mathfrak{D}}\left|\sum_{\lambda'\in\Lambda}\widehat{g}(\xi+\lambda')\overline{\widehat{\varphi}(\xi+\lambda')}d\xi\right|^2\\ &= \int_{\mathfrak{D}}|[\widehat{g},\widehat{\varphi}_n]_o(\xi)|^2d\xi. \end{aligned} \tag{3.5}$$

By the Cauchy Schwarz's inequality, we have $|[\widehat{g},\widehat{\varphi}_n]_0(\xi)|^2\leq[\widehat{g},\widehat{g}]_{-s}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s(\xi)$ for almost every $\xi\in\mathbb{K}$. Since $t>s$ and $[\widehat{\varphi},\widehat{\varphi}]_t\in L^\infty(\mathbb{K})$, it follows that

$$[\widehat{\varphi},\widehat{\varphi}]_s(\xi)\leq[\widehat{\varphi},\widehat{\varphi}]_t(\xi).$$

Therefore, $[\widehat{\varphi},\widehat{\varphi}]_s\in L^\infty(\mathbb{K})$ and thus we deduce from (3.5) that

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda}|\langle g,\varphi_n(\xi-\lambda)\rangle|^2 \le \sum_{n=1}^{r}\int_{\mathfrak{D}}[\widehat{g},\widehat{g}]_{-s}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s(\xi)d\xi$$

$$\le \sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\int_{\mathfrak{D}}[\widehat{g},\widehat{g}]_{-s}(\xi)d\xi$$

$$= \sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}}|\widehat{g}(\xi)|^2(1+\|\xi\|_2^2)^{-s}d\xi$$

$$= \sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\|g\|^2_{H^{-s}(\mathbb{K})}.$$

□

Lemma 3.4. *Let* $0 \neq s < t$, *and* $\widehat{b^\ell}(\xi) = \left(\widehat{b}^\ell_{n,m}(\xi)\right)_{n,m=1}^{r}$, $1 \le \ell \le qN-1$ *be a sequence of* $r\times r$ *order matrices of* $\mathbb{N}_0$*-periodic measurable functions on* $\mathbb{K}$*, define*

$$\Omega_{s,t}(\xi) = \sum_{j=0}^{\infty}(qN)^{-2js}(1+\|\xi\|_2^2)^s\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}^\ell_{n,m}((\mathfrak{p}^{-1}N)^{j+1}\xi)|^2(1+\|(\mathfrak{p}^{-1}N)^{j+1}\|_2^2)^{-t},\ \ \xi\in\mathbb{K}.$$

If there exists a non-negative number $\alpha > -s$ *and a positive constant* C *such that*

$$\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}^\ell_{n,m}(\cdot)|^2 \le C\min(1,\|.\|_2^{2\alpha}),\ \ a.e\ on\ \mathbb{K} \tag{3.6}$$

then $\Omega_{s,t}\in L^\infty(\mathbb{K})$.

Proof. Let us consider the two cases $s>0$ and $s<0$ separately. Suppose $s>0$. Since $t>s$, by Lemma 3.3, we have

$$\Omega_{s,t}(\xi) \le \sum_{j=0}^{\infty}(qN)^{-2js}\left(1+o_1^2\|\xi\|_2^2\right)^s\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}^\ell_{n,m}(\mathfrak{p}^{-j-1})|^2\left(1+(qN)^{-2j-2}o_2^2\|\xi\|^2\right)^{-t}. \tag{3.7}$$

By Lemma 3.3, there exists a positive constant C' such that

$$B_{s,t}(\xi) = \sum_{j=0}^{\infty}(qN)^{-2js}\left(1+o_1^2\|\xi\|^2\right)^s\left(1+(qN)^{-2j-2}o_2^2\|\xi\|^2\right)^{-t} \le C^t,\ \ \forall\,\xi\in\mathbb{K}. \tag{3.8}$$

This implies that $\Omega_{s,t}(\xi) \le C'C$, $\forall\,\xi\in\mathbb{K}$, *i.e.*, $\Omega_{s,t}\in L^\infty(\mathbb{K})$. Suppose $s<0$ without loss of generality, we assume that $s<t<0$. By Lemma 3.3, we have

$$\begin{aligned}\Omega_{s,t}(\xi) &\le \sum_{j=0}^{\infty}(qN)^{-2js}\left(1+o_2^2\|\xi\|_2^2\right)^s\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}^\ell_{n,m}((\mathfrak{p}^{-1}N)^{j+1})|^2\left(1+(qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t}\\ &= \ominus_{s,t}(\xi)\end{aligned}$$

For $o_1\|\xi\|\le 1$ and $j\ge 0$, we have

$$\left(1+(qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t} \le 2^{-t}\ \text{ and }\ \left(1+o_2^2\|\xi\|^2\right)^s \le 1.$$

Since $\alpha \geq 0$, $\alpha + s > 0$, by Lemma 3.3 and Equation (3.6), we have the following estimate

$$\begin{aligned}
\ominus_{s,t}(\xi) &\leq 2^{-t}\sum_{j=0}^{\infty}(qN)^{-2js}\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}_{n,m}^{\ell}((\mathfrak{p}^{-1}N)^{j+1}\xi)^2 \\
&\leq 2^{-t}C\sum_{j=0}^{\infty}(qN)^{-2js}\|(\mathfrak{p}^{-1}N)^{j+1}\xi\|_2^{2\alpha} \\
&\leq 2^{-t}C(qN)^{-2\alpha}\sum_{j=0}^{\infty}m^{-2j(\alpha+s)}(o_1\|\xi\|)^{2\alpha} \\
&\leq 2^{-t}C(qN)^{-2\alpha}\sum_{j=0}^{\infty}(qN)^{-2j(\alpha+s)} \\
&= \frac{2^{-t}C(qN)^{-2\alpha}}{1-(qN)^{-2(\alpha+s)}} < \infty.
\end{aligned}$$

For $o_1\|\xi\| > 1$, there exists $J \in \mathbb{N}_0$ such that $(qN)^J \leq o_1\|\xi\| < (qN)^{J+1}$. Then for $j = 0, 1, ..., J$, we have

$$\begin{aligned}
(1+(qN)^{-2j-2}o_1^2\|\xi\|^2)^{-t} &\leq \left(1+(qN)^{2(J-j)}\right)^{-t} \\
&= (qN)^{-2(J-j)t}\left((qN)^{-2(J-j)}+1\right)^{-t} \\
&\leq 2^{-t}(qN)^{-2(J-j)t}
\end{aligned}$$

$$(1+o_2^2\|\xi\|^2)^s \leq \left(1+o_2^2o_1^{-2}(qN)^{2J}\right)^s \leq o_2^{2s}o_1^{-2s}(qN)^{2Js}.$$

Write $\ominus_{s,t}(\xi) = \ominus_{s,t}^1(\xi) + \ominus_{s,t}^2(\xi)$, where

$$\ominus_{s,t}^1(\xi) = \sum_{j=0}^{J}(qN)^{-2js}\left(1+o_2^2\|\xi\|^2\right)^s\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}_{n,m}^{\ell}((\mathfrak{p}^{-1}N)^{j+1}\xi)|^2\left(1+(qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t},$$

$$\ominus_{s,t}^2(\xi) = \sum_{j=J+1}^{\infty}(qN)^{-2js}\left(1+o_2^2\|\xi\|^2\right)^s\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}_{n,m}^{\ell}((\mathfrak{p}^{-1}N)^{j+1}\xi)|^2\left(1+(qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t}.$$

Then by $(qN)^J \leq o_1\|\zeta\| < (qN)^{J+1}$ and $J \in \mathbb{N}_0$, it follows from $s < t < 0$ that

$$\begin{aligned}
\ominus_{s,t}^1(\xi) &= \sum_{j=0}^{J}(qN)^{-2js}\left(1+o_2^2\|\xi\|^2\right)^s\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}_{n,m}^{\ell}((\mathfrak{p}^{-1}N)^{j+1}\xi)|^2\left(1+(qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t} \\
&\leq Co_2^{2s}o_1^{-2s}2^{-t}\sum_{j=0}^{J}(qN)^{-2(J-j)(t-s)} \leq Co_2^{2s}o_1^{-2s}2^{-t}\sum_{j=0}^{\infty}(qN)^{-2j(t-s)} \\
&= o_2^{2s}o_1^{-2s}2^{-t}\frac{1}{1-(qN)^{-2(t-s)}} < \infty.
\end{aligned}$$

Since $(qN)^j \leq o_1\|\xi\| < (qN)^{J+1}$, we have for $j \geq J+1$

$$\left(1+(qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t} \leq \left(1+(qN)^{2(J-j)}\right)^{-t} \leq 2^{-t}$$

and

$$\left(1+o_2^2\|\xi\|^2\right)^s \leq \left(1+o_2^2o_1^{-2}(qN)^{2J}\right)^s \leq o_2^{2s}o_1^{-2s}(qN)^{2Js}.$$

Since $\alpha \geq 0,\ \alpha + s > 0$, by Lemma 3.3 and Equation (3.6), we have

$$\begin{aligned}
\ominus^2_{s,t}(\xi) &= \sum_{j=J+1}^{\infty} (qN)^{-2js}\left(1 + o_2^2\|\xi\|^2\right)^s \sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}((\mathfrak{p}^{-1}N)^{j+1}\xi)|^2 \left(1 + (qN)^{-2j-2}o_1^2\|\xi\|^2\right)^{-t} \\
&\leq 2^{-t}o_2^{2s}o_1^{-2s}C \sum_{j=J+1}^{\infty} (qN)^{-2(j-J)s} \sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-1}N^{j+1}\xi)|^2 \\
&\leq 2^{-t}o_2^{2s}o_1^{-2s}C \sum_{j=J+1}^{\infty} (qN)^{-2(j-J)s}\|\mathfrak{p}^{-j-1}\xi\|_2^{2\alpha} \\
&\leq 2^{-t}o_2^{2s}o_1^{-2s}C \sum_{j=J+1}^{\infty} (qN)^{-2(j-J)s}(qN)^{-2\alpha(j+1)}(o_1\|\xi\|)^{2\alpha} \\
&\leq 2^{-t}o_2^{2s}o_1^{-2s}C \sum_{j=J+1}^{\infty} (qN)^{-2(j-J)(\alpha+s)} \\
&= 2^{-t}o_2^{2s}o_1^{-2s}C \sum_{j=1}^{\infty} q^{-2j(\alpha+s)} \\
&= 2^{-t}o_2^{2s}o_1^{-2s}C \sum_{j=1}^{\infty} q^{-2j(\alpha+s)} \\
&= 2^{-t}o_2^{2s}o_1^{-2(\alpha+s)}C \frac{q^{-2}(\alpha+s)}{1-q^{-2s}} < \infty.
\end{aligned}$$

Therefore, for the case $s < 0$, we conclude that $\Omega_{s,t} \in L^\infty(\mathbb{K})$. □

Now we proceed to prove the main result of this paper.

Theorem 3.5. *Let $\varphi = (\varphi_1, \varphi_2, ..., \varphi_r)^T \in (H^s(\mathbb{K}))^r$ be a $\mathfrak{p}$-refinable function vector satisfying the refinable equation, and let $\widehat{b^\ell}(\xi) = (\widehat{b^\ell_{n,m}}(\xi))^r_{n,m}$, $1 \leq \ell \leq qN-1$ be a sequence of $r \times r$ order matrices of $\mathbb{N}_0$-periodic measurables functions on $\mathbb{K}$, $\psi_\ell = (\psi^\ell_1, \psi^\ell_2, ..., \psi^\ell_r)^T$, $1 \leq \ell \leq qN-1$ be the wavelet function vectors defined by* (2.1)*, and $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_{qN-1})$ be the nonuniform multi-wavelet system defined by* (2.3)*. Assume that*

(i) $[\widehat{\varphi}_n, \widehat{\varphi}_n]_t \in L^\infty(\mathbb{K})$ *for some $t > s$ with $n = 1, 2, ..., r$.*

(ii) *There exists a non-negative number $\alpha > -s$ and a positive constant C such that*

$$\sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\xi)|^2 \leq C\min(1, \|\xi\|_2^{2\alpha}), \ \ a.e \ \ on \ \ \mathbb{K}.$$

Then $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_{qN-1})$ is a NUMWBS in $H^s(\mathbb{K})$.

Proof. For the case $s = 0$, we take $0 < s_0 < \min\{t, \alpha\}$, then the condition (i) and (ii) hold for $s = s_0$. Therefore, the conclusion holds for $s = 0$ if it holds for $s = s_0$. So, in order to finish the proof, we need to prove the conclusion holds for $s \neq 0$. By Lemma 3.2, it is enough to prove that there exists a positive constant B such that

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda} |\langle g, \varphi_{n;0,\lambda}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda} |\langle g, \psi^{\ell,s}_{n;j,\lambda}\rangle|^2 \leq B\|g\|_{H^{-s}(\mathbb{K})} \ \ for \ \ g \in H^{-s}(\mathbb{K}). \tag{3.9}$$

For the first part, by Lemma 3.3, we have

$$\sum_{n=1}^{r}\sum_{\lambda\in\Lambda} |\langle g, \varphi^{\ell,s}_{n;0,\lambda}\rangle|^2 \leq \sum_{n=1}^{r} \|[\widehat{\varphi}_n, \widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\|g\|^2_{H^{-s}(\mathbb{K})} \ \ for \ \ g \in H^{-s}(\mathbb{K}). \tag{3.10}$$

Next, we check the second part. For $g \in H^{-s}(\mathbb{K})$, compute

$$\sum_{\lambda\in\Lambda} |\langle g, \psi^{\ell,s}_{n;j,\lambda}\rangle|^2 = (qN)^{-j(1+2s)} \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{K}} \widehat{g}(\xi)\overline{\widehat{\psi^{\ell}_n}(\mathfrak{p}^{-1}N\xi)}\chi_\lambda((\mathfrak{p}^{-1}N)^j\xi)d\xi \right|$$

$$= (qN)^{-j(1+2s)} \sum_{\lambda\in\Lambda} \left| \sum_{\lambda'\in\Lambda} \int_{\mathfrak{D}} \widehat{g}(\mathfrak{p}^j(\xi+\lambda'))\overline{\widehat{\psi^{\ell}_n}(\xi+\lambda')}\chi_\lambda(\xi)d\xi \right|^2$$

$$= (qN)^{-j(1+2s)} \sum_{\lambda\in\Lambda} \left| \sum_{\lambda'\in\Lambda} \int_{\mathfrak{D}} \widehat{g}((\mathfrak{p}^{-1}N)^{-j}(\xi+\lambda'))\overline{\widehat{\psi^{\ell}_n}(\xi+\lambda')}d\xi \right|^2$$

$$= (qN)^{j(1-2s)} \int_{\mathfrak{D}} \left| [\widehat{g}((\mathfrak{p}^{-1}N)^{-j}\xi), \widehat{\psi^{\ell}_n}(\xi)]_0(\xi) \right|^2 d\xi.$$

We can write each component of ψ_ℓ

$$\psi^{\ell}_n(\cdot) = \sum_{m=1}^{r} \widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-1}N\xi)\widehat{\varphi}_n(\mathfrak{p}^{-1}N\xi) \;\; for\; n = 1, 2, ..., r \; and \; 1 \le \ell \le qN-1$$

and it follows that

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle g, \psi^{\ell,s}_{n;j,\lambda}\rangle|^2 &= (qN)^{J(1-2s)} \int_{\mathfrak{D}} \left| \sum_{\lambda\in\Lambda}\sum_{m=1}^{r} \widehat{g}(\mathfrak{p}^{-1}N(\xi+\lambda))\overline{\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-1}N(\zeta+\lambda))\widehat{\varphi}_n(\mathfrak{p}^{-1}N(\xi+\lambda))} \right|^2 d\xi \\
&= (qN)^{j(1-2s)} \int_{\mathfrak{D}} \left| \sum_{\gamma\in\Gamma_{\mathfrak{p}}}\sum_{m=1}^{r} \overline{\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-1}N\xi+\gamma)} \left[\widehat{g}((\mathfrak{p}^{-1}N)^{j+1}\xi), \widehat{\varphi}_n\right]_0 (\mathfrak{p}^{-1}\xi+\gamma) \right|^2 d\xi \\
&\le (qN)^{(j+1)-2s} \sum_{\gamma\in\Gamma_{\mathfrak{p}}} \int_{\mathfrak{D}} \left| \sum_{m=1}^{r} \overline{\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-1}N\xi+\gamma)} \left[\widehat{g}((\mathfrak{p}^{-1}N)^{j+1}), \widehat{\varphi}_n\right]_0 (\mathfrak{p}^{-1}N\xi+\gamma) \right|^2 d\xi \\
&\le (qN)^{(j+2)-2js} \int_{\mathfrak{D}} \sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\xi)|^2 \left[\widehat{g}((\mathfrak{p}^{-1}N)^{j-1} \cdot \widehat{g}((\mathfrak{p}^{-1}N)^{j+1})\right]_{-t}(\xi) \left[\widehat{\varphi}_n, \widehat{\varphi}_n\right]_t(\xi)d\xi \\
&\le (qN)^{(j+2)-2js} \max_{1\le n\le r}\{\| \left[\widehat{\varphi}_n, \widehat{\varphi}_n\right]_t \|_{L^\infty(\mathbb{K})}\} \int_{\mathfrak{D}} \sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\xi)|^2 \left[\widehat{g}((\mathfrak{p}^{-1}N)^{j+1}(\xi)), \widehat{g}(\mathfrak{p}^{-1}N)^{j+1}\xi)\right]_{-t}(\xi)d\xi \\
&\le (qN)^{(j+2)-2js} \max_{1\le n\le r}\{\| \left[\widehat{\varphi}_n, \widehat{\varphi}_n\right]_t \|_{L^\infty(\mathbb{K})}\} \int_{\mathbb{K}} \sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\xi)|^2 |\widehat{g}((\mathfrak{p}^{-1}N)^{j+1}(\xi))|^2 \left(1+\|\xi\|_2^2\right)^{-t} d\xi \\
&= (qN)^{1-2js} \max_{1\le n\le r}\{\|[\widehat{\varphi}_n, \widehat{\varphi}_n]_t\|_{L^\infty(\mathbb{K})}\} \sum_{\mathbb{K}}\sum_{m=1}^{r} |\widehat{b}_{n,m}((\mathfrak{p}^{-1}N)^{j+1}\xi)|^2 |\widehat{g}(\xi)|^2 |\widehat{g}(\xi)|^2 (1+\|(\mathfrak{p}^{-1}N)^{-j-1}\xi\|_2^2)^{-t} d\xi.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
\sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda} \left|\langle g, \psi^{\ell,s}_{n;j,\lambda}\rangle\right|^2 &\le qN \max_{1\le n\le r}\{\|[\widehat{\varphi}_n, \widehat{\varphi}_n]_t\|_{L^\infty(\mathbb{K})}\} \int_{\mathbb{K}} |\widehat{g}(\xi)|^2 \left(1+\|\xi\|_2^2\right)^{-s} \\
&\times \sum_{j=0}^{\infty}(qN)^{-2js}\left(1+\|\xi\|_2^2\right)^{s} \sum_{\ell=1}^{qN-1}\sum_{n=1}^{r}\sum_{m=1}^{r} \left|\widehat{b}^{\ell}_{n,m}((\mathfrak{p}^{-1}N)^{j+1}\xi)\right|^2 \left(1+\|(\mathfrak{p}^{-1}N)^{j+1}\xi\|_2^2\right)^{-t} d\xi.
\end{aligned}$$

By Lemma 3.4, it follows that

$$\sum_{n=1}^{r}\sum_{\ell=1}^{qN-1}\sum_{j=0}^{\infty}\sum_{\lambda\in\Lambda}|\langle g,\psi_{n;j,\lambda}\rangle|^2 \le (qN)\max_{1\le n\le r}\{\|[\widehat{\varphi}_n,\widehat{\varphi}_n]\|\|\Omega_{s,t}\|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}}|\widehat{g}(\xi)|^2\left(1+\|\xi\|_2^2\right)^{-s}$$
$$= (qN)\max_{1\le n\le r}\{\|[\widehat{\varphi}_n,\widehat{\varphi}_n]\|\|\Omega_{s,t}\|_{L^\infty(\mathbb{K})}\|g\|_{H^{-s}(\mathbb{K})}.$$

Consequently, (3.9) holds with

$$B=\sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}+(qN)\max_{1\le n\le r}\{\|[\widehat{\varphi}_n,\widehat{\varphi}_n]\|\|\Omega_{s,t}\|_{L^\infty(\mathbb{K})}\}\|\Omega_{s,t}\|_{L^\infty(\mathbb{K})}. \tag{3.11}$$

The proof is completed. □

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