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Bilateral mock theta functions of order eleven and their Lerch representations

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Abstract

We use bilateral basic hypergeometric series to obtain some bilateral mock theta functions and show that these functions are related to the basic hypergeometric series ${}_6\Phi_5$. Also Ramanujan's characterization of mock theta functions is satisfied by these functions. We also express them in terms of the Lerch's transcendental function $f(x, \xi; q, p)$.

Keywords: Mock theta functions, bilateral mock theta functions, Lerch transcendent, hypergeometric series, characteristic property, unit circle

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1. Introduction

Ramanujan discovered certain nice functions and referred to them in his last communication to G. H. Hardy in January 1920 under the name of mock theta functions (*cf.* [21]). In this letter he gave a list of seventeen such functions and without assigning any reason designated them as of third, fifth and seventh order. To quote his exact words, "Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2\pi im/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes... The question is: - is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is $O(1)$ at all the points $e^{2\pi im/n}$? The answer is it is not necessarily so. When it is not so I call the function mock theta function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconcievable to construct a theta function to cut out the singularities of the original function."

More precisely, a mock theta function $f(q)$ according to Ramanujan, is a function defined by means of a q-series which is convergent for $|q| < 1$ and which meets the following three conditions (*cf.* [22]).

(A) Infinitely many roots of unity are exponential singularities;

(B) If ξ is some root of unity, then corresponding to it there exists a theta function $\theta_\xi(q)$ such that $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.

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(C) f is not the sum of two functions, one of which is a theta function and the other a function that is bounded radially towards all roots of unity.

There is no standard definition of order of a mock theta function as yet. As per the definition proposed by Agarwal [1], if the mock theta function is expressible in terms of ${}_{r+1}\Phi_r$ series then it is of order $2r + 1$. Some other definitions of order are used in literature as well (*cf.* [7]).

Several other mathematicians found functions purporting to satisfy the above conditions after that. A list is given below.

(1) Watson discovered three functions of third order in 1936 (*cf.* [30]).

(2) Andrews et al discovered seven functions of sixth order in 1991 (*cf.* [5]).

(3) Y. S. Choi discovered four functions of tenth order in 1999 (*cf.* [9]).

(4) Gordon and McIntosh discovered eight functions of eighth order in 2000 (*cf.* [13]).

(5) Hikami dicovered four functions (one each of second and fourth order and two of eighth order) in 2005 and 2006 (*cf.* [16, 17]).

(6) McIntosh discovered three functions of second order in 2007 (*cf.* [19]).

(7) Berndt et al discovered two functions in 2007 (*cf.* [7]).

(8) Bringmann et al discovered two functions in 2011 (*cf.* [8]).

(9) Andrews discovered four functions in 2012 (*cf.* [6]).

Griffin et al. [15] have demonstrated that the mock theta functions given by Ramanujan indeed satisfy the three conditions (A), (B) and (C). As on date, for the remaining functions listed above, no one has proved that they meet all the three conditions. Property B has been established for all of them though.

Given a mock theta function expressed in terms of a series from 0 to ∞, we define the corresponding bilateral mock theta function by considering the analogous bilateral series, i.e. the same series but which now ranges from $-\infty$ to ∞. Watson [31] initiated the study of bilateral mock theta functins by investigating the properties of the bilateral forms for four of the ten mock theta functions of order five given by Ramanujan and showed that they too meet Property B. Also he has expressed them in terms of the transcendental function $f(x, \xi; q, p)$ given by M. Lerch [18]. S. D. Prasad [20] in 1970 has defined the bilateral forms of the five generalized third order mock theta functions. The bilateral form of sixth order mock theta functions given by Andrews was studied by A. Gupta [14]. Bhaskar Srivastava [26]-[29] has studied bilateral mock theta functions of order five, eight, two and those stemming from the mock theta functions given by Andrews [6] and Bringmann et al. [8].

This paper is divided as follows: In subsection 1.1 we give definitions. In subsection 2.1 we define the following eight functions, namely

$$f_{0,c_5}(q) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{5n(n-1)}{2}} q^n}{(-q;q)_n}, \tag{1.1}$$

$$f_{1,c_5}(q) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{5n(n-1)}{2}} q^{2n}}{(-q;q)_n}, \tag{1.2}$$

$$F_{0,c_5}(q^2) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{5n(n-1)} q^{2n}}{(q;q^2)_n}, \tag{1.3}$$

$$F_{1,c_5}(q^4) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{5n(2n-2)} q^{8n}}{(q^6;q^4)_n}, \tag{1.4}$$

$$\Psi_{0,c_5}(q) = \sum_{-\infty}^{\infty}(-1)^n q^{2n^2+6n}(-q;q)_n, \tag{1.5}$$

$$\Phi_{1,c_5}(q^2) = \sum_{-\infty}^{\infty}(-1)^n q^{4n^2+8n}(-q;q^2)_n, \tag{1.6}$$

$$\Phi_{0,c_5}(q^2) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{5n^2}}{(-q;q^2)_n}, \tag{1.7}$$

$$\Psi_{1,c_5}(q) = \sum_{-\infty}^{\infty}(-1)^{n+1} \frac{q^{\frac{5n(n+1)}{2}}}{2(-q;q)_n}. \tag{1.8}$$

In subsection 2.2, we have expressed the above functions as the limiting cases of a basic hypergeometric series ${}_6\Phi_5$ on a single base q, q^2 or q^4. In subsection 2.3, we have shown that these functions possess Property B of the mock theta functions defined by Ramanujan and therefore are genuine bilateral mock theta functions. In subsection 2.4 the above functions are shown to be expressible in terms of the Lerch's transcendent.

1.1. Preliminaries

Suppose n is an integer and q, z are complex numbers. If $n \geq 0$ we define $(z)_n = (z;q)_n = \prod_{i=0}^{n-1}(1 - q^i z)$ if $n \leq 0$ and $(z)_{-n} = (z;q)_{-n} = \frac{(-z)^{-n} q^{\frac{n(n+1)}{2}}}{(\frac{q}{z};q)_n}$ and more generally $(z_1, z_2, \cdots, z_r; q)_n = (z_1)_n (z_2)_n \cdots (z_r)_n$.

For $\left|q^k\right| < 1$ let us define $(z;q^k)_n = (1-z)(1-zq^k)\cdots(1-zq^{k(n-1)})$, $n \geq 1$ $(z;q^k)_0 = 1$ and $(z;q^k)_\infty = \lim_{n\to\infty}(z;q^k)_n = \prod_{i\geq 0}(1-q^{ki}z)$ and even more generally,

$$(z_1, z_2, \cdots, z_r; q^k)_\infty = (z_1;q^k)_\infty \cdots (z_r;q^k)_\infty.$$

A basic hypergeometric series ${}_{r+1}\Phi_r$ on base q^k is defined as

$${}_{r+1}\Phi_r\left[\begin{matrix} a_1, a_2, & \cdots, & a_{r+1} \\ b_1, b_2, & \cdots, & b_r \end{matrix} ; \ q^k; z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{r+1}; q^k)_n z^n}{(q^k;q^k)_n (b_1, b_2, \cdots, b_r; q^k)_n}, \quad (|z| < 1)$$

and a bilateral basic hypergeometric series ${}_r\Psi_r$ is defined as

$${}_r\Psi_r\left[\begin{matrix} a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_r \end{matrix} ; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \cdots, a_r; q)_n \, z^n}{(b_1, \cdots, b_r; q)_n}, \quad \left(\left|\frac{b_1 \cdots b_r}{a_1 \cdots a_r}\right| < |z| < 1\right).$$

The Lerch transcendental function $f(x, \xi; q, p)$ is defined by:

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty} \frac{(pq)^{n^2}(x\xi)^{-2n}}{(-p\xi^{-2};p^2)_n} \tag{1.9}$$

and by

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty}(-\xi^2 p; p^2)_n q^{n^2} x^{2n}. \tag{1.10}$$

2. Main results

2.1. Eight bilateral mock theta functions

We define the bilateral mock theta functions $f_{0,c_5}(q)$, $f_{1,c_5}(q)$, $F_{0,c_5}(q^2)$, $F_{1,c_5}(q^4)$, $\Psi_{0,c_5}(q)$, $\Phi_{1,c_5}(q^2)$, $\Phi_{0,c_5}(q^2)$, $\Psi_{1,c_5}(q)$ using Slater's transformation which is given on page 142 in [12] between ${}_5\Psi_5$:

$$\frac{\left(b_1,\ldots,b_5,\frac{q}{a_1},\ldots,\frac{q}{a_5},dz,\frac{q}{dz};q\right)_\infty}{\left(c_1,\ldots,c_5,\frac{q}{c_1},\ldots,\frac{q}{c_5};q\right)_\infty}\ {}_5\Psi_5\left[\begin{array}{ccc} a_1, & \ldots & ,a_5\\ b_1, & \ldots & ,b_5\end{array};q;z\right]$$
$$=\frac{q}{c_1}\frac{\left(\frac{c_1}{a_1},\ldots,\frac{c_1}{a_5},\frac{qb_1}{c_1},\ldots,\frac{qb_5}{c_1},\frac{dc_1z}{q},\frac{q^2}{dc_1z};q\right)_\infty}{\left(c_1,\frac{q}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_5},\frac{qc_2}{c_1},\ldots,\frac{qc_5}{c_1};q\right)_\infty}\ {}_5\Psi_5\left[\begin{array}{ccc}\frac{qa_1}{c_1}, & \ldots & ,\frac{qa_5}{c_1}\\ \frac{qb_1}{c_1}, & \ldots & ,\frac{qb_5}{c_1}\end{array};q;z\right]+\text{ idem}(c_1,\ldots,c_5), \quad (2.1)$$

where $d=\frac{a_1\cdots a_5}{c_1\cdots c_5}$ and $|\frac{b_1\cdots b_5}{a_1\cdots a_5}|<|z|<1$ and idem$(c_1,\ldots,c_5)$ means that the preceding expression is repeated with $c_1,\ldots c_5$ interchanged.

Now taking $a_1,\ldots,a_5\to\infty$, $b_1=-q$, $b_2=\cdots=b_5=0$, $z=\frac{q}{a_1\cdots a_5}$ in (2.1) we have

$$\frac{\left(-q,\frac{q}{c_1\cdots c_5},c_1\cdots c_5;q\right)_\infty}{\left(c_1,\ldots,c_5,\frac{q}{c_1},\ldots,\frac{q}{c_5},q\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{\frac{5n(n-1)}{2}}q^n}{(-q;q)_n}$$
$$=\frac{q}{c_1}\frac{\left(\frac{-q^2}{c_1},\frac{1}{c_2\cdots c_5},qc_2\cdots c_5;q\right)_\infty}{\left(c_1,\frac{q}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_5},\frac{qc_2}{c_1},\ldots,\frac{qc_5}{c_1};q\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{\frac{5n(n+1)}{2}}q^n}{c_1^{5n}\left(\frac{-q^2}{c_1};q\right)_n}+\text{ idem}(c_1,\ldots,c_5). \quad (2.2)$$

Now taking $a_1,\ldots,a_5\to\infty$, $b_1=-q$, $b_2=\cdots=b_5=0$, $z=\frac{q^2}{a_1\cdots a_5}$ in (2.1) we have

$$\frac{\left(-q,\frac{q^2}{c_1\cdots c_5},\frac{c_1\cdots c_5}{q};q\right)_\infty}{\left(c_1,\ldots,c_5,\frac{q}{c_1},\ldots,\frac{q}{c_5},q\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{\frac{5n(n-1)}{2}}q^{2n}}{(-q;q)_n}$$
$$=\frac{q}{c_1}\frac{\left(\frac{-q^2}{c_1},\frac{q}{c_2\cdots c_5},c_2\cdots c_5;q\right)_\infty}{\left(c_1,\frac{q}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_5},\frac{qc_2}{c_1},\ldots,\frac{qc_5}{c_1};q\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{\frac{5n(n+1)}{2}}q^{2n}}{c_1^{5n}\left(\frac{-q^2}{c_1};q\right)_n}+\text{ idem}(c_1,\ldots,c_5). \quad (2.3)$$

Now taking $a_1,\ldots,a_5\to\infty$, $b_1=q$, $b_2=\cdots=b_5=0$, $z=\frac{q^2}{a_1\cdots a_5}$ in (2.1) and base changed to q^2 we have

$$\frac{\left(q,\frac{q^2}{c_1\cdots c_5},c_1\cdots c_5;q^2\right)_\infty}{\left(c_1,\ldots,c_5,\frac{q^2}{c_1},\ldots,\frac{q^2}{c_5},q^2\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{5n(n-1)}q^{2n}}{(q;q^2)_n}$$
$$=\frac{q^2}{c_1}\frac{\left(\frac{q^3}{c_1},\frac{1}{c_2\cdots c_5},q^2c_2\cdots c_5;q^2\right)_\infty}{\left(c_1,\frac{q^2}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_7},\frac{q^2c_2}{c_1},\ldots,\frac{q^2c_5}{c_1};q^2\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{5n(n+1)}q^{2n}}{c_1^{5n}\left(\frac{q^3}{c_1};q^2\right)_n}+\text{ idem}(c_1,\ldots,c_5). \quad (2.4)$$

Now taking $a_1,\ldots,a_5\to\infty$, $b_1=q^6$, $b_2=\cdots=b_5=0$, $z=\frac{q^8}{a_1\cdots a_5}$ in (2.1) and base changed to q^4 we have

$$\frac{\left(q^6,\frac{q^8}{c_1\cdots c_5},\frac{c_1\cdots c_5}{q^4};q^4\right)_\infty}{\left(c_1,\ldots,c_5,\frac{q^4}{c_1},\ldots,\frac{q^4}{c_5},q^4\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{5n(2n-2)}q^{8n}}{(q^6;q^4)_n}$$
$$=\frac{q^4}{c_1}\frac{\left(\frac{q^{10}}{c_1},\frac{q^4}{c_2\cdots c_5},c_2\cdots c_5;q^4\right)_\infty}{\left(c_1,\frac{q^4}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_5},\frac{q^4c_2}{c_1},\ldots,\frac{q^4c_5}{c_1};q^4\right)_\infty}\sum_{-\infty}^{\infty}\frac{q^{5n(2n+2)}q^{8n}}{c_1^{5n}\left(\frac{q^{10}}{c_1};q^4\right)_n}+\text{ idem}(c_1,\ldots,c_5). \quad (2.5)$$

Now taking $a_1, \ldots, a_4 \to \infty$, $a_5 = -q$, $b_1 = \cdots = b_5 = 0$, $z = \frac{-q^8}{a_1 \cdots a_4}$ in (2.1) we have

$$\frac{\left(-1, \frac{q^9}{c_1\cdots c_5}, \frac{c_1\cdots c_5}{q^8}; q\right)_\infty}{\left(c_1, \ldots, c_5, \frac{q}{c_1}, \ldots, \frac{q}{c_5}, q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n q^{2n^2+6n}(-q;q)_n$$
$$= \frac{q}{c_1} \frac{\left(\frac{-c_1}{q}, \frac{q^8}{c_2\cdots c_5}, \frac{c_2\cdots c_5}{q^7}; q\right)_\infty}{\left(c_1, \frac{q}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_5}, \frac{qc_2}{c_1}, \ldots, \frac{qc_5}{c_1}; q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{2n^2+10n}}{c_1^{4n}} \left(-\frac{q^2}{c_1}; q\right)_n + \text{ idem}(c_1, \ldots, c_5). \quad (2.6)$$

Now taking $a_1, \ldots, a_4 \to \infty$, $a_5 = -q$, $b_1 = \cdots = b_5 = 0$, $z = \frac{-q^{12}}{a_1 \cdots a_4}$ in (2.1) and base changed to q^2 we have

$$\frac{\left(-q, \frac{q^{13}}{c_1\cdots c_5}, \frac{c_1\cdots c_5}{q^{11}}; q^2\right)_\infty}{\left(c_1, \ldots, c_5, \frac{q^2}{c_1}, \ldots, \frac{q^2}{c_5}, q^2\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n q^{4n^2+8n}(-q;q^2)_n$$
$$= \frac{q^2}{c_1} \frac{\left(\frac{-c_1}{q}, \frac{q^{11}}{c_2\cdots c_5}, \frac{c_2\cdots c_5}{q^9}; q^2\right)_\infty}{\left(c_1, \frac{q^2}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_5}, \frac{q^2c_2}{c_1}, \ldots, \frac{q^2c_5}{c_1}; q^2\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{4n^2+16n}}{c_1^{4n}} \left(-\frac{q^3}{c_1}; q^2\right)_n + \text{ idem}(c_1, \ldots, c_5). \quad (2.7)$$

Now taking $a_1, \ldots, a_5 \to \infty$, $b_1 = -q$, $b_2 = \cdots = b_5 = 0$, $z = \frac{q^5}{a_1 \cdots a_5}$ in (2.1) and base changed to q^2 we have

$$\frac{\left(-q, \frac{q^5}{c_1\cdots c_5}, \frac{c_1\cdots c_5}{q^3}; q^2\right)_\infty}{\left(c_1, \ldots, c_5, \frac{q^2}{c_1}, \ldots, \frac{q^2}{c_5}, q^2\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{5n^2}}{(-q;q^2)_n}$$
$$= \frac{q^2}{c_1} \frac{\left(\frac{-q^3}{c_1}, \frac{q^3}{c_2\cdots c_5}, \frac{c_2\cdots c_5}{q}; q^2\right)_\infty}{\left(c_1, \frac{q^2}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_5}, \frac{q^2c_2}{c_1}, \ldots, \frac{q^2c_5}{c_1}; q^2\right)_\infty} \sum_{-\infty}^{\infty} \frac{q^{5n(n+2)}}{c_1^{5n}\left(-\frac{q^3}{c_1}; q^2\right)_n} + \text{ idem}(c_1, \ldots, c_5). \quad (2.8)$$

Now taking $a_1, \ldots, a_5 \to \infty$, $b_1 = -1$, $b_2 = \cdots = b_5 = 0$, $z = \frac{1}{a_1 \cdots a_5}$ in (2.1) we have

$$\frac{\left(-1, \frac{1}{c_1\cdots c_5}, qc_1\cdots c_5; q\right)_\infty}{\left(c_1, \ldots, c_5, \frac{q}{c_1}, \ldots, \frac{q}{c_5}, q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^{n+1} \frac{q^{\frac{5n(n+1)}{2}}}{2(-q;q)_n}$$
$$= \frac{q}{c_1} \frac{\left(\frac{-q}{c_1}, \frac{1}{c_2\cdots c_5 q}, q^2c_2\cdots c_5; q\right)_\infty}{\left(c_1, \frac{q}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_5}, \frac{qc_2}{c_1}, \ldots, \frac{qc_5}{c_1}; q\right)_\infty} \sum_{-\infty}^{\infty} \frac{q^{\frac{5n(n+1)}{2}}}{c_1^{5n}\left(-\frac{q}{c_1}; q\right)_n} + \text{ idem}(c_1, \ldots, c_5). \quad (2.9)$$

Using the infinite series in Equations (2.2) to (2.9) we define the bilateral mock theta functions. They are given by Equations (1.1) to (1.8).

2.2. *Bilateral mock theta functions as the limiting case of a basic hypergeometric series*

Bilateral mock theta functions defined in subsection 2.1 have the following relation with the basic hypergeometric series ${}_6\Phi_5$:

$$f_{0,c_5}(q) = \sum_{-\infty}^{\infty} (-1)^n \frac{q^{\frac{5n(n-1)}{2}} q^n}{(-q;q)_n}$$
$$= \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{1}{t}, & -\frac{1}{t}, & \ldots & , -\frac{1}{t} \\ -q, & 0, & \ldots & , 0 \end{matrix} ; q; -t^5 q\right] - 2q^4 \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} \frac{q}{t}, & \ldots & , -\frac{q}{t}, & -q \\ 0, & \ldots & , 0, & 0 \end{matrix} ; q; -t^4 q^4\right],$$

$$\begin{aligned} f_{1,c_5}(q) &= \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{5n(n-1)}{2}} q^{2n}}{(-q;q)_n} \\ &= \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{1}{t}, & -\frac{1}{t}, & \dots & ,-\frac{1}{t} \\ -q, & 0, & \dots & ,0 \end{matrix} ;q;-t^5q^2\right] - 2q^3 \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{q}{t}, & \dots & ,-\frac{q}{t}, & -q \\ 0, & \dots & ,0, & 0 \end{matrix} ;q;-t^4q^3\right], \end{aligned}$$

$$\begin{aligned} F_{0,c_5}(q^2) &= \sum_{-\infty}^{\infty}(-1)^n \frac{q^{5n(n-1)} q^{2n}}{(q;q^2)_n} \\ &= \lim_{t\to 0} {}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{1}{t}, & -\frac{1}{t}, & \dots & ,-\frac{1}{t} \\ q, & 0, & \dots & ,0 \end{matrix} ;q^2;-t^5q^2\right] \\ &+ \left(q^7 - q^8\right) \lim_{t\to 0} {}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & q^3 \\ 0, & \dots & ,0, & 0 \end{matrix} ;q^2;t^4q^7\right], \end{aligned}$$

$$\begin{aligned} F_{1,c_5}(q^4) &= \sum_{-\infty}^{\infty}(-1)^n \frac{q^{5n(2n-2)} q^{8n}}{(q^6;q^4)_n} \\ &= \lim_{t\to 0} {}_6\Phi_5(q^4)\left[\begin{matrix} -\frac{1}{t}, & -\frac{1}{t}, & \dots & ,-\frac{1}{t} \\ q^6, & 0, & \dots & ,0 \end{matrix} ;q^4;-t^5q^8\right] \\ &+ \left(q^{14} - q^{12}\right) \lim_{t\to 0} {}_6\Phi_5(q^4)\left[\begin{matrix} -\frac{q^4}{t}, & \dots & ,-\frac{q^4}{t}, & q^2 \\ 0, & \dots & ,0, & 0 \end{matrix} ;q^4;t^4q^{14}\right], \end{aligned}$$

$$\begin{aligned} \Psi_{0,c_5}(q) &= \sum_{-\infty}^{\infty}(-1)^n q^{2n^2+6n}(-q;q)_n \\ &= \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{q}{t}, & \dots & ,-\frac{q}{t}, & -q \\ 0, & \dots & ,0, & ,0 \end{matrix} ;q;-t^4q^4\right] - \frac{1}{2q^4} \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{q}{t}, & -\frac{q}{t}, & \dots & ,-\frac{q}{t} \\ -q, & 0, & \dots & ,0 \end{matrix} ;q;-\frac{t^5}{q^4}\right], \end{aligned}$$

$$\begin{aligned} \Phi_{1,c_5}(q^2) &= \sum_{-\infty}^{\infty}(-1)^n q^{4n^2+8n}(-q;q^2)_n \\ &= \lim_{t\to 0} {}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & -q \\ 0, & \dots & ,0, & 0 \end{matrix} ;q^2;-t^4q^4\right] \\ &- \frac{1}{(q^3+q^4)} \lim_{t\to 0} {}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t} \\ -q^3, & 0, & \dots & ,0 \end{matrix} ;q^2;-\frac{t^5}{q^3}\right], \end{aligned}$$

$$\begin{aligned} \Phi_{0,c_5}(q^2) &= \sum_{-\infty}^{\infty}(-1)^n \frac{q^{5n^2}}{(-q;q^2)_n} \\ &= \lim_{t\to 0} {}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t} \\ -q, & 0, & \dots & ,0 \end{matrix} ;q^2;-\frac{t^5}{q^5}\right] \\ &- \left(q^4 + q^5\right) \lim_{t\to 0} {}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & -q^3 \\ 0, & \dots & ,0, & 0 \end{matrix} ;q^2;-t^4q^4\right], \end{aligned}$$

$$\begin{aligned} \Psi_{1,c_5}(q) &= \sum_{-\infty}^{\infty}(-1)^{n+1} \frac{q^{\frac{5n(n+1)}{2}}}{2(-q;q)_n} \\ &= \frac{-1}{2}\left(\lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{q}{t}, & -\frac{q}{t}, & \dots & ,-\frac{q}{t} \\ -q, & 0, & \dots & ,0 \end{matrix} ;q;-t^5\right] + \lim_{t\to 0} {}_6\Phi_5\left[\begin{matrix} -\frac{q}{t}, & \dots & ,-\frac{q}{t}, & -q \\ 0, & \dots & ,0, & 0 \end{matrix} ;q;-t^4\right]\right). \end{aligned}$$

2.3. Characteristic property of bilateral mock theta functions

Property B of a mock theta function is regarded as characteristic property which we reiterate as follows: Corresponding to any root of unity of the form $\xi = e^{\pi i \frac{h}{k}}$ (with h and k integers) there exists a theta function $\theta_\xi(q)$ such that the difference betwen the given mock theta function and $\theta_\xi(q)$ is bounded when q approaches ξ radially. The goal of this section is to show that the functions defined by Equations (1.1) to (1.8) satisfy Property B and hence are genuine bilateral mock theta functions.

We say that a point $e^{\pi i \frac{h}{k}}$ is a point of the first category if h is even and k is odd, is a point of the second category if h and k are both odd and is a point of the third category if h is odd and k is even.

Theorem 2.1. *For approach to a point of the first category radially* $\Phi_{0,c_5}(q^2) = O(1)$.

Proof. We have,

$$\begin{aligned}\Phi_{0,c_5}(q^2) &= \sum_{-\infty}^{\infty}(-1)^n\frac{q^{5n^2}}{(-q;q^2)_n}\\ &= \sum_{0}^{\infty}(-1)^n\frac{q^{5n^2}}{(-q;q^2)_n} + \sum_{1}^{\infty}(-1)^{-n}\frac{q^{5n^2}}{(-q;q^2)_{-n}}\\ &= \sum_{0}^{\infty}(-1)^n\frac{q^{5n^2}}{(-q;q^2)_n} - (q^4+q^5)\lim_{t\to 0}{}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & \ldots & ,-\frac{q^2}{t}, & -q^3 \\ 0, & \ldots & ,0, & 0\end{matrix}\;;q^2;-t^4q^4\right].\end{aligned} \tag{2.10}$$

Now let

$$\begin{aligned}T_{0,5}(q^2) &= \sum_{0}^{\infty}(-1)^n\frac{q^{5n^2}}{(-q;q^2)_n}\\ &= \sum_{0}^{\infty}\frac{(-1)^n q^{5n^2}}{\prod_{r=1}^{n}(1+q^{2r-1})}.\end{aligned}$$

Let $q = \rho e^{\pi i(\frac{h}{k})}$, $R(\rho) > 0$ and $\rho \to 1-$ so that

$$T_{0,5}(q^2) = \sum_{0}^{\infty}(-1)^n\frac{\rho^{5n^2}e^{\pi i(\frac{h}{k})5n^2}}{\prod_{r=1}^{n}(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)})}. \tag{2.11}$$

Putting $n = uk+v$, we can partition the above sum in the residue classes mod k so that, we have

$$\begin{aligned}T_{0,5}(q^2) &= \sum_{v=0}^{k-1}\sum_{u=0}^{\infty}(-1)^{uk+v}\frac{\rho^{5(uk+v)^2}e^{\pi i(\frac{h}{k})5(uk+v)^2}}{\prod_{r=1}^{uk+v}(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)})}\\ &= \sum_{v=0}^{k-1}\sum_{u=0}^{\infty}a_{v,u}.\end{aligned} \tag{2.12}$$

So,

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| = \frac{\rho^{5k(2uk+2v+k)}}{\prod_{r=uk+v+1}^{uk+k+v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right|}. \tag{2.13}$$

We now use an inequality given by Andrews and Hickerson which states that $|1+Rz| \le \sqrt{\frac{R}{R'}}|1+R'z|$ for $0 < R' \le R \le 1$ and $|z| = 1$ (*cf.* [5]). Our goal is to estimate the denominator in (2.13). So,

$$\begin{aligned}
\prod_{r=uk+v+1}^{uk+k+v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right| &= \prod_{r=1}^{k}\left|1+\rho^{2r+2uk+2v-1}e^{\pi i(\frac{h}{k})(2r+2uk+2v-1)}\right| \\
&= \prod_{r=1}^{k}\left|1+\rho^{2r+2uk+2v-1}e^{\pi i(\frac{h}{k})(2r+2v-1)}\right| \\
&\geq \prod_{r=1}^{k}\rho^{r+uk-1}\left|1+\rho^{2v+1}e^{\pi i(\frac{h}{k})(2r+2v-1)}\right| \\
&\left(R' = \rho^{2r+2uk+2v-1},\ R = \rho^{2v+1}\right) \\
&= \rho^{\frac{k(2uk+k-1)}{2}}\prod_{r=1}^{k}\left|1+\rho^{2v+1}e^{\pi i(\frac{h}{k})(2r+2v-1)}\right| \\
&= \rho^{\frac{k(2uk+k-1)}{2}}\left(1+\rho^{k(2v+1)}\right) \\
&\text{(since } 1+\rho^{2v+1}e^{\pi i(\frac{h}{k})(2r+2v-1)} \text{ runs through the roots of } [(x-1)^k-\rho^{k(2v+1)}]) \\
&\geq \rho^{\frac{k}{2}(2uk+k-1)}. \qquad (2.14)
\end{aligned}$$

Hence from Equations (2.13) and (2.14) we get

$$\begin{aligned}
\left|\frac{a_{v,u+1}}{a_{v,u}}\right| &\leq \frac{\rho^{5k(2uk+2v+k)}}{\rho^{\frac{k(2uk+k-1)}{2}}} \\
&\leq \rho^{k(9uk+10v+\frac{9k}{2}+\frac{1}{2})} \\
&\leq \epsilon < 1. \qquad (2.15)
\end{aligned}$$

Hence $\sum_u a_{v,u}$ is uniformly convergent.

$$\begin{aligned}
|T_{0,5}(q^2)| &\leq \sum_{v=0}^{k-1}\sum_{u=0}^{\infty}\epsilon^u|a_{v,0}| \\
&= \frac{1}{1-\epsilon}\sum_{v=0}^{k-1}|a_{v,0}| \\
&= \frac{\sum_{v=0}^{k-1}\left|(-1)^v\rho^{5v^2}e^{\pi i(\frac{h}{k})5v^2}\right|}{(1-\epsilon)\prod_{r=1}^{v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right|} \\
&\leq \frac{\sum_{v=0}^{k-1}\rho^{5v^2}}{(1-\epsilon)\prod_{r=1}^{v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right|} \\
&= O(1) \qquad (2.16)
\end{aligned}$$

for fixed k as $\rho \to 1-$. Since the second function on the right in Equation (2.10) is also a bounded function of q since ${}_6\Phi_5$ is convergent for $|q| < 1$, hence $\Phi_{0,c_5}(q^2) = O(1)$ when q approaches a point of the first category radially. □

Theorem 2.2. *For approach to a point of the second category radially* $\Phi_{0,c_5}(-q^2) = O(1)$.

Proof. If q approaches a point of the second category radially, then simultaenously $-q$ approaches a point of first category radially. Hence from the proof of Theorem 2.1 we conclude that $\Phi_{0,c_5}(-q^2) = O(1)$. □

Similarly it can also be proved that

(1) For approach to a point of the first category radially $f_{0,c_5}(q) = O(1)$, $f_{1,c_5}(q) = O(1)$, $F_{0,c_5}(q^2) = O(1)$, $F_{1,c_5}(q^4) = O(1)$, $\Psi_{1,c_5}(q) = O(1)$ and

(2) For approach to a point of the second category radially $f_{0,c_5}(-q) = O(1)$, $f_{1,c_5}(-q) = O(1)$, $F_{0,c_5}(-q^2) = O(1)$, $F_{1,c_5}(-q^4) = O(1)$, $\Psi_{1,c_5}(-q) = O(1)$.

Theorem 2.3. *For approach to a point of the third category radially* $\Phi_{1,c_5}(q^2) = O(1)$ *and* $\Psi_{0,c_5}(q) = O(1)$.

Proof. Suppose $q = \rho e^{\pi i(\frac{h}{k})}$ where h is odd, k is even and $0 \le \rho \le 1$ is a point on or within the unit circle. Note that when q approaches a point of the third category radially, then $\rho \to 1-$. We consider the case $\Phi_{1,c_5}(q^2) = O(1)$. The proof that $\Psi_{0,c_5}(q) = O(1)$ is similar.

$$\begin{aligned}
\Phi_{1,c_5}(q^2) &= \sum_{-\infty}^{\infty}(-1)^n q^{4n^2+8n}\left(-q;q^2\right)_n \\
&= \sum_{0}^{\infty}(-1)^n q^{4n^2+8n}\left(-q;q^2\right)_n + \sum_{1}^{\infty}(-1)^n q^{4n^2-8n}(-q;q^2)_{-n} \\
&= \sum_{0}^{\infty}(-1)^n q^{4n^2+8n}\left(-q;q^2\right)_n - \frac{1}{(q^3+q^4)}\lim_{t\to 0}{}_6\Phi_5(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & -\frac{q^2}{t}, & \dots & , -\frac{q^2}{t} \\ -q^3, & 0, & \dots & ,0 \end{matrix}\; ;q^2; -\frac{t^5}{q^3}\right].
\end{aligned} \tag{2.17}$$

Now let

$$\begin{aligned}
k_{1,5}\left(q^2\right) &= \sum_{0}^{\infty}(-1)^n q^{4n^2+8n}\left(-q;q^2\right)_n \\
&= \sum_{0}^{\infty}(-1)^n q^{4n^2+8n}\prod_{r=1}^{n}\left(1+q^{2r-1}\right).
\end{aligned}$$

Let $q = \rho e^{\pi i(\frac{h}{k})}$ and $\rho \to 1-$ (where h is odd and k is even) so that

$$k_{1,5}(q^2) = \sum_{0}^{\infty}(-1)^n \rho^{4n^2+8n} e^{\pi i(\frac{h}{k})(4n^2+8n)}\prod_{r=1}^{n}\left(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right).$$

Putting $n = 2uk + v$, we have

$$\begin{aligned}
k_{1,5}(q^2) &= \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty}\rho^{4(2uk+v)^2+8(2uk+v)}e^{\pi i(\frac{h}{k})(4(2uk+v)^2+8(2uk+v))}\prod_{r=1}^{2uk+v}\left(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right) \\
&= \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty} a_{v,u} \quad \text{(say)}.
\end{aligned} \tag{2.18}$$

Therefore

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| = \rho^{16k(2uk+k+v+1)} \times \prod_{r=2uk+v+1}^{2uk+v+2k}\left|(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)})\right|. \tag{2.19}$$

Further we calculate

$$\begin{aligned}
\prod_{r=2uk+v+1}^{2uk+v+2k}\left|\left(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right)\right| &= \prod_{r=1}^{2k}\left|1+\rho^{(4uk+2v+2r-1)}e^{\pi i(\frac{h}{k})(4uk+2v+2r-1)}\right| \\
&= \prod_{r=1}^{2k}\left|1+\rho^{(4uk+2v+2r-1)}e^{\pi i(\frac{h}{k})(2v+2r-1)}\right| \\
&= \prod_{r=1}^{2k}\left[1+2\rho^{(4uk+2v+2r-1)}\cos(2v+2r-1)\frac{h\pi}{k}+\rho^{(8uk+4v+4r-2)}\right]^{\frac{1}{2}}.
\end{aligned}$$

Since when $\beta \leq \alpha \leq 1$ we have

$$\frac{1+2\alpha\cos\theta+\alpha^2}{\alpha} \leq \frac{1+2\beta\cos\theta+\beta^2}{\beta}$$

hence we get,

$$\prod_{r=1}^{2k}\left|1+\rho^{(2r+4uk+2v-1)}e^{\pi i(\frac{h}{k})(2v+2r-1)}\right| \leq \prod_{r=1}^{2k}\left[\rho^{2r-4k}\left(1+2\rho^{(4uk+2v+4k-1)}\cos\frac{h\pi}{k}(2v+2r-1)+\rho^{(8uk+4v+8k-2)}\right)\right]^{\frac{1}{2}}$$
$$=\rho^{-k(2k-1)}\prod_{r=1}^{2k}\left|\left(1+\rho^{(4k+4uk+2v-1)}e^{\pi i(\frac{h}{k})(2v+2r-1)}\right)\right|.$$

Now as r runs through the values $1, 2, \ldots, 2k$ the points $e^{\pi i(\frac{h}{k})(2v+2r-1)}$ assume the positions $1, e^{\frac{\pi i}{k}}, e^{\frac{2\pi i}{k}}, \ldots, e^{\frac{(2k-1)\pi i}{k}}$ respectively.

Hence

$$\prod_{r=1}^{2k}\left|\left(1+\rho^{(4k+4uk+2v-1)}e^{\pi i(\frac{h}{k})(2v+2r-1)}\right)\right| = \prod_{r=0}^{2k-1}\left|\left(1+\rho^{(4k+4uk+2v-1)}e^{i(\frac{r\pi}{k})}\right)\right|$$
$$= 1-\rho^{2k(4uk+2v-1+4k)}.$$

Thus

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| \leq \rho^{16k(2uk+v+k+1)}\rho^{-k(2k-1)}(1-\rho^{2k(4uk+2v+4k-1)}) \tag{2.20}$$
$$\leq \rho^{16k(2uk+v+k+1)}\rho^{-k(2k-1)} \tag{2.21}$$
$$\leq \rho^{32uk^2+16vk+14k^2+17k} \tag{2.22}$$
$$\leq \epsilon < 1, \tag{2.23}$$

where $0 < \epsilon < 1$.

Hence $\sum_u a_{v,u}$ is uniformly convergent.

Also

$$|k_{1,5}(q^2)| \leq \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty}\epsilon^u|a_{v,0}| = \frac{1}{1-\epsilon}\sum_{v=0}^{2k-1}|a_{v,0}|$$
$$= \frac{1}{1-\epsilon}\sum_{v=0}^{2k-1}\left|(-1)^v\rho^{4v^2+8v}e^{\pi i(\frac{h}{k})(4v^2+8v)}\right| \times \prod_{r=1}^{v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right|$$
$$\leq \frac{1}{1-\epsilon}\sum_{v=0}^{2k-1}\rho^{4v^2+8v} \times \prod_{r=1}^{v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right| = O(1) \tag{2.24}$$

for fixed k as $\rho \to 1-$.

Hence $k_{1,5}(q^2)$ is bounded as q approaches a point of the third category radially. Also the second function on the right in Equation (2.17) is a bounded function of q for $|q| < 1$. Hence $\Phi_{1,5}(q^2)$ is uniformly convergent and bounded when q approaches a point of third category radially.

Similarly it can be proved that $\Psi_{0,c_5}(q) = O(1)$ for approach to $|q| = 1$ along the radius of third category (i.e. h odd and k even). □

Thus Theorems 2.1, 2.2 and 2.3 confirm that the bilateral mock theta functions defined in subsection 2.1 satisfy Property B.

2.4. Representation of bilateral mock theta functions as Lerch transcendents

We define Lerch's transcendental function by:

$$f(x,\xi,q,p) = \sum_{n=-\infty}^{\infty} \frac{(pq)^{n^2}(x\xi)^{-2n}}{(-p\xi^{-2};p^2)_n}.$$

This is also equivalent to

$$f(x,\xi,q,p) = \sum_{n=-\infty}^{\infty} (-\xi^2 p;p^2)_n q^{n^2} x^{2n}.$$

The bilateral mock theta functions defined in subsection 2.1 can be expressed in terms of the Lerch transcendent by means of the following lemma.

Lemma 2.4. *For* $\epsilon = \pm 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{\alpha n^2} q^{\beta n}}{(\epsilon q^{\gamma};q^{\delta})_n} = f\left(i(-\epsilon)^{-1/2} q^{\frac{2\gamma-2\beta-\delta}{4}}, (-\epsilon)^{1/2} q^{\frac{\delta-2\gamma}{4}}; q^{\frac{2\alpha-\delta}{2}}, q^{\frac{\delta}{2}}\right)$$

and

$$\sum_{n=-\infty}^{\infty} (-1)^n (-q;q^{\gamma})_n q^{\alpha n^2} q^{\beta n} = f\left(iq^{\frac{\beta}{2}}, q^{\frac{2-\gamma}{4}}; q^{\alpha}, q^{\frac{\gamma}{2}}\right).$$

Proof. The proof follows from direct substitution and use of basic hypergeometric transformations. □

For example we have $f_{1,c_5}(q) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{5n(n-1)}{2}} q^{2n}}{(-q;q)_n} = f\left(iq^{1/2}, q^{-1/4}; q^2, q^{1/2}\right)$ if we let $\alpha = 5/2, \beta = -1/2, \epsilon = -1$, $\gamma = \delta = 1$ and $\Phi_{1,c_5}(q^2) = \sum_{-\infty}^{\infty}(-1)^n q^{4n^2+8n}(-q;q^2)_n = f\left(iq^4, 1; q^4, q\right)$ if we let $\alpha = 4, \beta = 8, \gamma = 2$ in the above lemma. Similarly all the functions defined by Equations (1.1) to (1.8) can be recast as Lerch transcendents.

3. Conclusion

We may label the functions defined by Equations (1.1) to (1.8) as bilateral mock theta functions of order eleven. This is in view of their satisfying Ramanujan's Property B and the definition of order proposed by Agarwal. The fact that these functions can be recast as Lerch transcendents may aid in finding their relations with ordinary theta functions. Similarly expressing these functions in terms of Hecke type series may also be of interest.

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