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Homogeneous $\bar{q}$-blossoming and Bézier curves

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Abstract

Homogeneous $\bar{q}$-blossom is introduced by altering the diagonal property of classical homogeneous blossom. We apply this new blossom to define two parameter family of Bernstein basis functions and Bézier curves. A special case of homogeneous $\bar{q}$-blossom gives infinitely many de Casteljau type algorithms for classical Bézier curves. An analogue of Marsden's identity is also derived by applying homogeneous $\bar{q}$-blossom. Properties and identities of new Bernstein basis functions and Bézier curves including affine invariance, linear precision and end point interpolation derived. De Casteljau type evaluation algorithm is used to develop a subdivision procedure for (q_1, q_2)-Bézier curves. Finally, it is shown that the control polygons generated by recursive midpoint subdivision converge uniformly to the original (q_1, q_2)-Bézier curve.

Keywords: q-blossom, homogeneous $\bar{q}$-blossom, (q_1, q_2)-Bernstein basis functions, (q_1, q_2)-Bézier curves, de Casteljau algorithm, subdivision

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1. Introduction

It is well known that one way to develop some aspects of Bernstein basis functions and Bézier curves is using polynomial blossoming techniques. Polynomial blossom (or polar form) of a polynomial P is a unique symmetric and multiaffine function $p(x_1, \ldots, x_n)$ that reduces to P along the diagonal. That is blossoms satisfy the following three axioms:

- Symmetry:

$$p(x_1, \ldots, x_n) = p(x_{\sigma_1}, \ldots, x_{\sigma_n}),$$

  where $(\sigma_1, \ldots, \sigma_n)$ is any permutation of the set $\{1, \ldots, n\}$.

- Multi-affine:

$$p(x_1, \ldots, (1-\alpha)x_k + \alpha y_k, \ldots, x_n) = (1-\alpha)p(x_1, \ldots, x_k, \ldots, x_n) + \alpha p(x_1, \ldots, y_k, \ldots, x_n).$$

- Diagonal:

$$p(x, \ldots, x) = P(x).$$

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A degree n polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ is homogenized by multiplying each term x^k by y^{n-k}. Homogenization yields a new polynomial of homogeneous degree n in two variables,

$$P(x,y) = \sum_{k=0}^{n} a_k y^{n-k} x^k.$$

Let $p(x_1, \ldots, x_n)$ be the blossom of $P(x)$. We homogenize $p(x_1, \ldots, x_n)$ with respect to each variable x_k by (x_k, y_k), $k = 1, \ldots, n$. That is the blossom of a degree n homogeneous polynomial $P(x, y)$ is the homogenization

$$p((x_1, y_1), \ldots, (x_n, y_n))$$

of the blossom $p(x_1, \ldots, x_n)$. In $p(x_1, \ldots, x_n)$ each variable x_k appears at most to the first power in every term. Similarly, in $p((x_1, y_1), \ldots, (x_n, y_n))$ every term has the factor x_k or y_k but not both. The blossom of a homogeneous polynomial $P(x, y)$ is the unique symmetric, multi-linear polynomial $p((x_1, y_1), \ldots, (x_n, y_n))$ such that

$$p((x, y), \ldots, (x, y)) = P(x, y).$$

Note that by homogenizing $p(x_1, \ldots, x_n)$ the multi-affine property is replaced by the multi-linear property. Multi-linear means that the polynomial $p((x_1, y_1), \ldots, (x_n, y_n))$ is a linear function in each pair of variables. That is,

$$\begin{aligned} p((x_1, y_1), \ldots, a(x_k, y_k) + b(u_k, v_k), \ldots, (x_n, y_n)) =& a p((x_1, y_1), \ldots, (x_k, y_k), \ldots, (x_n, y_n)) \\ &+ b p((x_1, y_1), \ldots, (u_k, v_k), \ldots, (x_n, y_n)). \end{aligned}$$

There are many variants of classical blossom, each leads us to either a generalization of classical Bernstein bases and Bézier curves or Bernstein bases and Bézier curves for different spaces. Replacing the multi-affine property by pseudo-affine property leads to blossom of Chebyshev systems [1, 11, 12]. Replacing the multi-affine property with multi-barycentric property leads to blossom for trigonometric polynomials [7, 10]. Using similar techniques in [7] blossom for non-polynomial spaces of homogeneous degree n defined in [2]. h-blossom is constructed by replacing the diagonal property with h-diagonal property, see [17] and references therein. Replacing diagonal property with q-diagonal property leads to q-blossom for polynomials [18]. h-blossom and q-blossom are called quantum blossom and these two tecniques of blossoming unified by another variant of quantum blossom called (q, h)-blossom [5]. One of the recent paper used a general notion of polynomial blossoming to obtain novel polynomial Bernstein bases and Bézier curves, see [6]. These novel Bernstein bases and Bézier curves give quantum Bernstein bases and Bézier curves as a special ceses. [14] develops a new variant of quantum blossom to study the properties and identities of (p, q)-Bézier curves over the interval $[0, 1]$. The homogeneous quantum blossom with one parameter, called homogeneous q-blossom, is developed in [4] by homogenization of each variable of the q-blossom.

We want to extend the notion of quantum blossom to non-polynomial Bernstein bases and Bézier curves. Non-polynomial Bernstein bases and Bézier curves obtained from homogeneous blossom (see [2]). Hence our goal is introducing a new variant of homogeneous quantum blossom with two parameter, called homogeneous $\bar{q}$-blossom, which generalizes the homogeneous q-blossom. Here and throughout the paper the notation $\bar{q}$ indicates that the blossom is with two parameter, namely $\bar{q} = (q_1, q_2)$.

To keep the length of the paper manageable, this paper is about homogeneous $\bar{q}$-blossoming of polynomial spaces. Homogeneous $\bar{q}$-blossoming of non-polynomial spaces will be discussed in a forthcoming paper. Naturally, it is expected that our new homogeneous $\bar{q}$-blossom should satisfy outcomes of q-blossom. Indeed, our homogeneous $\bar{q}$-blossom not only satisfy the outcomes, but also generalize the q-blossom. Using homogeneous $\bar{q}$-blossom, we are able to introduce two parameter family of Bernstein bases and Bézier curves which generalize q-Bernstein bases and q-Bézier curves. In addition, we see that our generalization leads to infinitely many de Casteljau type algorithms for classical Bézier curves. We also see that our homogeneous $\bar{q}$-blossoming leads us to a generalization of (p, q)-Bernstein basis and (p, q)-Bézier curves (see, [9, 13, 14]). That is, unlike the previous variants of blossoming techniques, our homogeneous $\bar{q}$-blossoming techniques can be applied to (p, q)-Bézier curves over arbitrary interval $[a, b]$. Moreover, the subdivision procedure is developed to the (p, q)-Bézier curves for the first time.

We proceed in the following fashion. In Section 2 we construct homogeneous $\bar{q}$-blossom for polynomials of homogeneous degree n. After proving the existence and uniqueness of homogeneous $\bar{q}$-blossom, we show that there are infinitely many recursive blossoming algorithm each starts with specific homogeneous $\bar{q}$-blossom values. As a result of these blossoming algorithms, we show that there are $n!$ different recursive evaluation algorithms for polynomials of degree n. We define (q_1, q_2)-Bernstein basis functions and (q_1, q_2)-Bézier curves in Section 3, and we apply our homogeneous $\bar{q}$-blossom to study their properties. We show that every polynomial of degree n is a (q_1, q_2)-Bézier curve and there are $n!$ de Casteljau type recursive evaluation algorithms for these curves. We also derive an analogue of Marsden's identity for (q_1, q_2)-Bernstein basis functions. Section 4 is about identities and properties of (q_1, q_2)-Bernstein bases and (q_1, q_2)-Bézier curves, including affine invariance, linear precision, end point interpolation. In Section 5 we use the de Casteljau type evaluation algorithm to develop a subdivision procedure for (q_1, q_2)-Bézier curves. We also show that the control polygons generated by recursive midpoint subdivision converge uniformly to the original (q_1, q_2)-Bézier curve.

2. Homogeneous $\bar{q}$-blossoming

We are now going to construct $\bar{q}$-blossoming for homogeneous polynomials. We will then apply these $\bar{q}$-blossoms to obtain two parameter family of Bézier curves and study their properties.

Definition 2.1. Let $P(x, y)$ be a polynomial of homogeneous degree n. The homogeneous $\bar{q}$-blossom of $P(x, y)$ is the unique symmetric, multi-linear function $p((x_1, y_1), \ldots, (x_n, y_n))$ that reduces to $P(x, y)$ along the (q_1, q_2)-diagonal. That is, $p((x_1, y_1), \ldots, (x_n, y_n))$ is the unique function that satisfies the following three axioms:

(a) Symmetry Property: For every permutation $(\sigma_1, \ldots, \sigma_n)$ of the set $\{1, \ldots, n\}$

$$p((x_{\sigma_1}, y_{\sigma_1}), \ldots, (x_{\sigma_n}, y_{\sigma_n})) = p((x_1, y_1), \ldots, (x_n, y_n)).$$

(b) Multi-linear Property: For $i = 1, \ldots, n$,

$$\begin{aligned} p((x_1, y_1), \ldots, a(x_i, y_i) + b(u_i, v_i), \ldots, (x_n, y_n)) =& ap((x_1, y_1), \ldots, (x_i, y_i), \ldots, (x_n, y_n)) \\ &+ bp((x_1, y_1), \ldots, (u_i, v_i), \ldots, (x_n, y_n)). \end{aligned}$$

(c) Diagonal Property: p is equal to P on the (q_1, q_2)-diagonal i.e.

$$p((q_1^{n-1}x, q_2^{n-1}y), (q_1^{n-2}x, q_2^{n-2}y), \ldots, (q_1 x, q_2 y), (x, y)) = P(x, y).$$

Theorem 2.2. *Let* $P(x, y) = \sum_{k=0}^{n} c_k x^k y^{n-k}$ *be a polynomial of homogeneous degree n and let* $\mathbf{e}_n = (e_1, \ldots, e_n)$ *be a multi-index with* $e_i \in \{0, 1\},\ i = 1, 2, \ldots, n$ *and* $|\mathbf{e}_n| = \sum_{i=1}^{n} e_i$. *If*

$$\sum_{|\mathbf{e}_n|=k} \prod_{i=1}^{n} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} \neq 0, \quad k = 0, 1, \ldots, n,$$

then there exist a unique function p that is symmetric, multi-linear and reduces to P along the (q_1, q_2)*-diagonal.*

Proof. Let $\varphi_{n,k}((x_1, y_1), \ldots, (x_n, y_n)) = \sum_{|\mathbf{e}_n|=k} \left(\prod_{i=1}^{n} x_i^{e_i} y_i^{1-e_i} \right)$, where $\mathbf{e}_n = (e_1, \ldots, e_n)$ is a multi-index with $e_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$ and $|\mathbf{e}_n| = \sum_{i=1}^{n} e_i$. The function $\varphi_{n,k}$ is symmetric since the summation runs over all permutations of $(\underbrace{1, \ldots, 1}_{k \text{ terms}}, \underbrace{0, \ldots, 0}_{n-k \text{ terms}})$. Moreover, since each term in $\varphi_{n,k}$ is homogeneous of degree one in each pair (x_i, y_i), the function $\varphi_{n,k}$ is muti-linear. Evaluating $\varphi_{n,k}$ along (q_1, q_2)-diagonal gives

$$\varphi_{n,k}((q_1^{n-1}x, q_2^{n-1}y), (q_1^{n-2}x, q_2^{n-2}y), \ldots, (q_1 x, q_2 y), (x, y)) = \left(\sum_{|\mathbf{e}_n|=k} \prod_{i=1}^{n} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} \right) x^k y^{n-k}.$$

Hence, assuming that $\sum_{|\mathbf{e}_n|=k}\prod_{i=1}^{n}(q_1^{n-i})^{e_i}(q_2^{n-i})^{1-e_i} \neq 0$, the homogeneous $\bar{q}$-blossom of the basis functions $x^k y^{n-k}$, $k = 0, 1, \ldots, n$ are

$$\gamma_{n,k}((x_1, y_1), \ldots, (x_n, y_n)) = \frac{\varphi_{n,k}((x_1, y_1), \ldots, (x_n, y_n))}{\sum_{|\mathbf{e}_n|=k}\prod_{i=1}^{n}(q_1^{n-i})^{e_i}(q_2^{n-i})^{1-e_i}}$$

and the homogeneous $\bar{q}$-blossom of P is

$$p((x_1, y_1), \ldots, (x_n, y_n)) = \sum_{k=0}^{n} c_k \gamma_{n,k}((x_1, y_1), \ldots, (x_n, y_n)).$$

Existence is proved. To prove uniqueness, suppose that the polynomial P has two distinct homogeneous $\bar{q}$-blossom p and g. Since every symmetric multi-linear polynomial of homogeneous degree n has unique representation in terms of the functions $\gamma_{n,k},\ k = 0, 1, \ldots, n$, there are constants c_k and $d_k,\ k = 0, 1, \ldots, n$ such that

$$p((x_1, y_1), \ldots, (x_n, y_n)) = \sum_{k=0}^{n} c_k \gamma_{n,k}((x_1, y_1), \ldots, (x_n, y_n))$$

and

$$g((x_1, y_1), \ldots, (x_n, y_n)) = \sum_{k=0}^{n} d_k \gamma_{n,k}((x_1, y_1), \ldots, (x_n, y_n)).$$

Evaluating p and g along (q_1, q_2)-diagonal gives

$$\sum_{k=0}^{n} c_k x^k y^{n-k} = \sum_{k=0}^{n} d_k x^k y^{n-k}.$$

Hence $c_k = d_k$. Therefore the homogeneous $\bar{q}$-blossom is unique. □

In [4] another variant of homogeneous quantum blossom (with one variable), called homogeneous q-blossom has been developed by homogenization of the corresponding multi-affine blossom. In homogeneous $\bar{q}$-blossom our approach rely on altering the diagonal property of the classical homogeneous blossom. Although the approaches are different, there is a relation between homogeneous q-blossom and homogeneous $\bar{q}$-blossom. This relation is as follows:

Start from the classical q-blossom $\tilde{p}(x_1, x_2, \ldots, x_n; q)$ introduced in [18] and homogenize each variable to obtain homogeneous q-blossom $\tilde{p}((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n); q)$ that developed in [4]. Finally, setting $q = \frac{q_1}{q_2}$ and multiplying the result by $q_2^{-(n-1)n/2}$ gives homogeneous $\bar{q}$-blossom. That is

$$q_2^{-(n-1)n/2}\tilde{p}\left((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n); \frac{q_1}{q_2}\right) = p((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)).$$

But we prefer homogeneous $\bar{q}$-blossom. Because, as we will see later in the following sections, homogeneous $\bar{q}$-blossom generalizes the results in [18]. Our approach also gives new results for classical Bézier curves.

In the above proof we have assumed that $\sum_{|\mathbf{e}_n|=k}\prod_{i=1}^{n}(q_1^{n-i})^{e_i}(q_2^{n-i})^{1-e_i} \neq 0$. Now we are going to find the values of q_1 and q_2 which satisfies the assumption. First, let us define (q_1, q_2)-integer $[n]_{q_1,q_2}$ by

$$[n]_{q_1,q_2} := \frac{q_1^n - q_2^n}{q_1 - q_2} = \sum_{k=0}^{n-1} q_1^k q_2^{n-k-1}.$$

We also define (q_1, q_2)-factorial $[n]_{q_1,q_2}!$ by

$$[n]_{q_1,q_2}! := \begin{cases} [n]_{q_1,q_2} \cdot [n-1]_{q_1,q_2} \cdots [1]_{q_1,q_2} & \text{for } n \geqslant 1 \\ 1 & \text{for } n = 0. \end{cases}$$

Finally we define (q_1, q_2)-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2}$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2} = \frac{[n]_{q_1,q_2}!}{[k]_{q_1,q_2}![n-k]_{q_1,q_2}!},$$

$k = 0, 1, \ldots, n$. Note that the (q_1, q_2)-binomial coefficient is in fact (p, q)-binomial coefficient. For more information about (p, q)-binomial coefficients see [8, 9, 13].

The (q_1, q_2)-binomial coefficient satisfies the following pascal type identities:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2} = q_1^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q_1,q_2} + q_2^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q_1,q_2} \tag{2.1}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2} = q_2^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q_1,q_2} + q_1^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q_1,q_2}. \tag{2.2}$$

Lemma 2.3. *Let* $\mathbf{e}_n = (e_1, \ldots, e_n)$ *be a multi index with* $e_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$ *and* $|\mathbf{e}_n| = \sum_{i=1}^n e_i$. *Then*

$$\sum_{|\mathbf{e}_n|=k} \prod_{i=1}^{n} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} = q_1^{k(k-1)/2} q_2^{(n-k)(n-k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2}, \ k = 0, 1, \ldots, n. \tag{2.3}$$

Proof. Clearly our assumption is true for $n = 1$, since

$$\sum_{|\mathbf{e}_1|=0} \prod_{i=1}^{1} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} = \sum_{|\mathbf{e}_n|=1} \prod_{i=1}^{1} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} = 1.$$

Now suppose that our assumption is true for $n - 1$, that is

$$\sum_{|\mathbf{e}_{n-1}|=k} \prod_{i=1}^{n-1} (q_1^{n-i-1})^{e_i} (q_2^{n-i-1})^{1-e_i} = q_1^{k(k-1)/2} q_2^{(n-k-1)(n-k-2)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q_1,q_2}, \ \ k = 0, 1, \ldots, n-1. \tag{2.4}$$

Splitting the summation in Eq. (2.3) and rearranging gives

$$\begin{aligned}
\sum_{|\mathbf{e}_n|=k} \prod_{i=1}^{n} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} &= \sum_{\substack{|\mathbf{e}_{n-1}|=k \\ e_n=0}} \prod_{i=1}^{n} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} + \sum_{\substack{|\mathbf{e}_{n-1}|=k-1 \\ e_n=1}} \prod_{i=1}^{n} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} \\
&= \sum_{|\mathbf{e}_{n-1}|=k} \left(q_1^0\right)^0 \left(q_2^0\right)^1 \left(\prod_{i=1}^{n-1} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} \right) + \sum_{|\mathbf{e}_{n-1}|=k-1} \left(q_1^0\right)^1 \left(q_2^0\right)^0 \left(\prod_{i=1}^{n-1} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} \right) \\
&= \sum_{|\mathbf{e}_{n-1}|=k} \prod_{i=1}^{n-1} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} + \sum_{|\mathbf{e}_{n-1}|=k-1} \prod_{i=1}^{n-1} (q_1^{n-i})^{e_i} (q_2^{n-i})^{1-e_i} \\
&= \sum_{|\mathbf{e}_{n-1}|=k} q_1^{\sum_{j=0}^{n-1} e_i} q_2^{\sum_{j=0}^{n-1} 1-e_i} \prod_{i=1}^{n-1} (q_1^{n-i-1})^{e_i} (q_2^{n-i-1})^{1-e_i} + \sum_{|\mathbf{e}_{n-1}|=k-1} q_1^{\sum_{j=0}^{n-1} e_i} q_2^{\sum_{j=0}^{n-1} 1-e_i} \prod_{i=1}^{n-1} (q_1^{n-i-1})^{e_i} (q_2^{n-i-1})^{1-e_i} \\
&= q_1^k q_2^{n-k-1} \sum_{|\mathbf{e}_{n-1}|=k} \prod_{i=1}^{n-1} (q_1^{n-i-1})^{e_i} (q_2^{n-i-1})^{1-e_i} + q_1^{k-1} q_2^{n-k} \sum_{|\mathbf{e}_{n-1}|=k-1} \prod_{i=1}^{n-1} (q_1^{n-i-1})^{e_i} (q_2^{n-i-1})^{1-e_i} \\
&= q_1^{k(k-1)/2} q_2^{(n-k)(n-k-1)/2} \left\{ q_1^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q_1,q_2} + q_2^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q_1,q_2} \right\} \\
&= q_1^{k(k-1)/2} q_2^{(n-k)(n-k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2}.
\end{aligned}$$

Hence Eq. (2.3) is true for all n. □

Corollary 2.4. $\sum_{|\mathbf{e}_n|=k}\prod_{i=1}^{n}(q_1^{n-i})^{e_i}(q_2^{n-i})^{1-e_i}=0$ *if and only if at least one of the following conditions are satisfied.*

i. $q_1=0$

ii. $q_2=0$

iii. $q_1=-q_2$, *n is even and k is odd.*

Proof. Conditions *i.* and *ii.* follows from Eq. (2.3). Now consider that the (q_1,q_2)-integers and (q_1,q_2)-binomial coefficients as a function of q_1 while q_2 is a constant. Since $[m]_{q_1,q_2}=\sum_{k=0}^{m-1}q_1^kq_2^{m-k-1}=0$ only when $q_1=-q_2$ and m is even, using similar arguments in [18], we see that the (q_1,q_2)-binomial coefficient $\begin{bmatrix} n\\ k\end{bmatrix}_{q_1,q_2}$ has a real root $q_1=-q_2$ with multiplicity

$$\lfloor n/2\rfloor-\lfloor k/2\rfloor-\lfloor (n-k)/2\rfloor=\begin{cases}1, & \text{if } n \text{ is even and } k \text{ is odd}\\ 0, & \text{otherwise.}\end{cases}$$

□

From now on, the restriction stated in Corollary 2.4 apply whenever we speak of the homogeneous $\bar{q}$-blossom.

The following theorem gives recursive blossoming algorithm which uses specific homogeneous $\bar{q}$-blossom values to find homogeneous $\bar{q}$-blossom of any polynomial of homogeneous degree n.

Theorem 2.5. *Let P be a polynomial of homogeneous degree n and let p be the homogeneous $\bar{q}$-blossom of P. For a fixed non-zero c, set*

$$b_k^0=p((q_1^{n-1}a,q_2^{n-1}c),\ldots,(q_1^ka,q_2^kc),(q_1^{k-1}b,q_2^{k-1}c),\ldots,(q_1b,q_2c),(b,c)),$$

$k=0,1,\ldots,n$ *and define recursively*

$$b_k^{r+1}((x_1,y_1),\ldots,(x_{r+1},y_{r+1}))=\alpha_{r,k}b_k^r((x_1,y_1),\ldots,(x_r,y_r))+\beta_{r,k}b_{k+1}^r((x_1,y_1),\ldots,(x_r,y_r)) \quad (2.5)$$

for $r=0,1,\ldots,n-1$, $k=0,1,\ldots,n-r-1$ *where*

$$\alpha_{r,k}=\frac{q_2^k\,x_{r+1}c-q_1^k\,b\,y_{r+1}}{q_1^{r+k}q_2^k\,a\,c-q_1^k\,q_2^{r+k}\,b\,c}\ \ \textit{and}\ \ \beta_{r,k}=\frac{q_1^{r+k}\,a\,y_{r+1}-q_2^{r+k}\,x_{r+1}c}{q_1^{r+k}q_2^k\,a\,c-q_1^k\,q_2^{r+k}\,b\,c}.$$

Then

$$b_k^r((x_1,y_1),\ldots,(x_r,y_r))=p((q_1^{n-1}a,q_2^{n-1}c),\ldots,(q_1^{r+k}a,q_2^{r+k}c),(x_1,y_1),\ldots,(x_r,y_r),(q_1^{k-1}b,q_2^{k-1}c),\ldots,(b,c)).$$

In particular

$$b_0^n((x_1,y_1),\ldots,(x_n,y_n))=p((x_1,y_1),\ldots,(x_n,y_n)).$$

Proof. Note that the solution of the system of the linear equations

$$\alpha\begin{pmatrix} q_1^{r+k}a\\ q_2^{r+k}c\end{pmatrix}+\beta\begin{pmatrix} q_1^kb\\ q_2^kc\end{pmatrix}=\begin{pmatrix} x_{r+1}\\ y_{r+1}\end{pmatrix}$$

is $\alpha=\alpha_{r,k}$ and $\beta=\beta_{r,k}$. Hence the result follows easily by induction on r and using the symmetry and multi-linear property of homogeneous $\bar{q}$-blossom. □

In Theorem 2.5 for each value of $c\neq 0$, there is one new recursive evaluation algorithm. In total there are infinitely many blossoming algorithm to evaluate homogeneous $\bar{q}$-blossom of P. Note also that for each value of c, we have different starting points in the blossoming algorithm.

Since homogeneous $\bar{q}$-blossom is symmetric, for any permutation $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ of $\{1,2,\ldots,n\}$, if we substitute $x_r=q_1^{\sigma_r-1}x$ and $y_r=q_2^{\sigma_r-1}c$ in Eq. (2.5) we obtain a recursive evaluation algorithm for $P(x,c)$. There are $n!$ number of permutations of $\{1,2,\ldots,n\}$. Thus we have the following corollary.

Corollary 2.6. *Let P be a polynomial of homogeneous degree n and let p be the homogeneous $\bar{q}$-blossom of P. There are $n!$ recursive evaluation algorithm for $P(x, c)$, where c is a fixed non-zero value. These algorithms are defined as follows: For any permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $\{1, 2, \ldots, n\}$ and a fixed non-zero c, set*

$$b_k^0 = p((q_1^{n-1}a, q_2^{n-1}c), \ldots, (q_1^k a, q_2^k c), (q_1^{k-1}b, q_2^{k-1}c), \ldots, (q_1 b, q_2 c), (b, c)),$$

$k = 0, 1, \ldots, n$ and define recursively

$$b_k^{r+1}(x) = \alpha_{r,k}(x) b_k^r(x) + \beta_{r,k}(x) b_{k+1}^r(x) \tag{2.6}$$

for $r = 0, 1, \ldots, n-1$, $k = 0, 1, \ldots, n-r-1$ where

$$\alpha_{r,k}(x) = \frac{q_1^{\sigma_{r+1}-1} q_2^k\, x - q_1^k\, q_2^{\sigma_{r+1}-1}\, b}{q_1^{r+k} q_2^k\, a - q_1^k\, q_2^{r+k}\, b} \;\; \text{and} \;\; \beta_{r,k} = \frac{q_1^{r+k}\, q_2^{\sigma_{r+1}-1}\, a - q_1^{\sigma_{r+1}-1} q_2^{r+k} x}{q_1^{r+k} q_2^k\, a - q_1^k\, q_2^{r+k}\, b}.$$

Then

$$b_k^r(x) = p((q_1^{n-1}a, q_2^{n-1}c), \ldots, (q_1^{r+k}a, q_2^{r+k}c), (q_1^{\sigma_1-1}x, q_2^{\sigma_1-1}c), \ldots, (q_1^{\sigma_r-1}x, q_2^{\sigma_r-1}c), (q_1^{k-1}b, q_2^{k-1}c), \ldots, (b, c)).$$

In particular $b_0^n(x) = P(x, c)$.

3. Two parameter family of Bézier curves

Now we give a theorem which will lead us to definition of two parameter Bernstein bases and Bézier curves.

Theorem 3.1. *Let $P(x)$ be a polynomial of degree n and let $p((x_1, y_1), \ldots, (x_n, y_n))$ be the homogeneous $\bar{q}$-blossom of its homogenization $P(x, y)$. Then*

$$p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (x, 1)) = \sum_{k=0}^{m} P_{m,k}(x) \tilde{B}_k^{n,m}(x), \tag{3.1}$$

$m = 1, \ldots, n$, where

$$P_{m,k}(x) = p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1^m x, q_2^m), (q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{n-m+k}a, q_2^{n-m+k}), (q_1^{k-1}b, q_2^{k-1}), \ldots, (b, 1))$$

and

$$\tilde{B}_k^{n,m}(x) = \begin{bmatrix} m \\ k \end{bmatrix}_{q_1,q_2} \frac{\displaystyle\prod_{i=0}^{k-1} (q_1^{n-m+i}a - q_2^{n-m+i}x) \cdot \prod_{i=0}^{m-k-1} (q_1^i x - q_2^i b)}{\displaystyle\prod_{i=n-m}^{n-1} (q_1^i a - q_2^i b)}.$$

Proof. From multi-linear property of homogeneous $\bar{q}$-blossom, we have

$$\begin{aligned} p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (x, 1)) =& \frac{x - b}{q_1^{n-1}a - q_2^{n-1}b} p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (q_1^{n-1}a, q_2^{n-1})) \\ &+ \frac{q_1^{n-1}a - q_2^{n-1}x}{q_1^{n-1}a - q_2^{n-1}b} p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (b, 1)) \\ =& P_{1,0}(x) \tilde{B}_0^{n,1}(x) + P_{1,1}(x) \tilde{B}_1^{n,1}(x). \end{aligned}$$

Hence Eq. (3.1) is true for $m = 1$. Now suppose that our hypothesis is true for a fixed value $m < n$. That is

$$p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (x, 1)) = \sum_{k=0}^{m} P_{m,k}(x) \tilde{B}_k^{n,m}(x). \tag{3.2}$$

Applying multi-linear property of homogeneous $\bar{q}$-blossom to $P_{m,k}(x)$ gives

$$P_{m,k}(x) = \alpha_{m,k}(x)P_{m+1,k}(x) + \beta_{m,k}(x)P_{m+1,k+1}(x), \tag{3.3}$$

where $\alpha_{m,k}(x) = \dfrac{q_1^m q_2^k x - q_1^k q_2^m b}{q_1^{n-m+k-1} q_2^k a - q_1^k q_2^{n-m+k-1} b}$ and $\beta_{m,k}(x) = \dfrac{q_1^{n-m+k-1} q_2^m a - q_1^m q_2^{n-m+k-1} x}{q_1^{n-m+k-1} q_2^k a - q_1^k q_2^{n-m+k-1} b}$. Substituting Eq. (3.3) in Eq. (3.2), we obtain

$$\begin{aligned}
p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (x, 1)) &= \sum_{k=0}^{m} P_{m,k}(x)\tilde{B}_k^{n,m}(x) \\
&= \sum_{k=0}^{m} \left(\alpha_{m,k}(x)P_{m+1,k}(x) + \beta_{m,k}(x)P_{m+1,k+1}(x)\right)\tilde{B}_k^{n,m}(x) \\
&= \sum_{k=0}^{m} \alpha_{m,k}(x)P_{m+1,k}(x)\tilde{B}_k^{n,m}(x) + \sum_{k=0}^{m} \beta_{m,k}(x)P_{m+1,k+1}(x)\tilde{B}_k^{n,m}(x) \\
&= \sum_{k=0}^{m} \alpha_{m,k}(x)P_{m+1,k}(x)\tilde{B}_k^{n,m}(x) + \sum_{k=1}^{m+1} \beta_{m,k-1}(x)P_{m+1,k}(x)\tilde{B}_{k-1}^{n,m}(x) \\
&= P_{m+1,0}(x)\alpha_{m,0}(x)\tilde{B}_0^{n,m}(x) + \sum_{k=1}^{m} P_{m+1,k}(x)\left(\alpha_{m,k}(x)\tilde{B}_k^{n,m}(x) + \beta_{m,k-1}(x)\tilde{B}_{k-1}^{n,m}(x)\right) \\
&\quad + P_{m+1,m+1}(x)\beta_{m,m}(x)\tilde{B}_m^{n,m}(x).
\end{aligned}$$

Since $\alpha_{m,0}(x)\tilde{B}_0^{n,m}(x) = \tilde{B}_0^{n,m+1}(x)$, $\alpha_{m,k}(x)\tilde{B}_k^{n,m}(x) + \beta_{m,k-1}(x)\tilde{B}_{k-1}^{n,m}(x) = \tilde{B}_k^{n,m+1}(x)$ and $\beta_{m,m}(x)\tilde{B}_m^{n,m}(x) = \tilde{B}_{m+1}^{n,m+1}(x)$, we have

$$p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1 x, q_2), (x, 1)) = \sum_{k=0}^{m+1} P_{m+1,k}(x)\tilde{B}_k^{n,m+1}(x)$$

which completes the proof. □

One of important results of Theorem 3.1 is that it leads us to definition of two parameter family of Bernstein basis functions which we call (q_1, q_2)-Bernstein basis functions over arbitrary interval $[a, b]$. We define (q_1, q_2)-Bernstein basis functions over arbitrary interval $[a, b]$ as follows:

$$B_k^n(x; [a, b]; q_1, q_2) := \tilde{B}_k^{n,n}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_{q_1,q_2} \frac{\prod_{i=0}^{k-1}(q_1^i a - q_2^i x) \cdot \prod_{i=0}^{n-k-1}(q_1^i x - q_2^i b)}{\prod_{i=0}^{n-1}(q_1^i a - q_2^i b)}, \tag{3.4}$$

$k = 0, 1, \ldots, n$. Clearly, Eq. (3.4) is well defined when $\prod_{i=0}^{n-1}(q_1^i a - q_2^i b) \neq 0$. Some special cases of (q_1, q_2)-Bernstein basis functions are

- If $q_1 = q$, $q_2 = p$, $a = 0$ and $b = 1$, Eq. (3.4) is (p, q)-Bernstein basis functions given in [9, 13].
- If $q_1 = q$, $q_2 = 1$, $a = 0$ and $b = 1$, Eq. (3.4) gives q-Bernstein basis functions over interval $[0, 1]$, see [15, 16].
- If $q_1 = q$, and $q_2 = 1$ Eq. (3.4) gives q-Bernstein basis functions over arbitrary interval, see [18].
- If $q_1 = q_2 = 1$, Eq. (3.4) reduces to classical Bernstein basis functions defined on $[a, b]$, see [3].

By using pascal type identities (2.1) and (2.2), it can be easily shown that (q_1, q_2)-Bernstein basis functions satisfies the following recurrence relations:

$$B_k^n(x;[a,b];q_1,q_2) = q_1^{n-k}\left(\frac{q_1^{k-1}a - q_2^{k-1}x}{q_1^{n-1}a - q_2^{n-1}b}\right)B_{k-1}^{n-1}(x;[a,b];q_1,q_2) + q_2^{k}\left(\frac{q_1^{n-k-1}x - q_2^{n-k-1}b}{q_1^{n-1}a - q_2^{n-1}b}\right)B_k^{n-1}(x;[a,b];q_1,q_2) \quad (3.5)$$

and

$$B_k^n(x;[a,b];q_1,q_2) = q_2^{n-k}\left(\frac{q_1^{k-1}a - q_2^{k-1}x}{q_1^{n-1}a - q_2^{n-1}b}\right)B_{k-1}^{n-1}(x;[a,b];q_1,q_2) + q_1^{k}\left(\frac{q_1^{n-k-1}x - q_2^{n-k-1}b}{q_1^{n-1}a - q_2^{n-1}b}\right)B_k^{n-1}(x;[a,b];q_1,q_2). \quad (3.6)$$

As in their q-analogue, (q_1, q_2)-Bernstein basis functions are scale invariant. That is

$$B_k^n(wx;[wa,wb];q_1,q_2) = B_k^n(x;[a,b];q_1,q_2) \text{ for all } w \neq 0. \quad (3.7)$$

Note that we have defined (q_1, q_2)-Bernstein basis functions via homogeneous $\bar{q}$-blossoming. Hence the restriction on q_1 and q_2 values for homogeneous $\bar{q}$-blossoming also valid for (q_1, q_2)-Bernstein basis functions. In addition the restriction will be valid for two parameter family of Bézier curves which we are going to define in the subsequent text.

We define two parameter family of Bézier curves, called (q_1, q_2)-Bézier curves, as follows:
A (q_1, q_2)-Bézier curve of degree n over interval $[a, b]$ defined by

$$P(x) = \sum_{k=0}^{n} \mathbf{b}_k B_k^n(x;[a,b];q_1,q_2). \quad (3.8)$$

The coefficients $\mathbf{b}_k \in \mathbb{R}^d,\ d = 2, 3,\ k = 0, 1, \ldots n$, are called control points of (q_1, q_2)-Bézier curve $P(x)$.

Note that since $P(x) = P(x, 1)$, we can obtain $P(x)$ from its homogeneous $\bar{q}$-blossom by setting $(x_k, y_k) = (q_1^{n-k}x, q_2^{n-k})$, $k = 1, \ldots, n$. That is

$$P(x) = p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1x, q_2), (x, 1)). \quad (3.9)$$

Hence we have the following important result of Theorem 3.1:

Corollary 3.2. *Every polynomial $P(x)$ of degree n is a (q_1, q_2)-Bézier curve of the form*

$$P(x) = \sum_{k=0}^{n} p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{k}a, q_2^{k}), (q_1^{k-1}b, q_2^{k-1}), \ldots, (q_1b, q_2), (b, 1))B_k^n(x;[a,b];q_1,q_2), \quad (3.10)$$

where $p((x_1, y_1), \ldots, (x_n, y_n))$ is the homogeneous $\bar{q}$-blossom of $P(x, y)$, the homogenization of $P(x)$.

Proof. Follows easily from Theorem 3.1 and Corollary 2.6. □

From now on, to avoid the confusion, unless we explicitly indicate otherwise, we will use capital bold letters for polynomials and capital letters for homogeneous polynomials. But, even they are polynomials of degree n, we will not change the notation of (q_1, q_2)-Bernstein bases.

Applying Corollary 2.6 we obtain $n!$ de Casteljau type evaluation algorithm for (q_1, q_2)-Bézier curves. These algorithms are given in the following theorem.

Theorem 3.3. *Let* $\mathbf{P}$ *be a (q_1, q_2)-Bézier curve and let p be the homogeneous $\bar{q}$-blossom of its homogenization P. There are $n!$ de Casteljau type recursive evaluation algorithm of* $\mathbf{P}$. *These algorithms are as follows: For any permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of the set $\{1, 2, \ldots, n\}$, set*

$$\mathbf{b}_k^0 = p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{k}a, q_2^{k}), (q_1^{k-1}b, q_2^{k-1}), \ldots, (q_1b, q_2), (b, 1)),$$

$k = 0, 1, \ldots, n$ and define recursively

$$\mathbf{b}_k^{r+1}(x) = \alpha_{r,k}(x)\mathbf{b}_k^r(x) + \beta_{r,k}(x)\mathbf{b}_{k+1}^r(x) \quad (3.11)$$

for $r = 0, 1, \ldots, n-1$, $k = 0, 1, \ldots, n-r-1$ *where*

$$\alpha_{r,k}(x) = \frac{q_1^{\sigma_{r+1}-1} q_2^k\, x - q_1^k\, q_2^{\sigma_{r+1}-1}\, b}{q_1^{r+k} q_2^k\, a - q_1^k\, q_2^{r+k}\, b} \quad \text{and} \quad \beta_{r,k} = \frac{q_1^{r+k}\, q_2^{\sigma_{r+1}-1}\, a - q_1^{\sigma_{r+1}-1} q_2^{r+k} x}{q_1^{r+k} q_2^k\, a - q_1^k\, q_2^{r+k}\, b}.$$

Then

$$\mathbf{b}_k^r(x) = p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{r+k}a, q_2^{r+k}), (q_1^{\sigma_1-1}x, q_2^{\sigma_1-1}), \ldots, (q_1^{\sigma_r-1}x, q_2^{\sigma_r-1}), (q_1^{k-1}b, q_2^{k-1}), \ldots, (b, 1)).$$

In particular $\mathbf{b}_0^n(x) = \mathbf{P}(x)$.

Proof. Follows from Corollary 2.6 with $c = 1$. □

Remark 3.4. For any non-zero value c, result of recursive evaluation algorithms in Corollary 2.6 gives the polynomial $P(x, c)$ in (q_1, q_2)-Bézier form. That is

$$P(x, c) = \sum_{k=0}^{n} p((q_1^{n-1}a, q_2^{n-1}c), \ldots, (q_1^k a, q_2^k c), (q_1^{k-1}b, q_2^{k-1}c), \ldots, (q_1 b, q_2 c), (b, c)) B_k^n(x; [a, b]; q_1, q_2)\,. \tag{3.12}$$

Corollary 3.5. *The* (q_1, q_2)*-Bernstein basis functions of degree n on the interval* $[a, b]$ *form a basis for the polynomials of degree n.*

Corollary 3.6. *The control points of* (q_1, q_2)*-Bézier curve on the interval* $[a, b]$ *are unique.*

Corollary 3.7 (Dual functional property of homogeneous $\bar{q}$-blossom)**.** *Let* **P** *be a* (q_1, q_2)*-Bézier curve of degree n on the interval* $[a, b]$ *and let p be the homogeneous* $\bar{q}$*-blossom of its homogenization P. Then the control points of* (q_1, q_2)*-Bézier curve are given by*

$$\mathbf{b}_k = p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^k a, q_2^k), (q_1^{k-1}b, q_2^{k-1}), \ldots, (q_1 b, q_2), (b, 1)), \quad k = 0, 1, \ldots, n. \tag{3.13}$$

For any permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $\{1, 2, \ldots, n\}$ a new de Casteljau type evaluation algorithm for q-Bézier curve is given in [18] by

$$b_k^{r+1}(x) = (1 - \tilde{\beta}_{r,k}(x)) b_k^r(x) + \tilde{\beta}_{r,k}(x) b_{k+1}^r(x),$$

where $\tilde{\beta}_{r,k}(x) = \frac{q^r a - q^{\sigma_{r+1}-1-k} x}{q^r a - b}$. Setting $q = \frac{q_1}{q_2}$ in $\tilde{\beta}_{r,k}(x)$ and rearranging gives

$$\tilde{\beta}_{r,k}(x) = q_2^{-(\sigma_{r+1}-1-k)} \beta_{r,k}(x)$$

which suggests that for each $q_2 \neq 1$, de Casteljau type evaluation algorithms given in Theorem 3.3 are different from the algorithms given in [18]. Note also that, the relation between $\tilde{\beta}_{r,k}(x)$ and $\beta_{r,k}(x)$ and the relation between homogeneous q-blossom and homogeneous $\bar{q}$-blossom are different. This difference shows that our results can not be obtained from the results given in [18] by a change of variable.

Now we give a (q_1, q_2) analogue of a well known identity.

Theorem 3.8 (Marsden type identity)**.**

$$\frac{\prod_{i=0}^{n-1} \left(q_1^i t - q_2^i x\right)}{\prod_{i=0}^{n-1} \left(q_1^i a - q_2^i b\right)} = \sum_{k=0}^{n} (-1)^k (q_1 q_2)^{\frac{k(k-1)}{2}} \frac{B_{n-k}^n\left(x; \left[\left(\frac{q_1}{q_2}\right)^{n-1} a, b\right]; q_1^{-1}, q_2^{-1}\right) B_k^n(t; [a, b]; q_1, q_2)}{q_2^{k(n-1)} \begin{bmatrix} n \\ n-k \end{bmatrix}_{q_1, q_2}}. \tag{3.14}$$

Proof. Let $P(x,y) = \dfrac{\prod_{i=0}^{n-1}\left(q_1^i x - q_2^i y\right)}{\prod_{i=0}^{n-1}\left(q_1^i a - q_2^i b\right)}$. Clearly P is a polynomial of homogeneous degree n and the homogeneous $\bar{q}$-blossom of P is

$$p((x_1,y_1),\ldots,(x_n,y_n)) = \frac{\prod_{i=1}^{n}(x_i - y_i)}{\prod_{i=0}^{n-1}\left(q_1^i a - q_2^i b\right)}.$$

Then by Remark 3.4, we have

$$\begin{aligned} P(x,c) &= \sum_{k=0}^{n} p((q_1^{n-1}a, q_2^{n-1}c),\ldots,(q_1^k a, q_2^k c),(q_1^{k-1}b, q_2^{k-1}c),\ldots,(q_1 b, q_2 c),(b,c)) B_k^n(x;[a,b];q_1,q_2) \\ &= \sum_{k=0}^{n} \frac{\prod_{i=1}^{n-k}\left(q_1^{n-i}a - q_2^{n-i}c\right)\cdot \prod_{i=n-k+1}^{n}\left(q_1^{n-i}b - q_2^{n-i}c\right)}{\prod_{i=0}^{n-1}\left(q_1^i a - q_2^i b\right)} B_k^n(x;[a,b];q_1,q_2) \\ &= \sum_{k=0}^{n}(-1)^k (q_1q_2)^{\frac{k(k-1)}{2}} \frac{B_{n-k}^n\left(c;\left[\left(\frac{q_1}{q_2}\right)^{n-1}a, b\right]; q_1^{-1}, q_2^{-1}\right) B_k^n(x;[a,b];q_1,q_2)}{q_2^{k(n-1)}\begin{bmatrix} n \\ n-k\end{bmatrix}_{q_1,q_2}}. \end{aligned} \tag{3.15}$$

Because

$$\prod_{i=1}^{n-k}\left(q_1^{n-i}a - q_2^{n-i}c\right) = \prod_{i=0}^{n-k-1}\left(q_1^{n-i-1}a - q_2^{n-i-1}c\right) = (q_2^{n-1})^{n-k}\prod_{i=0}^{n-k-1}\left(q_1^{-i}\left(\frac{q_1^{n-1}}{q_2^{n-1}}a\right) - q_2^{-i}c\right),$$

$$\prod_{i=n-k+1}^{n}\left(q_1^{n-i}b - q_2^{n-i}c\right) = \prod_{i=0}^{k-1}\left(q_1^i b - q_2^i c\right) = (-1)^k (q_1q_2)^{\frac{k(k-1)}{2}}\prod_{i=0}^{k-1}\left(q_1^{-i}c - q_2^{-i}b\right)$$

and

$$\prod_{i=0}^{n-1}\left(q_1^i a - q_2^i b\right) = \prod_{i=0}^{n-1}\left(q_1^{n-i-1}a - q_2^{n-i-1}b\right) = \left(q_2^{n-1}\right)^n \prod_{i=0}^{n-1}\left(q_1^{-i}\left(\frac{q_1^{n-1}}{q_2^{n-1}}a\right) - q_2^{-i}b\right).$$

Thus, since left hand side of Eq. (3.14) is $P(t,x)$, Theorem 3.8 follows by substituting $x = t$ and $c = x$ in Eq. (3.15). □

We end this section with a new result on classical Bézier curves. Since $[n]_{q,q} = q^{n-1}n$, it can easily shown that (q_1,q_2)-Bernstein bases and (q_1,q_2)-Bézier curves reduced to classical Bernstein bases and Bézier curves for $q_1 = q_2 = q$. In that case infinitely many de Casteljau type algorithms for classical Bézier curves derived form Theorem 3.3. These algorithms are given in the following theorem.

Theorem 3.9. *For any permutation $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ of $\{1,2,\ldots,n\}$, and any value of $q \neq 0$, set*

$$\mathbf{b}_k^0 := \mathbf{b}_k = p((q^{n-1}a, q^{n-1}),\ldots,(q^k a, q^k),(q^{k-1}b, q^{k-1}),\ldots,(qb,q),(b,1)),$$

$k = 0,1,\ldots,n$ and define recursively

$$\mathbf{b}_k^{r+1}(x) = \frac{q^{\sigma_{r+1}-1}(b-x)}{q^{r+k}(b-a)}\mathbf{b}_k^r(x) + \frac{q^{\sigma_{r+1}-1}(x-a)}{q^k(b-a)}\mathbf{b}_{k+1}^r(x)$$

for $r = 0, 1, \dots, n-1$, $k = 0, 1, \dots, n-r-1$. *Then*

$$\mathbf{b}_0^n(x) = \sum_{k=0}^{n} \mathbf{b}_k \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n}.$$

Proof. Follows easily from Theorem 3.3 by setting $q_1 = q_2 = q$. □

4. Properties and identities for (q_1, q_2)-Bernstein bases and (q_1, q_2)-Bézier curves

Here we are going to derive properties and identities for (q_1, q_2)-Bernstein basis functions and (q_1, q_2)-Bézier curves.

Proposition 4.1.

$$\sum_{k=0}^{n} B_k^n(x; [a,b]; q_1, q_2) = 1. \tag{4.1}$$

Proof. Homogenization of $\mathbf{P}(x) = 1$ is $P(x,y) = y^n$. The homogeneous $\bar{q}$-blossom of $P(x,y) = y^n$ is

$$p((x_1, y_1), \dots, (x_n, y_n)) = \prod_{i=1}^{n} \frac{y_i}{q_2^{n-i}}.$$

Hence Propostion 4.1 follows from dual functional property of (q_1, q_2)-Bézier curves, since

$$p((q_1^{n-1} a, q_2^{n-1}), \dots, (q_1^k a, q_2^k), (q_1^{k-1} b, q_2^{k-1}), \dots, (q_1 b, q_2), (b, 1)) = 1, \; k = 0, 1, \dots, n.$$

□

The following proposition is a consequence of Proposition 4.1.

Proposition 4.2. (q_1, q_2)*-Bézier curves are invariant under affine transformations.*

Proof. Let $\mathbf{P}$ be a (q_1, q_2)-Bézier curve with control points $\mathbf{b}_k$, $k = 0, 1, \dots n$, A be linear transformation and v be a vector. Then by partition of unity property of (q_1, q_2)-Bernstein bases

$$\sum_{k=0}^{n} (A\,\mathbf{b}_k + v)\, B_k^n(x; [a,b]; q_1, q_2) = A \sum_{k=0}^{n} \mathbf{b}_k B_k^n(x; [a,b]; q_1, q_2) + v \sum_{k=0}^{n} B_k^n(x; [a,b]; q_1, q_2) = A\mathbf{P}(x) + v.$$

□

Proposition 4.3.

$$x = \sum_{k=0}^{n} \left(a - \frac{q_2^{n-k} [k]_{q_1,q_2}}{[n]_{q_1,q_2}} (b-a) \right) B_k^n\left(x; [a,b]; q_1, q_2\right). \tag{4.2}$$

Proof. Homogenization of $\mathbf{P}(x) = x$ is $P(x,y) = xy^{n-1}$. The homogeneous $\bar{q}$-blossom of $P(x,y) = xy^{n-1}$ is

$$p((x_1, y_1), \dots, (x_n, y_n)) = \frac{\displaystyle\sum_{i=1}^{n} x_i \prod_{\substack{j=1 \\ j \neq i}}^{n} y_j}{\displaystyle\sum_{i=1}^{n} q_1^{n-i} \prod_{\substack{j=1 \\ j \neq i}}^{n} q_2^{n-j}}.$$

Thus by dual functional property, we have

$$p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^k a, q_2^k), (q_1^{k-1}b, q_2^{k-1}), \ldots, (q_1 b, q_2), (b,1)) = \frac{\sum_{i=1}^{n-k} q_1^{n-i} a \prod_{\substack{j=1 \\ j\neq i}}^{n} q_2^{n-j} + \sum_{i=n-k+1}^{n} q_1^{n-i} b \prod_{\substack{j=1 \\ j\neq i}}^{n} q_2^{n-j}}{\sum_{i=1}^{n} q_1^{n-i} \prod_{\substack{j=1 \\ j\neq i}}^{n} q_2^{n-j}}$$

$$= \frac{\sum_{i=1}^{n-k} q_1^{n-i} q_2^{i-n} a + \sum_{i=n-k+1}^{n} q_1^{n-i} q_2^{i-n} b}{\sum_{i=1}^{n} q_1^{n-i} q_2^{i-n}}$$

$$= \frac{q_2^{1-n} \sum_{i=1}^{n-k} q_1^{n-i} q_2^{i-1} a + q_2^{1-n} \sum_{i=n-k+1}^{n} q_1^{n-i} q_2^{i-1} b}{q_2^{1-n} \sum_{i=1}^{n} q_1^{n-i} q_2^{i-1}}$$

$$= \frac{\sum_{i=1}^{n-k} q_1^{n-i} q_2^{i-1} a + \sum_{i=n-k+1}^{n} q_1^{n-i} q_2^{i-1} b}{\sum_{i=1}^{n} q_1^{n-i} q_2^{i-1}}.$$

Thus Eq. (4.2) follows since

$$\sum_{i=1}^{n-k} q_1^{n-i} q_2^{i-1} a = \left([n]_{q_1,q_2} - q_2^{n-k}[k]_{q_1,q_2}\right) a, \quad \sum_{i=n-k+1}^{n} q_1^{n-i} q_2^{i-1} b = q_2^{n-k}[k]_{q_1,q_2} b$$

and

$$\sum_{i=1}^{n} q_1^{n-i} q_2^{i-1} = [n]_{q_1,q_2}.$$

□

Proposition 4.4. *Let* $\mathbf{P}(x) = \sum_{k=0}^{n} \mathbf{b}_k B_k^n(x; [a,b]; q_1, q_2)$ *be a* (q_1, q_2)*-Bézier curve. Then*

$$\mathbf{b}_0 = \mathbf{P}(a) \text{ and } \mathbf{b}_n = \mathbf{P}(b). \tag{4.3}$$

Proof. Eq. (4.3) follows from dual functional property of (q_1, q_2)-Bézier curves and (q_1, q_2)-diagonal property of homogeneous $\bar{q}$-blossom. □

5. Subdivision of (q_1, q_2)-Bézier curves

We are going to give subdivision algorithms for (q_1, q_2)-Bézier curves.

Theorem 5.1. *Let* $\mathbf{b}_k,\ k = 0, 1, \ldots, n$ *be the control points for a* (q_1, q_2)*-Bézier curve* $\mathbf{P}$ *of degree n. Control points for the curve* $\mathbf{P}$ *over the subintervals* $[a, t]$ *and* $[t, b]$ *are given by*

$$\mathbf{L}_i = p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^i a, q_2^i), (q_1^{i-1}t, q_2^{i-1}), \ldots, (t, 1)) \tag{5.1}$$

and

$$\mathbf{R}_i = p((q_1^{n-1}t, q_2^{n-1}), \ldots, (q_1^i t, q_2^i), (q_1^{i-1}b, q_2^{i-1}), \ldots, (b, 1)). \tag{5.2}$$

Moreover, the explicit representation of $\mathbf{L}_i$ *and* $\mathbf{R}_i$ *are*

$$\mathbf{L}_i = \sum_{j=0}^{i} \mathbf{b}_j B_j^i(t; [a, b]; q_1, q_2) \ \text{ and } \ \mathbf{R}_i = \sum_{j=0}^{n-i} \mathbf{b}_{i+j} B_j^{n-i}(t; [a, b]; q_1, q_2).$$

Proof. Eq. (5.1) and Eq. (5.2) follows from dual functional property of (q_1, q_2)-Bézier curves. From Theorem 2.5 the homogeneous $\bar{q}$-blossom of $b_k^r(x, y)$ is

$$b_k^r((x_1, y_1), \ldots, (x_r, y_r)) = p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{r+k}a, q_2^{r+k}), (x_1, y_1), \ldots, (x_r, y_r), (q_1^{k-1}b, q_2^{k-1}), \ldots, (b, 1)). \tag{5.3}$$

Applying (q_1, q_2)-diagonal property of homogeneous $\bar{q}$-blossom in Eq. (5.3) gives

$$\begin{aligned} b_k^r(x, c) =& b_k^r((q_1^{r-1}x, q_2^{r-1}c), \ldots, (x, c)) \\ =& p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{r+k}a, q_2^{r+k}), (q_1^{r-1}x, q_2^{r-1}c), \ldots, (x, c), (q_1^{k-1}b, q_2^{k-1}), \ldots, (b, 1)) \end{aligned} \tag{5.4}$$

Hence

$$\begin{aligned} b_0^i(t, 1) =& p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^i a, q_2^i), (q_1^{i-1}t, q_2^{i-1}), \ldots, (t, 1)) \\ =& \mathbf{L}_i \end{aligned}$$

and

$$\begin{aligned} b_i^{n-i}(q_1^i t, q_2^i) =& p((q_1^{n-1}t, q_2^{n-1}), \ldots, (q_1^i t, q_2^i), (q_1^{i-1}b, q_2^{i-1}), \ldots, (b, 1)) \\ =& \mathbf{R}_i. \end{aligned}$$

On the other hand, since $\mathbf{b}_k^r$ is a polynomial of degree r, its homogenization b_k^r is a polynomial of homogeneous degree r. Hence by Remark 3.4, $b_k^r(x, c)$ can be written in (q_1, q_2)-Bézier form over arbitrary interval, say $[\tilde{a}, \tilde{b}]$. That is

$$b_k^r(x, c) = \sum_{j=0}^{r} b_k^r((q_1^{r-1}\tilde{a}, q_2^{r-1}c), \ldots, (q_1^j \tilde{a}, q_2^j c), (q_1^{j-1}\tilde{b}, q_2^{j-1}c), \ldots, (q_1\tilde{b}, q_2 c), (\tilde{b}, c)) B_j^r(x; [\tilde{a}, \tilde{b}]; q_1, q_2). \tag{5.5}$$

Now setting $[\tilde{a}, \tilde{b}] = [a, b]$ in Eq. (5.5) and using Eq. (5.3) gives

$$\begin{aligned} b_0^i(t, 1) &= \sum_{j=0}^{i} b_0^i((q_1^{i-1}a, q_2^{i-1}), \ldots, (q_1^j a, q_2^j), (q_1^{j-1}b, q_2^{j-1}), \ldots, (q_1 b, q_2), (b, 1)) B_j^i(t; [a, b]; q_1, q_2) \\ &= \sum_{j=0}^{i} p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^j a, q_2^j), (q_1^{j-1}b, q_2^{j-1}), \ldots, (q_1 b, q_2), (b, 1)) B_j^i(t; [a, b]; q_1, q_2) \\ &= \sum_{j=0}^{i} b_j B_j^i(t; [a, b]; q_1, q_2). \end{aligned}$$

Hence

$$\mathbf{L}_i = \sum_{j=0}^{i} b_j B_j^i(t; [a, b]; q_1, q_2).$$

Similarly, setting $[\tilde{a}, \tilde{b}] = [q_1^i a, q_1^i b]$ in Eq. (5.5) and using Eq. (5.3) and scale invariance property of (q_1, q_2)-Bernstein basis functions gives

$$\begin{aligned} b_i^{n-i}(q_1^i t, q_2^i) &= \sum_{j=0}^{n-i} b_i^{n-i}((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{i+j}a, q_2^{i+j}), (q_1^{i+j-1}b, q_2^{i+j-1}), \ldots, (q_1^i b, q_2^i))B_j^{n-i}(q_1^i t; [q_1^i a, q_1^i b]; q_1, q_2) \\ &= \sum_{j=0}^{n-i} p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{i+j}a, q_2^{i+j}), (q_1^{i+j-1}b, q_2^{i+j-1}), \ldots, (q_1 b, q_2), (b, 1))B_j^{n-i}(t; [a, b]; q_1, q_2) \\ &= \sum_{j=0}^{n-i} b_{i+j} B_j^{n-i}(t; [a, b]; q_1, q_2). \end{aligned}$$

Thus

$$\mathbf{R}_i = \sum_{j=0}^{n-i} b_{i+j} B_j^{n-i}(t; [a, b]; q_1, q_2).$$

□

As in q-blossoms, the points $\mathbf{L}_i$ in the proof of Theorem 5.1 can be obtained from intermediate points of de Casteljau type algorithm in Theorem 3.3 with identity permutation. Similarly, the points $\mathbf{R}_i$ can be obtained from intermediate points of de Casteljau type algorithm in Theorem 3.3 with the permutation $\sigma_r = n + 1 - r$ (see [18]).

5.1. Recursive subdivision

We can now construct a recursive subdivision algorithm for a (q_1, q_2)-Bézier curve $\mathbf{P}$ defined over the interval $[a, b]$ as follows. First take $t = \frac{a+b}{2}$, the midpoint of a and b, then subdivide $\mathbf{P}$ into two curve segments. Iteratively subdivide each curve segment at the midpoint of its corresponding subinterval. Then after each iteration the number of curve segments doubles and at the mth iteration we will have 2^m curve segments corresponding to the subintervals $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, 2^m - 1$, where $t_i = a + \frac{i}{2^m}(b - a)$. The control polygons for all these curve segments are a piecewise linear approximation to the original (q_1, q_2)-Bézier curve $\mathbf{P}$.

Theorem 5.2. *The control polygons generated by midpoint recursive subdivision converge uniformly to the original (q_1, q_2)-Bézier curve.*

Proof. Consider a (q_1, q_2)-Bézier curve $\mathbf{P}$ defined over the interval $[a, b]$ and let p be the homogeneous $\bar{q}$-blossom of its homogenization P. If we subdivide the curve $\mathbf{P}$ at a point $t \in [a, b]$, then by Theorem 5.1 the control points of the left segment are

$$\mathbf{L}_i = p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^i a, q_2^i), (q_1^{i-1}t, q_2^{i-1}), \ldots, (t, 1)),\ i = 0, 1, \ldots, n.$$

Therefore by the multi-linear property of the homogeneous $\bar{q}$-blossom we have

$$\begin{aligned} \mathbf{L}_{i+1} - \mathbf{L}_i =& p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{i+1}a, q_2^{i+1}), (q_1^i(t - a), 0), (q_1^{i-1}t, q_2^{i-1}), \ldots, (t, 1)) \\ =& (t - a)p((q_1^{n-1}a, q_2^{n-1}), \ldots, (q_1^{i+1}a, q_2^{i+1}), (q_1^i, 0), (q_1^{i-1}t, q_2^{i-1}), \ldots, (t, 1)). \end{aligned}$$

Set

$$M = \max_{0 \leqslant i \leqslant n-1} |p((q_1^{n-1}x, q_2^{n-1}), \ldots, (q_1^{i+1}x, q_2^{i+1}), (q_1^i, 0), (q_1^{i-1}y, q_2^{i-1}), \ldots, (y, 1))|,$$

$x, y \in [a, b]$. Then for $i = 0, 1, \ldots, n - 1$, $|\mathbf{L}_{i+1} - \mathbf{L}_i| \leqslant M|t - a| < M|b - a|$. So $\sum_{i=0}^{n-1} |\mathbf{L}_{i+1} - \mathbf{L}_i| < nM|b - a|$. Similar arguments show that $\sum_{i=0}^{n-1} |\mathbf{R}_{i+1} - \mathbf{R}_i| < nM|b - a|$, where $\mathbf{R}_i,\ i = 0, 1, \ldots, n$ are control points of the right segment.

Now let $\tilde{\mathbf{P}}$ be a segment of the original (q_1, q_2)-Bézier curve $\mathbf{P}$ constructed after m iterations of midpoint subdivision and let $L(x)$ denote the corresponding control polygon. Then $\tilde{\mathbf{P}}$ is the restriction of $\mathbf{P}$ over a subinterval $[\tilde{a}, \tilde{b}] \subset [a, b]$ of length $\frac{b-a}{2^m}$ and $\tilde{\mathbf{P}}$ and $L(x)$ coincide at the endpoints $\tilde{a}$ and $\tilde{b}$. Hence for any $x \in [\tilde{a}, \tilde{b}]$,

$$\begin{aligned}
|\tilde{\mathbf{P}}(x) - L(x)| = |\mathbf{P}(x) - L(x)| &\leqslant |\mathbf{P}(x) - \tilde{\mathbf{P}}(\tilde{a})| + |L(\tilde{a}) - L(x)| \\
&\leqslant |t - a| \max_{\tau \in [\tilde{a}, \tilde{b}]} |\mathbf{P}'(x)| + nM|\tilde{b} - \tilde{a}| \\
&\leqslant \frac{\tilde{M}(b-a)}{2^m},
\end{aligned}$$

where $\tilde{M} = \max_{\tau \in [\tilde{a}, \tilde{b}]} |\mathbf{P}'(x)| + nM$. Therefore the control polygons generated by recursive midpoint subdivision converge uniformly to the original (q_1, q_2)-Bézier curve. □

In the following figures, levels 1, 3, and 6 recursive subdivision apllied to same control polygon with two different set of q_1, q_2 values.

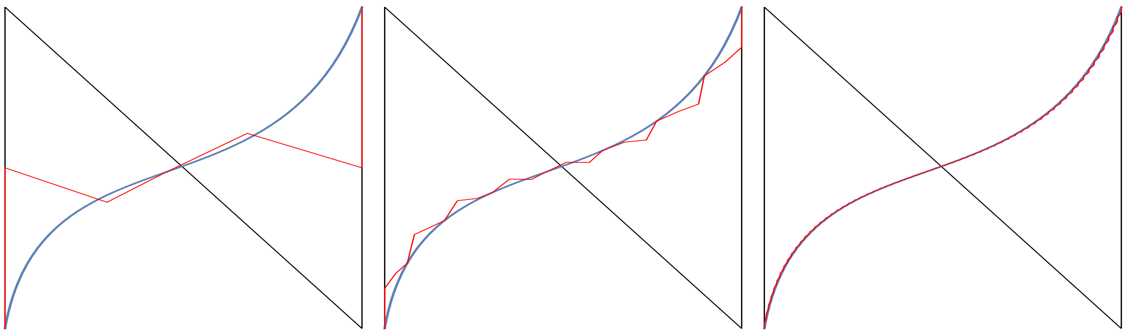

Figure 1. Qubic (q_1, q_2)-Bézier curve with $q_1 = 0.5$, $q_2 = 0.6$. From left to right levels 1, 3, and 6 of midpoint subdivision are displayed.

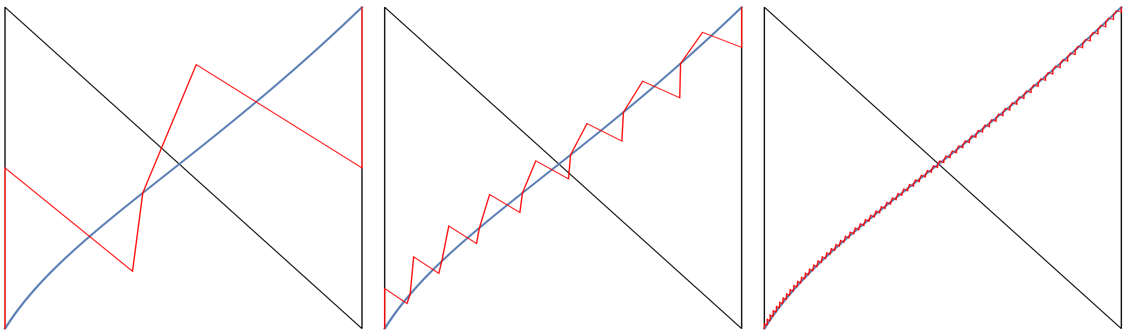

Figure 2. Cubic (q_1, q_2)-Bézier curve with $q_1 = -0.5$ and $q_2 = -2$. From left to right levels 1, 3, and 6 of midpoint subdivision are displayed.

6. Conclusion

We constructed homogeneous $\bar{q}$-blossom as an extension of existing q-blossom for polynomial spaces. We defined two parameter family of Bernstein bases and Bézier curves over arbitrary interval. Properties and identities for (q_1, q_2)-Bernstein bases and (q_1, q_2)-Bézier curves obtained by applying homogeneous $\bar{q}$-blossom. We are now extending the theory of homogeneous quantum blossom for non-polynomial spaces.

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