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Diagonal M-contractive maps on ordered metric spaces

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Abstract

A (Matkowski type) functional extension – to the realm of ordered metric spaces – is given for the diagonal fixed point result in Ćirić and Prešić (Acta Math. Univ. Comenian. 76, 143–147, 2007) involving Prešić iterative processes.

Keywords: Ordered metric space, Diagonal fixed point, Prešić iterative process, Admissible function, Matkowski functional contraction

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1. Introduction

Let (X, d) be a metric space. Further, let $S \in \mathcal{F}(X)$ be a self-map of X. (Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ denotes the class of all functions from A to B; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$). Denote $\mathrm{Fix}(S) = \{x \in X; x = Sx\}$; each point of this set is referred to as *fixed* under S. Concerning the determination of such points, a basic result in this area is the 1922 one due to Banach [2]. Call the subset Y of X, *almost singleton* (in short: *asingleton*), provided $[y_1, y_2 \in Y$ implies $y_1 = y_2]$; and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Then, let us say that S is *Banach* $(d; \alpha)$*-contractive* (where $\alpha \geq 0$), if

$$d(Sx, Sy) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

Theorem 1.1. *Assume that S is Banach $(d; \alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let X be d-complete. Then,*

(11-a) *S is fix-singleton:* $\mathrm{Fix}(S) = \{z\}$*, for some $z \in X$*

(11-b) *S is a strong Picard operator:* $\lim_n S^n x = z$*, $\forall x \in X$.*

This result (referred to as: *Banach fixed point theorem*; in short: (B-fpt)) found some basic applications to the operator equations theory. As a consequence, many extensions for it were proposed. From the perspective of our present exposition, the *diagonal* ones are of interest; precisely, given $k \geq 1$, these consist in the initial self-map S being viewed as the *diagonal part* of a mapping $T : X^k \to X$; i,e.,

$$Sx = T(x^k),\ x \in X,$$

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where, for each $x \in X$, we denoted

x^k = the element $(z_0, ..., z_{k-1}) \in X^k$, with $(z_i = x; 0 \le i \le k-1)$.

In this context, denote

$\text{Fixd}(T) = \text{Fix}(S)\,(= \{z \in X; z = Sz\})$;

each element of this set will be called a *diagonal fixed point* of T.

The existence and uniqueness of such points is to be discussed in the context below. Denote for simplicity

X^∞=the class of all sequences $(x_n; n \ge 0)$ over X.

Then, given $k \ge 1$, denote

X^k=the class of all k-tuples $(y_0, ..., y_{k-1})$ over X.

For any sequence (u_n) and any couple $i, j \in N$ with $i < j$, let us put

$u[i;j] = (u_i, ..., u_j)$ (the (i, j)-*segment* of (u_n));
clearly, $u[i;j]$ is an element of X^{j-i+1}.

Let us say that $(u_n; n \ge 0)$ is a *k-iterative sequence*, provided

(iter) u_n=the above one, $0 \le n \le k-1$; $u_n = T(u_{n-k}, ..., u_{n-1}), n \ge k$;

for simplicity, we will denote it as $(u_n = T^n U_0; n \ge 0)$, where $U_0 := u[0; k-1] = (u_0, ..., u_{k-1})$ is the $(0, k-1)$-segment (in X^k) of our sequence. The class of all these objects will be denoted as $X^\infty(k - iter)$.

The determination (via such iterative sequences) of the introduced diagonal points is to be performed over the directions below, comparable with the ones in Turinici [27, Paper 1-4], and having as (non-diagonal) origin the related developments in Rus [20, Chapter 2, Section 2.2]:

pp-0) Let us say that T is *fixd-asingleton*, if $\text{Fixd}(T) = \text{Fix}(S)$ is an asingleton; and *fixd-singleton*, provided $\text{Fixd}(T) = \text{Fix}(S)$ is a singleton.

pp-1) Let us say that the k-iterative sequence (u_n) has the *Prešić property* (modulo (d, T)) when (u_n) is d-Cauchy. If $(u_n) \in X^\infty(k - iter)$ is generic here, we then say that T is a *Prešić operator* (modulo d).

pp-2) Let us say that the k-iterative sequence (u_n) has the *strong Prešić property* (modulo (d, T)) when (u_n) is d-convergent and $z := \lim_n(u_n)$ is an element of $\text{Fixd}(T) = \text{Fix}(S)$. If $(u_n) \in X^\infty(k - iter)$ is generic here, we then say that T is a *strong Prešić operator* (modulo d).

Sufficient conditions for such properties involve metrical contractions. Given $\Gamma := (\gamma_0, ..., \gamma_{k-1}) \in R_+^k$, let us say that T is *Prešić* $(d; \Gamma)$*-contractive*, provided

(P-contr) $d(T(x_0, ...x_{k-1}), T(x_1, ..., x_k)) \le \gamma_0 d(x_0, x_1) + ... + \gamma_{k-1} d(x_{k-1}, x_k)$, for each $(x_0, ..., x_k) \in X^{k+1}$.

Concerning the regularity condition imposed upon the vector Γ, assume that

(norm-sub) $\Gamma = (\gamma_0, ..., \gamma_{k-1})$ is *norm subunitary*:
$|\Gamma| := \gamma_0 + ... + \gamma_{k-1} < 1$.

The following 1965 fixed point result obtained by Prešić [16] is the first contribution in this area.

Theorem 1.2. *Suppose that T is Prešić $(d; \Gamma)$-contractive, where $\Gamma = (\gamma_1, ..., \gamma_k) \in R_+^k$ is norm subunitary. In addition, let X be d-complete. Then,*

(12-a) *T is fixd-singleton:* $\text{Fixd}(T) = \text{Fix}(S) = \{z\}$, *for some $z \in X$.*

(12-b) *T is a strong Prešić operator (modulo d): for each k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to* $\text{Fixd}(T) = \text{Fix}(S)$.

Concerning the imposed conditions, note that for each $x, y \in X$,

$$d(Sx, Sy) = d(T(x^k), T(y^k)) \leq d(T(x^k), T(x^{k-1}, y)) + ... + d(T(x, y^{k-1}), T(y^k)) \leq \gamma_0 d(x, y) + ... + \gamma_{k-1} d(x, y) = |\Gamma| d(x, y).$$

As a consequence of this,

S is Banach $(d; |\Gamma|)$-contractive, where $|\Gamma| < 1$;

which, along with Banach fixed point theorem, yields

$\mathrm{Fixd}(T) = \mathrm{Fix}(S)$=singleton.

In other words, the existence and uniqueness part of Theorem 1.2 are directly assured by (B-fpt); so, its novelty consists in the underlying diagonal fixed point being approximated via (Prešić type) k-iterative processes.

In particular, when $k = 1$, Theorem 1.2 is just the Banach contraction principle. Hence, the question of extending it is not without interest. A basic contribution of this type is the 2007 one due to Ćirić and Prešić [5]; referred to as: *Ćirić-Prešić fixed point theorem* (in short: (CP-fpt)). Let us say that T is *Ćirić-Prešić $(d;\beta)$-contractive* (where $\beta \geq 0$), provided

(CP-contr) $d(T(x_0, ..., x_{k-1}), T(x_1, ..., x_k)) \leq \beta \max\{d(x_0, x_1), ..., d(x_{k-1}, x_k)\}$, for each $(x_0, ..., x_k) \in X^{k+1}$.

For technical reasons, we also define the concept

(str-nexp) S is *d-strictly-nonexpansive*:
$d(Sx, Sy) < d(x, y)$, $\forall x, y \in X$, $x \neq y$.

Theorem 1.3. *Assume that T is Ćirić-Prešić $(d;\beta)$-contractive, for a certain $\beta \in [0, 1[$. In addition, let X be d-complete. Then, the following conclusions hold:*

(13-a) *T is fixd-asingleton, whenever S is d-strictly-nonexpansive.*

(13-b) *T has diagonal fixed points:* $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$ *is nonempty.*

(13-c) *T is strong Prešić (modulo d): for each k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to* $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$.

Having these precise, it is our aim in the following to extend – from a functional perspective – this last class of results, over the quasi-ordered spaces realm. Note that similar methods are applicable to the statements in Abbas et al. [1] and Gholidahaneh et al. [8] based on diagonal versions of the contractive methods in Dutta and Choudhury [6]; further aspects will be delineated elsewhere.

2. Admissible functions

In the following, some preliminary facts involving (strongly) Matkowski admissible functions will be discussed.

Given a function $\varphi \in \mathcal{F}(R_+)$, call it *regressive* provided

$\varphi(0) = 0$ and $\varphi(t) < t$, for all $t \in R_+^0 :=]0, \infty[$;

the class of such functions will be denoted as $\mathcal{F}(re)(R_+)$. Further, let us introduce the sequential properties over $\mathcal{F}(re)(R_+)$

(M-a) $\varphi \in \mathcal{F}(re)(R_+)$ is *Matkowski admissible*:
for each $(t_n; n \geq 0)$ in R_+^0 with $(t_{n+1} \leq \varphi(t_n), \forall n)$, we have $\lim_n t_n = 0$.

(str-M-a) $\varphi \in \mathcal{F}(re)(R_+)$ is *strongly Matkowski admissible*:
for each $(t_n; n \geq 0)$ in R_+^0 with $(t_{n+1} \leq \varphi(t_n), \forall n)$, we have $\sum_n t_n < \infty$.

The former convention is taken from Matkowski [11]. The second one was formally introduced by Turinici [26]; but its origin goes back to Browder [3]. In particular, when $\varphi \in \mathcal{F}(re)(R_+)$ is increasing, then

(M-1) φ is Matkowski admissible iff $\lim_n \varphi^n(t) = 0$, for each $t \geq 0$;

(M-2) φ is strongly Matkowski admissible iff $\sum_n \varphi^n(t) < \infty$, $\forall t \geq 0$;

here, as usual, φ^n means the n-th *iterate* of φ, for each $n \geq 0$.

Clearly, each strongly Matkowski admissible function is Matkowski admissible too; but the reciprocal is not in general true.

For practical reasons, it would be useful to determine sufficient conditions under which this last property holds. Call the function $h \in \mathcal{F}(R_+^0)$, *int-normal* provided

(i-n-1) $h(.)$ is decreasing on R_+^0.

(i-n-2) $H(t) := \int_0^t h(\xi)d\xi < \infty$, for each $t > 0$.

Note that, by the former condition (i-n-1),

$\int_0^t h(\xi)d\xi := \lim_{s\to 0+} \int_s^t h(\xi)d\xi$ exists in $R_+ \cup \{\infty\}$, for each $t > 0$;

so, the latter condition (i-n-2) is meaningful. Moreover, by these definitions,

(H-1) $H(.)$ is strictly increasing on R_+^0 ($t_1 < t_2 \Longrightarrow H(t_1) < H(t_2)$).

(H-2) $H(.)$ is continuous on R_+^0 and $H(0+) := \lim_{t\to 0+} H(t) = 0$.

Given $\varphi \in \mathcal{F}(re)(R_+)$, let us associate it the function $g \in \mathcal{F}(R_+^0)$, as

$(g(t) = t/(t - \varphi(t)); t > 0)$; in short: $g = I/(I - \varphi)$.

Further, call $g \in \mathcal{F}(R_+^0)$, *int-subnormal* provided

$g(t) \leq h(t)$, $t \in R_+^0$, where $h \in \mathcal{F}(R_+^0)$ is int-normal.

The following int-subnormal type strongly Matkowski criterion is available.

Theorem 2.1. *Let the function $\varphi \in \mathcal{F}(re)(R_+)$ be such that*

the associated function $g = I/(I - \varphi)$ is int-subnormal.

Then, φ is strongly Matkowski admissible.

Proof. By the imposed condition,

$g(t) \leq h(t)$, $t \in R_+^0$, for some int-normal function $h \in \mathcal{F}(R_+^0)$.

Let the sequence $(t_n; n \geq 0)$ in R_+^0 be such that

$t_{n+1} \leq \varphi(t_n)$, for all $n \geq 0$; hence, $(t_n; n \geq 0)$ is strictly descending.

Further, let $i \geq 0$ be arbitrary fixed. By the above choice,

$t_i - \varphi(t_i) \leq t_i - t_{i+1}$; whence, $1 \leq (t_i - t_{i+1})/(t_i - \varphi(t_i))$.

Combining with the decreasing property of $h(.)$ yields (by the definition of $H(.)$)

$t_i \leq (t_i - t_{i+1})g(t_i) \leq (t_i - t_{i+1})h(t_i) \leq H(t_i) - H(t_{i+1})$.

Passing to limit in the relation between our extremal members, gives

$\lim_i t_i = 0$; whence, φ is Matkowski admissible.

On the other hand, from the same relation,

$\sum_{i\le n} t_i \le H(t_0) - H(t_{n+1})$, for each n;

wherefrom, by a limit process,

$\sum_n t_n \le H(t_0) - H(0+) < \infty$;

which tells us that the series $\sum_n t_n$ converges. □

Note that further statements of this type are valid; for a partial list of them, see Turinici [28, Section 13] and the references therein.

3. Main result

Let X be a nonempty set. Take a metric $d : X \times X \to R_+$; and let $(\le)$ be a *quasi-order* (reflexive and transitive relation) on X; the triple $(X, d, \le)$ will be referred to as a *quasi-ordered metric space*. As usual, we denote by $(\xrightarrow{d})$ the associated convergence structure on X:

$x_n \xrightarrow{d} x$ (also written: $\lim_n x_n = x$) if $d(x_n, x) \to 0$ as $n \to \infty$.

Let $(<)$ be the associated strict order

$x < y$ iff $x \le y$ and $x \ne y$;
clearly, $(<)$ is irreflexive but not in general transitive.

Given the subset Y of X, call it $(\le)$-*asingleton*, provided ($y_1, y_2 \in Y$ $y_1 \le y_2 \Longrightarrow y_1 = y_2$); and $(\le)$-*singleton*, provided it is in addition nonempty. Further, let the index $k \ge 1$ be fixed in the sequel; and take a mapping $T : X^k \to X$. Let $S : X \to X$ be the associated to T diagonal map, and $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$ stand for the class of diagonal fixed points of T. As before, our aim is to determine such points, via k-iterative processes. The basic setting of our problem is a quasi-order version of the preceding one. Denote for simplicity

$X^\infty(asc)$=the class of all ascending sequences $(x_n; n \ge 0)$ over X (in the sense: $i \le j$ implies $x_i \le x_j$).

Then, for the (already fixed index) $k \ge 1$, remember that we denoted

X^k=the class of all k-tuples $(y_0, ..., y_{k-1})$ over X.

For simplicity, let again d stand for the product metric on X^k:

$d((y_0, ..., y_{k-1}), (v_0, ..., v_{k-1})) = \max\{d(y_i, v_i); 0 \le i \le k-1\}$, $(y_0, ..., y_{k-1}), (v_0, ..., v_{k-1}) \in X^k$.

In this case, it is natural that the associated convergence structure on X^k be denoted in the same way as the one over X; namely,

for each sequence $(\zeta_n := (\zeta_n^0, ...\zeta_n^{k-1}); n \ge 0)$ in X^k and each element $\zeta := (\zeta_0, ..., \zeta_{k-1})$ in X^k, we put $\zeta_n \xrightarrow{d} \zeta$ (also written: $\lim_n \zeta_n = \zeta$) iff $\zeta_n^i \xrightarrow{d} \zeta^i$ (that is: $\lim_n \zeta_n^i = \zeta^i$), for all $i \in \{0, ..., k-1\}$.

Finally, let again $(\le)$ stand for the associated quasi-order in X^k:

$(y_0, ..., y_{k-1}) \le (v_0, ..., v_{k-1})$ iff $y_i \le v_i$, $i \in \{0, ..., k-1\}$.

Note that many properties of the quasi-ordered metric space $(X^k, d, \le)$ are deductible from the ones of the original structure $(X, d, \le)$; we do not give details.

As a completion of this, we introduce the convention

$X^k(asc)$=the class of all ascending k-tuples $(y_0, ..., y_{k-1})$ over X
(in the sense: $0 \le i \le j \le k-1$ implies $y_i \le y_j$).

For any ascending sequence (u_n) and any couple $i, j \in N$ with $i < j$, let us put

$u[i; j] = (u_i, ..., u_{j-1})$ (the (i, j)-*segment* of (u_n));
clearly, $u[i; j]$ is an element of $X^{j-i+1}(asc)$.

Given the ascending sequence $(u_n; n \ge 0)$, call it *k-iterative* provided

(iter) u_n=the above one, $0 \le n \le k-1$; $u_n = T(u_{n-k}, ..., u_{n-1}), n \ge k$;

for simplicity, we will denote it as $(u_n = T^n U_0; n \ge 0)$, where $U_0 := u[0; k-1] = (u_0, ..., u_{k-1})$ is the $(0, k-1)$-segment (in $X^k(asc)$) of our sequence. The class of all these objects will be denoted as $X^\infty(asc)(k - iter)$.

A natural question to be posed is that of indicating some minimal conditions under which iterative constructions of this type are effective. The following simple criterion is an appropriate answer for us.

Proposition 3.1. *Let the mapping $T : X^k \to X$ be such that*

(i) T is diagonally semi-progressive:
the family $X^k(T, \le)$ of all $(x_0, ..., x_{k-1}) \in X^k(asc)$ with $(x_0, ..., x_{k-1}) \le (x_1, ..., x_k)$, where $x_k = T(x_0, ..., x_{k-1})$, is nonempty.

(ii) T is increasing on X^k:
$(x_0, ..., x_{k-1}) \le (y_0, ..., y_{k-1})$ implies $T(x_0, ..., x_{k-1}) \le T(y_0, ..., y_{k-1})$.

Then, any k-iterative sequence $(u_n; n \ge 0)$ subjected to the condition $(u_0, ..., u_{k-1}) \in X^k(T, \le)$ is ascending.

Proof. Let the sequence (u_n) be as before. As T is diagonally semi-progressive, $(u_0, ..., u_{k-1}) \le (u_1, ..., u_k)$; so, combining with the increasing property of T,

$T(u_0, ..., u_{k-1}) \le T(u_1, ..., u_k)$; that is: $u_k \le u_{k+1}$.

As a consequence of this, $(u_1, ..., u_k) \le (u_2, ..., u_{k+1})$; so, combined with the same increasing property of T, one gets

$T(u_1, ..., u_k) \le T(u_2, ..., u_{k+1})$; that is: $u_{k+1} \le u_{k+2}$.

By an ordinary induction procedure we get the desired fact. □

Returning to our initial setting, the determination of our introduced diagonal points is to be performed upon the directions below, comparable with the ones in Turinici [27, Paper 1-4]:

opp-0) Let us say that T is *fixd-$(\le)$-asingleton*, if $\text{Fixd}(T) = \text{Fix}(S)$ is an $(\le)$-asingleton; and *fixd-$(\le)$-singleton*, provided $\text{Fixd}(T) = \text{Fix}(S)$ is a $(\le)$-singleton.

opp-1) Let us say that the ascending k-iterative sequence (u_n) has the *Prešić property* (modulo $(d, \le; T)$) when (u_n) is d-Cauchy. If $(u_n) \in X^\infty(asc)(k - iter)$ is generic here, we then say that T is a *Prešić operator* (modulo $(d, \le)$).

opp-2) Let us say that the ascending k-iterative sequence (u_n) has the *strong Prešić property* (modulo $(d, \le; T)$) when (u_n) is d-convergent and $z := \lim_n(u_n)$ is an element of $\text{Fixd}(T) = \text{Fix}(S)$. If $(u_n) \in X^\infty(asc)(k - iter)$ is generic here, we then say that T is a *strong Prešić operator* (modulo $(d, \le)$).

opp-3) Let us say that the ascending k-iterative sequence (u_n) has the *Bellman Prešić property* (modulo $(d, \le; T)$) when (u_n) is d-convergent, $z := \lim_n(u_n)$ is an element of $\text{Fixd}(T) = \text{Fix}(S)$, and $(u_n \le z, \forall n)$. If $(u_n) \in X^\infty(asc)(k - iter)$ is generic here, we then say that T is a *Bellman Prešić operator* (modulo $(d, \le)$).

In addition to these, we must impose the following regularity conditions:

Reg-1) Let us say that X is *$(d, \le)$-complete*, if for each sequence (z_n) in X: (z_n) is ascending d-Cauchy implies (z_n) is d-convergent ($z_n \xrightarrow{d} z$, for some $z \in X$).

Reg-2) Let us say that T is *$(d, \le)$-continuous*, if, for each sequence $(\zeta_n := (\zeta_n^0, ...\zeta_n^{k-1}); n \ge 0)$ in X^k and each $\zeta := (\zeta_0, ..., \zeta_{k-1})$ in X^k, one has: (ζ_n) is ascending in X^k and $\zeta_n \xrightarrow{d} \zeta$ imply $T\zeta_n \xrightarrow{d} T\zeta$.

Reg-3) Let us say that $(\leq)$ is $(d, \leq)$-*closed*, if, for each sequence (z_n) in X: (z_n) is ascending and $z_n \xrightarrow{d} z$ imply $(z_n \leq z, \forall n)$.

With these preliminaries, it is our aim in the following to give a functional version of the Ćirić-Prešić fixed point result over this class of quasi-ordered metric spaces. Some specific conventions are in order. Denote for each i, j with $i < j$

$$A(z_i, ..., z_j) = \max\{d(z_i, z_{i+1}), ..., d(z_{j-1}, z_j)\}, (z_i, ..., z_j) \in X^{j-i+1}.$$

Given $\varphi \in \mathcal{F}(R_+)$, let us say that T is $(d, \leq; \varphi)$-*contractive*, provided

(d-phi) $d(T(y_0, ..., y_{k-1}), T(y_1, ..., y_k)) \leq \varphi(A(y_0, ..., y_{k-1}, y_k))$, for all $(y_0, ..., y_{k-1}, y_k) \in X^{k+1}(asc)$.

Further, let us say that S is $(d, \leq)$-*strictly-nonexpansive*, provided

(s-nexp) $d(Sx, Sy) < d(x, y)$, whenever $x < y$.

Our main result in this exposition is

Theorem 3.2. *Suppose that T is $(d, \leq; \varphi)$-contractive, for some increasing strongly Matkowski admissible $\varphi \in \mathcal{F}(re)(R_+)$. In addition, let X be $(d, \leq)$-complete. Then, the following conclusions hold:*

(31-a) *If (in addition) S is $(d, \leq)$-strictly-nonexpansive, then*

T is fixd-$(\leq)$-asingleton; i.e.: $\text{Fixd}(T) = \text{Fix}(S)$ *is an $(\leq)$-asingleton.*

(31-b) *If (alternatively) T is $(d, \leq)$-continuous, then T is strong Prešić (modulo $(d, \leq)$): for each ascending k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to* $\text{Fixd}(T) = \text{Fix}(S)$.

(31-c) *If (alternatively) $(\leq)$ is $(d, \leq)$-closed, then T is Bellman Prešić (modulo $(d, \leq)$): for each ascending k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to* $\text{Fixd}(T) = \text{Fix}(S)$*, with in addition $(u_n \leq z, \forall n)$.*

Proof. The first part of this statement is immediate, as results from

Step 0. Let $z_1, z_2 \in X$ be a couple of points in $\text{Fixd}(T) = \text{Fix}(S)$ (i.e.: $z_1 = Sz_1$, $z_2 = Sz_2$), with $z_1 \leq z_2$; and – contrary to the written conclusion – suppose that $z_1 \neq z_2$; hence, $z_1 < z_2$. By the $(d, \leq)$-strict-nonexpansive property of S,

$$d(z_1, z_2) = d(Sz_1, Sz_2) < d(z_1, z_2); \text{ contradiction.}$$

Hence, necessarily, $z_1 = z_2$; and the claim follows.

It remains now to establish the remaining parts of the statement. There are several stages to be passed.

Step 1. Let the ascending k-iterative sequence (u_n) over X be given. Put for simplicity $(\rho_n = d(u_n, u_{n+1}); n \geq 0)$. Note that, by a previous convention, we have

$$A(u_i, ..., u_j) = \max\{\rho_i, ..., \rho_{j-1}\}, \text{ for each } i, j \text{ with } i < j.$$

For technical reasons, it would be useful to denote, for each $n \geq k$,

$$B_n = A(u_{n-k}, ..., u_{n-1}, u_n); \text{ i.e.: } B_n = \max\{\rho_{n-k}, ..., \rho_{n-1}\}.$$

Proposition 3.3. *Suppose that*

$$B_n = 0, \textit{ for some } n \geq k.$$

Then

(32-1) *The subsequence $(y_i := u_{n-k+i}; i \geq 0)$ is constant; that is,*

$$u_{n-k} = u_{n-k+1} = ... = a, \textit{ for some } a \in X.$$

(32-2) *In addition, we have*

$a = T(a^k) = Sa$;

hence, $a \in X$ is a diagonal fixed point of T.

Proof of Proposition 3.3. Let $n \geq k$ be such that $B_n = 0$. Then, by definition,

$u_{n-k} = ... = u_{n-1} = u_n = a$, for some $a \in X$.

Combining with the iterative procedure, we also get

$a = u_n = T(u_{n-k}, ..., u_{n-1}) = T(a^k)$;

so that, $a \in X$ is a diagonal fixed point of T. Finally, we have

$u_{n+1} = T(u_{n-k+1}, ..., u_{n-1}, u_n) = T(a^k) = a$,
$u_{n+2} = T(u_{n-k+2}, ..., u_n, u_{n+1}) = T(a^k) = a, ...$;

and, from this, $(u_{n-k+i} = a; i \geq 0)$. □

As a consequence of these remarks, it follows that, whenever

$B_n = 0$, for some $n \geq k$,

we are done; so, without loss, one may assume that

(str-pos) $B_n > 0$, for all $n \geq k$.

The following auxiliary fact concentrates the "deep" part of our argument.

Proposition 3.4. *Under these conditions, we have*
(33-1) $\rho_n \leq \varphi(B_n) < B_n$, *for each* $n \geq k$.
(33-2) $(B_{n+1} \leq B_n, \forall n \geq k)$*; whence,* $(B_{k+i}; i \geq 0)$ *is descending.*
(33-3) $\rho_{n+k} < B_{n+k} \leq \varphi(B_n) < B_n, \forall n \geq k$.

Proof of Proposition 3.4. i): Let $n \geq k$ be arbitrary fixed. By definition,

$A(u_{n-k}, ..., u_{n-1}, u_n) = B_n = \max\{\rho_{n-k}, ..., \rho_{n-1}\}$.

On the other hand, from our iterative construction,

$\rho_n = d(u_n, u_{n+1}) = d(T(u_{n-k}, ..., u_{n-1}), T(u_{n-k+1}, ..., u_{n-1}, u_n))$;

and this, by the contractive property (and φ=regressive), gives

$\rho_n \leq \varphi(A(u_{n-k}, ..., u_{n-1}, u_n)) = \varphi(B_n) < B_n$;

proving the desired fact.
ii): From the representation above one has, for each $n \geq k$,

$B_{n+1} = \max\{\rho_{n-k+1}, ..., \rho_{n-1}, \rho_n\} \leq \max\{B_n, \rho_n\}$;

and this, along with $\rho_n < B_n$, yields the written relation.
iii): By the obtained facts one gets (via φ=increasing), for each $n \geq k$

$\rho_{n+j} \leq \varphi(B_{n+j}) \leq \varphi(B_n), \forall j \in \{0, ..., k-1\}$.

This yields (by definition)

$B_{n+k} = \max\{\rho_n, ..., \rho_{n+k-1}\} \leq \varphi(B_n)$;

and proves our claim. □

Having these established, we may now pass to the final part of our argument.

Step 2. From the evaluations above, one has

$B_{(i+1)k} \leq \varphi^i(B_k)$, for all $i \geq 0$.

As a consequence, we get (again for all ranks $i \geq 0$)

$\rho_{ik} + ... + \rho_{(i+1)k-1} \leq kB_{(i+1)k} \leq k\varphi^i(B_k)$;

wherefrom (by adding these inequalities)

$\sum_n \rho_n \leq k \sum_i B_{(i+1)k} \leq k \sum_i \varphi^i(B_k) < \infty$;

telling us that the k-iterative sequence (u_n) is d-Cauchy. Combining with $(u_n; n \geq 0)$=ascending, yields (via X is $(d, \leq)$-complete)

$u_n \xrightarrow{d} z$ as $n \to \infty$, for some $z \in X$.

We must establish that the obtained limit z is a diagonal fixed point for T; i.e.: $z = T(z^k)(= Sz)$. There are two alternatives to discuss.

Alter 1. Suppose that T is $(d, \leq)$-continuous. By the very definition of our iterative process,

$u_{n+k} = T(u_n, ..., u_{n+k-1})$, $\forall n$.

On the other hand, the sequence $((u_n, ..., u_{n+k-1}); n \geq 0)$ in $X^k(asc)$ is ascending and $\lim_n(u_n, ..., u_{n+k-1}) = z^k$. Passing to limit as $n \to \infty$, gives (by the posed condition) $z = T(z^k) = Sz$; and the conclusion follows.

Alter 2. Suppose that $(\leq)$ is $(d, \leq)$-closed. Note that, as a consequence of this,

(ineq) $u_n \leq z$, for all n.

We have to establish that $z = T(z^k)(= Sz)$. To get such a conclusion, it will suffice proving that

$u_n \xrightarrow{d} Sz(= T(z^k))$, as $n \to \infty$.

This may be obtained as follows. By the triangle inequality, we have for each $n \geq k$

$$d(u_n, Sz) = d(T(u_{n-k}, ..., u_{n-1}), T(z^k)) \leq d(T(u_{n-k}, ..., u_{n-2}, u_{n-1}), T(u_{n-k+1}, ..., u_{n-1}, z)) + d(T(u_{n-k+1}, ..., u_{n-2}, u_{n-1}, z), T(u_{n-k+2}, ..., u_{n-1}, z^2)) + ... + d(T(u_{n-1}, z^{k-1}), T(z^k)).$$

On the other hand, from the choice of (u_n) and (ineq),

$(u_{n-k}, ..., u_{n-2}, u_{n-1}, z^k) \in X^{2k}(asc)$; hence, any $(i, i+k)$-segment of this (where $0 \leq i \leq k$) belongs to $X^{k+1}(asc)$.

This tells us that the contractive property is applicable upon the preceding relation; and yields

$d(u_n, Sz) \leq \varphi(\max\{\rho_{n-k}, ..., \rho_{n-2}, d(u_{n-1}, z)\}) + \varphi(\max\{\rho_{n-k+1}, ..., \rho_{n-2}, d(u_{n-1}, z)\}) + ... + \varphi(d(u_{n-1}, z))$;

wherefrom (via φ=increasing)

$d(u_n, Sz) \leq k\varphi(\max\{\rho_{n-k}, ..., \rho_{n-2}, d(u_{n-1}, z)\})$, $\forall n \geq k$.

Passing to limit as $n \to \infty$, we get

$\lim_n d(u_n, Sz) = 0$; so that (by uniqueness), $z = Sz(= T(z^k))$;

and this gives us the desired fact. The proof is thereby complete. □

Finally, note that Kannan type versions of these facts are obtainable as well via these methods; which means that the related statements in Păcurar [14, 15], Rao et al. [18], or Shukla et al. [23] are accessible in this way. On the other hand, multivalued extensions of these facts may be obtained under the lines in Shahzad and Shukla [21], Rajagopalan [17], and Latif et al. [10]. Finally, by a small modification of these methods one gets the results in Shukla et al. [22] obtained over the realm of relational metric spaces. We shall discuss all these elsewhere.

4. Trivial quasi-order versions

Let (X, d) be a metric space, and $k \geq 1$ be a natural number. Further, let $T : X^k \to X$ be a mapping, $S : X \to X$ be its associated diagonal map, and $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$ stand for the class of diagonal fixed points of T. The basic setting of our problem was already sketched. It is our aim in the following to give – along these structures – a functional version of the Ćirić-Prešić fixed point theorem (CP-fpt). Technically, this is possible by the developments above, by simply noting that the ambient metric space (X, d) is a quasi-ordered metric space if the quasi-order $(\leq)$ is the trivial one, $X \times X$. Precisely, the Bellman portion of the main result is of interest for us; because, under this setting,

$(\leq)$ is $(d, \leq)$-closed; hence, the Bellman Prešić property is identical with the strong Prešić property.

Denote for each i, j with $i < j$

$A(z_i, ..., z_j) = \max\{d(z_i, z_{i+1}), ..., d(z_{j-1}, z_j)\}$, $(z_i, ..., z_j) \in X^{j-i+1}$.

Given $\varphi \in \mathcal{F}(R_+)$, let us say that T is *(d, φ)-contractive*, provided

(d-phi) $d(T(y_0, ..., y_{k-1}), T(y_1, ..., y_k)) \leq \varphi(A(y_0, ..., y_{k-1}, y_k))$, for all $(y_0, ..., y_{k-1}, y_k) \in X^{k+1}$.

Further, let us remember that S is called *d-strictly-nonexpansive*, provided

(s-nexp) $d(Sx, Sy) < d(x, y)$, whenever $x \neq y$.

From the (quasi-ordered) main result we just proved, one gets the following trivial quasi-order type diagonal fixed point result with a practical meaning.

Theorem 4.1. *Suppose that T is (d, φ)-contractive, for some increasing strongly Matkowski admissible $\varphi \in \mathcal{F}(re)(R_+)$. In addition, let X be d-complete. Then, the following conclusions hold:*

(41-a) *If (in addition) S is d-strictly-nonexpansive, then*

T is fixd-asingleton: $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$ *is an asingleton.*

(41-b) *T is strong Prešić (modulo d): for each k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to* $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$.

In particular, when φ is linear and regressive

$\varphi(t) = \beta t$, $t \in R_+$, for some $\beta \in [0, 1[$,

the obtained result is just the one in Ćirić and Prešić [5]. On the other hand, when φ is continuous, the corresponding version of this result includes a related 1981 statement in Rus [19], refined in Turinici [27, Paper 1-4]. Some coincidence point extensions of these facts are possible, under the lines in George and Khan [7], Murthy [12], Pathak et al. [13] and Yeşilkaya [29]. For direct applications of these facts to convergence questions involving real sequences, we refer to Chen [4], Khan et al. [9], and the references therein.

5. Tasković approach

In the following, we show that the 2007 Ćirić-Prešić result [5] (subsumed to Theorem 1.3) is "almost" equivalent with a 1976 statement in Tasković [24].

Let the index $k \geq 1$ be fixed in the sequel. Take a function $f : R_+^k \to R_+$, and a vector $\Gamma := (\gamma_0, ..., \gamma_{k-1})$ in R_+^k. We say that f is *admissible*, provided

(adm-1) f is increasing (in its variables):
$(u_i \leq v_i, i \in \{0, ..., k-1\})$ imply $f(u_0, ..., u_{k-1}) \leq f(v_0, ..., v_{k-1})$.

(adm-2) f is *semi-homogeneous*:
$f(\lambda x_0, ..., \lambda x_{k-1}) \leq \lambda f(x_0, ..., x_{k-1})$, $\forall (x_0, ..., x_{k-1}) \in R_+^k$, $\forall \lambda \geq 0$.

Also, let us say that (f, Γ) is *regular*, when

(reg) the associated function $g : R_+ \to R_+$ introduced as
$(g(t) = f(\gamma_0 t, ..., \gamma_{k-1} t^k); t \in R_+)$ is continuous at $t = 1$.

(Note that, in the 2011 paper by Tasković [25], the properties (adm-1)+(adm-2)+(reg) are being referred to as: f has the *M-property*). Then, for an easy reference, let us introduce the couple of conditions

(tele-sub) (f, Γ) is *telescopic subunitary*:
$\alpha := f(\gamma_0, 0, ..., 0) + ... + f(0, ..., 0, \gamma_{k-1}) < 1$

(sub) (f, Γ) is *subunitary*: $\beta := f(\gamma_0, ..., \gamma_{k-1}) < 1$.

Having these precise, let us say that $T : X^k \to X$ is *Tasković $(d; f, \Gamma)$-contractive* (where $f \in \mathcal{F}(R^k_+, R_+)$, $\Gamma := (\gamma_0, ..., \gamma_{k-1}) \in R^k_+$), provided

(T-contr) $d(T(x_0, ..., x_{k-1}), T(x_1, ..., x_k)) \leq f(\gamma_0 d(x_0, x_1), ..., \gamma_{k-1} d(x_{k-1}, x_k))$, for each $(x_0, ..., x_k) \in X^{k+1}$.

The following 1976 statement in Tasković [24] is our starting point.

Theorem 5.1. *Assume that the mapping T is Tasković $(d; f, \Gamma)$-contractive, where $f \in \mathcal{F}(R^k_+, R_+)$ is a function and $\Gamma = (\gamma_0, ..., \gamma_{k-1})$ is a vector in R^k_+. In addition, let X be d-complete. Then, the following conclusions hold:*
(51-a) *If f is admissible and (f, Γ) is regular, telescopic subunitary, then T is fixd-asingleton, in the sense:* $\text{Fixd}(T) = \text{Fix}(S)$ *is an asingleton.*
(51-b) *If f is admissible and (f, Γ) is regular, subunitary, then T is strong Prešić (modulo d): for each k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to* $\text{Fixd}(T) = \text{Fix}(S)$.

Before passing to the discussion of this result, some technical remarks about the semi-homogeneous condition are in order. Given $f \in \mathcal{F}(R^k_+, R_+)$, call it *homogeneous*, provided

$f(\lambda x_0, ..., \lambda x_{k-1}) = \lambda f(x_0, ..., x_{k-1}), \forall (x_0, ..., x_{k-1}) \in R^k_+, \forall \lambda \geq 0.$

Clearly, any such function is semi-homogeneous. But, the reciprocal inclusion is also true; as it results from.

Proposition 5.2. *For each function $f \in \mathcal{F}(R^k_+, R_+)$, we have*

semi-homogeneous $\Longrightarrow$ homogeneous;
whence, semi-homogeneous $\Longleftrightarrow$ homogeneous.

Proof. Suppose that $f : R^k_+ \to R_+$ is semi-homogeneous; i.e.:

$(0 \leq) f(\lambda x_0, ..., \lambda x_{k-1}) \leq \lambda f(x_0, ..., x_{k-1}), \forall (x_0, ..., x_{k-1}) \in R^k_+, \forall \lambda \geq 0.$

Putting $\lambda = 0$ in this relation gives

$0 \leq f(0, ..., 0) \leq 0$; whence, $f(0, ..., 0) = 0$;

i.e.: the homogeneous property is fulfilled in case of $\lambda = 0$. It remains then to verify the underlying property in case of $\lambda > 0$. Denote for simplicity $\mu := 1/\lambda$; and let $(x_0, ..., x_{k-1}) \in R^k_+$ be arbitrary fixed. By the semi-homogeneous property, we have

$f(x_0, ..., x_{k-1}) = f(\mu\lambda x_0, ..., \mu\lambda x_{k-1}) \leq \mu f(\lambda x_0, ..., \lambda x_{k-1})$;
or, equivalently (cf. our notation): $\lambda f(x_0, ..., x_{k-1}) \leq f(\lambda x_0, ..., \lambda x_{k-1})$.

This, along with the semi-homogeneous inequality, gives

$f(\lambda x_0, ..., \lambda x_{k-1}) = \lambda f(x_0, ..., x_{k-1}), \forall (x_0, ..., x_{k-1}) \in R^k_+, \forall \lambda > 0;$

and proves our assertion. □

In this perspective, the following couple of results is in effect here. Take some function $f : R_+^k \to R_+$ and some vector $\Gamma := (\gamma_0, ..., \gamma_{k-1}) \in R_+^k$. We say that f is *quasi admissible*, provided

(qadm-1) f is increasing (in its variables): $(u_i \leq v_i, i \in \{0, ..., k-1\})$ imply $f(u_0, ..., u_{k-1}) \leq f(v_0, ..., v_{k-1})$.

(qadm-2) f is homogeneous: $f(\lambda x_0, ..., \lambda x_{k-1}) = \lambda f(x_0, ..., x_{k-1})$, $\forall (x_0, ..., x_{k-1}) \in R_+^k$, $\forall \lambda \geq 0$.

Theorem 5.3. *Assume that T is Tasković $(d; f, \Gamma)$-contractive, where $f \in \mathcal{F}(R_+^k, R_+)$ and $\Gamma = (\gamma_0, ..., \gamma_{k-1}) \in R_+^k$ are such that*

f is quasi admissible and (f, Γ) is telescopic subunitary.

In addition, let X be d-complete. Then, the following conclusions hold:

(52-a) *S is $(d; \alpha)$-contractive [$d(Sx, Sy) \leq \alpha d(x, y)$, $\forall x, y \in X$]; hence, S is fix-asingleton (or, equivalently: T is fixd-asingleton).*

(52-b) *S is strongly Picard (modulo d): for each $v_0 \in X$, the iterative sequence $(v_n = S^n v_0; n \geq 0)$ is d-convergent and $v := \lim_n(v_n)$ is a fixed point of S (or, equivalently: a diagonal fixed point of T).*

Theorem 5.4. *Assume that T is Tasković $(d; f, \Gamma)$-contractive, where $f \in \mathcal{F}(R_+^k, R_+)$ and $\Gamma = (\gamma_0, ..., \gamma_{k-1}) \in R_+^k$ are such that*

f is quasi admissible and (f, Γ) is subunitary.

In addition, let X be d-complete. Then,

T is strong Prešić (modulo d): for each k-iterative sequence (u_n) over X, $z := \lim_n u_n$ exists and belongs to $\mathrm{Fixd}(T) = \mathrm{Fix}(S)$.

Concerning the relationships between these results and the previous ones the following synthetic answer is available.

Proposition 5.5. *Under the above conventions,*

(52-1) *Theorem* 1.1 $\Longrightarrow$ *Theorem* 5.3 $\Longrightarrow$ *Theorem* 5.1 *(the first half)*

(52-2) *Theorem* 1.3 $\Longrightarrow$ *Theorem* 5.4 $\Longrightarrow$ *Theorem* 5.1 *(the second half)* $\Longrightarrow$ *Theorem* 1.3.

Proof. i): Two inclusions must be verified.

i-1): Let the premises of Theorem 5.3 be fulfilled. In particular, as (f, Γ) is telescopic subunitary,

$\alpha := f(\gamma_0, 0, ..., 0) + ... + f(0, ..., 0, \gamma_{k-1}) < 1$.

We claim that (under the precise notations)

S is $(d; \alpha)$-contractive: $d(Sx, Sy) \leq \alpha d(x, y)$, $\forall x, y \in X$.

In fact, let $x, y \in X$ be arbitrary fixed. From the Tasković contractive condition (and f=homogeneous), one derives

$$\begin{aligned} d(Sx, Sy) = d(T(x^k), T(y^k)) &\leq d(T(x^k), T(x^{k-1}, y)) + ... + d(T(x, y^{k-1}), T(y^k)) \\ &\leq f(\gamma_0 d(x, y), 0, ..., 0) + ... + f(0, ..., 0, \gamma_{k-1} d(x, y)) \\ &= (f(\gamma_0, 0, ..., 0) + ... + f(0, ..., 0, \gamma_{k-1}))d(x, y) \\ &= \alpha d(x, y); \end{aligned}$$

hence (52-a) follows. On the other hand, (52-b) is obtainable from this contractive condition, via Theorem 1.1; so that, the first inclusion follows.

i-2): Evident, via: admissible $\Longrightarrow$ quasi admissible.

ii): We have three inclusions to discuss.

ii-1): Let the premises of Theorem 5.4 be fulfilled. From the imposed properties, we have (under our notations)

$f(\gamma_0 t_0, ..., \gamma_{k-1} t_{k-1}) \leq \beta \max\{t_0, ..., t_{k-1}\}$, for each $(t_0, ..., t_{k-1}) \in R_+^k$.

As a consequence of this, the mapping T is Cirić-Prešić $(d;\beta)$-contractive; and this tells us that Theorem 1.3 implies Theorem 5.4.

ii-2): Clearly, Theorem 5.4 implies Theorem 5.1 (the second half), in view of: admissible $\Longrightarrow$ quasi admissible.

ii-3): Let the premises of Theorem 1.3 hold; and take the couple (f, Γ) as

$$f(t_0, ..., t_{k-1}) = \max\{t_0, ..., t_{k-1}\}, (t_0, ..., t_{k-1}) \in R_+^k; \Gamma = (\gamma_0, ..., \gamma_{k-1}) \in R_+^k, \text{ where } (\gamma_i = \beta; 0 \leq i \leq k-1).$$

Clearly, T is Tasković $(d; f, \Gamma)$-contractive; with, in addition,

f is admissible and (f, Γ) is regular, subunitary.

Putting these together, it results that Theorem 5.1 (the second half) implies Theorem 1.3. The proof is thereby complete. □

Summing up, the following equivalence relation holds

(eq-1) Theorem 5.1 (the second half)
is equivalent with Theorem 1.3;

which shows that the claim in Tasković [25] is retainable. On the other hand, by the same argument we do have

(eq-2) Theorem 5.1 (the second half)
is equivalent with Theorem 5.4.

Concerning this aspect, note that (by the involved concepts), the right to left inclusion is trivial. However, the left to right inclusion is not a trivial one; because, the regularity condition upon (f, Γ)

the associated map $g : R_+ \to R_+$ introduced as
$(g(t) = f(\gamma_0 t, ..., \gamma_{k-1} t^k); t \in R_+)$ is continuous at $t = 1$

is not deductible from the quasi admissible condition upon f. Further aspects may be found in Tasković [25].

6. Conclusion

In this paper, functional generalizations – to the realm of ordered metric spaces – are given for the 2007 diagonal fixed point result obtained by Ćirić and Prešić. In fact, similar extensions are obtainable as well for the diagonal versions of the contractive methods proposed by Dutta and Choudhury, as well as the diagonal versions of the contractive techniques having as starting point the well known contributions due to Nadler. Some of these directions may be interesting subjects of further studies in this area.

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