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Trigonometric functional equations on non-abelian semigroups

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Abstract

Let S be a semigroup, and let $\mu : S \longrightarrow \mathbb{C}$ be a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. We study the properties of the solutions of the functional equations

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) + 2f(y)g(x), \ \ x, y \in S,$$

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)f(y) + 2g(x)g(y), \ \ x, y \in S,$$

where σ is an involutive morphism. The solutions are expressed by means of solutions of d'Alembert's μ-functional equation and the functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\phi(y) + 2f(y)\phi(x), \ \ x, y \in S,$$

in which ϕ is a solution of d'Alembert's μ-functional equation. As an application we prove that, in a nilpotent group G which is generated by its squares, the solutions of the functional equation

$$f(xy) + \mu(y)f(xy^{-1}) = 2f(x)g(y) + 2f(y)g(x), \ \ x, y \in G$$

are abelian.
We also find the solutions of the functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\phi(y) + 2f(y)\psi(x), \ \ x, y \in S,$$

where σ is an involutive anti-automorphism, $f : S \longrightarrow \mathbb{C}$ is the unknown function and ϕ, ψ are non-zero solutions of d'Alembert's μ-functional equation. This enables us to solve the pexider functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2g_1(x)h_1(y) + 2\psi(x)h_2(y), \ \ x, y \in S$$

in which $f, g_1, h_1, h_2 : S \longrightarrow \mathbb{C}$ are the unknown functions and g_1 is even.

Keywords: Semigroup, Involutive automorphism, Involutive anti-automorphism, Multiplicative function, D'Alembert equation, Additive function, Quadratic equation, Pexider equation

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1. Introduction and notations

Addition formulas for trigonometric functions on a group G have the only term $f(xy)$ in the left side of the equation. The sine addition law:

$$f(xy) = f(x)g(y) + f(y)g(x),\ x, y \in G,$$

is an example. Another example is the functional equation

$$f(xy) = f(x)g(y) + f(y)g(x) + h(x)h(y),\ x, y \in G,$$

which was solved by Chung, Kannappan and Ng [2] in a group G, and by Ajebbar and Elqorachi [1] on semigroups generated by their squares.
Some papers study the more general case in which the left hand side $f(xy)$ has been replaced by the average $[f(xy) + f(x\sigma(y))]/2$. Among them the paper [3], in which Chung, Kannappan and Ng solved the two functional equations

$$f(x + y) + f(x + \sigma(y)) = 2f(x)g(y) + 2g(x)f(y),\ x, y \in G, \tag{1.1}$$

$$f(x + y) + f(x + \sigma(y)) = 2f(x)f(y) + 2g(x)g(y),\ x, y \in G \tag{1.2}$$

on abelian groups with the involution σ as the group inversion. Stetkær [6] solved (1.1) and (1.2) for a general involution.
In [5] Friis and Stetkær solved another more general functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y) + 2f(y)g(x) + 2h(x)h(y),\ x, y \in G,$$

where G is a topological abelian group and $\sigma : G \longrightarrow G$ is a continuous involutive automorphism of G.
Our contribution is to extend the existing results for (1.1) and (1.2) to semigroups, so the first purpose of the present paper is to study the functional equations

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) + 2g(x)f(y),\ x, y \in S, \tag{1.3}$$

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)f(y) + 2g(x)g(y),\ x, y \in S, \tag{1.4}$$

with σ is an involutive morphism (involutive automorphism or involutive anti-automorphism), and $f, g : S \longrightarrow \mathbb{C}$ are the unknown functions to be determined. The functional equations (1.3) and (1.4) have not been solved on general groups.
We express the solutions of (1.3) and (1.4) in terms of the solutions of d'Alembert's μ-functional equation

$$\phi(xy) + \mu(y)\phi(x\sigma(y)) = 2\phi(x)\phi(y),\ x, y \in S \tag{1.5}$$

and the solutions of the functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\phi(y) + 2f(y)\phi(x),\ x, y \in S, \tag{1.6}$$

in which ϕ is a solution of (1.5). The results obtained for the functional equation (1.3) and (1.4) can be compared to the ones of [7] because we formulate them in the same way. However, our formulas for the solutions of (1.3) and (1.4) are obtained in the non-abelian case.
We also prove that, in a nilpotent group which is generated by its squares, the solutions of the functional equation (1.3) with $\sigma(x) = x^{-1}$ are abelian.
The other purpose of this paper is to solve the functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2g_1(x)h_1(y) + 2\psi(x)h_2(y),\ x, y \in M, \tag{1.7}$$

where M is a monoid that need not be abelian, $\sigma : M \longrightarrow M$ is an anti-automorphism, and $\psi : M \longrightarrow \mathbb{C}$ is a non-zero prescribed solution of the d'Alembert μ-functional equation (1.5), while $f, g_1, h_1, h_2 : M \longrightarrow \mathbb{C}$ are the unknown

functions to be determined under the assumption that g_1 is even.
The core of our study of (1.7) is a discussion of a particular case

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\phi(y) + 2\psi(x)f(y),\ x, y \in S, \tag{1.8}$$

of (1.7), in which S is a semigroup and both ϕ and ψ are given non-zero different solutions of d'Alembert's μ-functional equation (1.5), and where $f : S \longrightarrow \mathbb{C}$ is the only unknown function. The motivation of this part is the recent paper, by Stetkær [9] in which he solved the functional equation

$$f(xy) = f(x)\chi(y) + \mu(x)f(y)\ , x, y \in G \tag{1.9}$$

and its Pexiderized version

$$f(xy) = g_1(x)h_1(y) + g(x)h_2(y)\ , x, y \in G, \tag{1.10}$$

where G is a group, $g, \chi, \mu : G \longrightarrow \mathbb{C}$ are given characters and $f, g_1, h_1, h_2 : G \longrightarrow \mathbb{C}$ are the unknown functions.
We wish to see what happens if we replace the two characters χ and μ of equation (1.9) by two solutions ϕ and ψ of d'Alembert's μ-functional equation (1.5), and replace the left hand side $f(xy)$ of (1.9) by the average $[f(xy) + f(x\sigma(y))]/2$, where σ is an involutive anti-automorphism. We were surprised by the simplicity of the forms of the solutions of (1.8). The solutions of (1.8) are of the form $\alpha(\phi - \psi)$, where $\alpha \in \mathbb{C}$, while, the solutions of (1.9) are $\alpha(\chi - \mu)$ plus an extra terms that originate from the commutator subgroup $[G, G]$. The treatment of the functional equation (1.7) can be compared with the one of the functional equation (1.10) in [9].

Notations. Throughout this paper S is a semigroup, M is a monoid, and G is a group. We denote by e the neutral element of M and G. The map $\sigma : S \longrightarrow S$ denotes an involutive morphism. That is σ is an involutive automorphism: $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$, or an involutive anti-automorphism: $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. We let $\mu : S \longrightarrow \mathbb{C}$ be a fixed multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. The multiplicative group $\mathbb{C}\backslash\{0\}$ is denoted $\mathbb{C}^*$.
We say that a complex-valued function f on S is abelian if $f(x_1x_2...x_n) = f(x_{\varrho(1)}x_{\varrho(2)}...x_{\varrho(n)})$ for all $x_1, x_2, ..., x_n \in S$, all permutation ϱ of n elements and all $n \in \mathbb{N}$.
If f is any function on S we let $f^*(x) := \mu(x)f(\sigma(x))$, $f^e = \frac{f+f^*}{2}$, $f^o = \frac{f+f^*}{2}$, $x \in S$. We say that f is even, if $f^e = f$, and that f is odd if $f^o = f$. Finally, we say that a semigroup S is generated by its squares if for all $x \in S$ there exist some $x_1, x_2, ..., x_n \in S$ such that $x = x_1^2x_2^2...x_n^2$.

2. Properties of the functional equation (1.3)

In this section we get information on the solutions of the functional equation (1.3), i.e.,

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) + 2g(x)f(y),\ x, y \in S,$$

where σ is an involutive morphism. No new phenomena occur when you pass from abelian groups to non-abelian semigroups.

Proposition 2.1. *Let S be a semigroup. Let $f, g : S \longrightarrow \mathbb{C}$ satisfy the functional equation* (1.3). *Assume furthermore that $f \neq 0$. Then there exists a constant $\alpha \in \mathbb{C}$ such that*

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y) + 2\alpha^2 f(x)f(y)\ \textit{for all}\ x, y \in S, \tag{2.1}$$

and we have $\phi_1 := g + \alpha f : S \longrightarrow \mathbb{C}$ and $\phi_2 := g - \alpha f : S \longrightarrow \mathbb{C}$ satisfy d'Alembert's μ-functional equation (1.5) *and*

$$g = \frac{\phi_1 + \phi_2}{2}. \tag{2.2}$$

Additionally we have the following.

(1) If $\phi_1 \neq \phi_2$, then $\alpha \neq 0$ and $f = \frac{1}{2\alpha}(\phi_1 - \phi_2)$.

(2) If $\phi_1 = \phi_2$, then $g = \phi_1$, and (f, ϕ_1) satisfies the functional equation (1.6).

Proof. If $g = 0$, then, with $\alpha = 0$ and $\phi_1 = \phi_2 = 0$, we can check easily that the identities (2.1), and (2.2) of Proposition 2.1 hold. The statement (2) is trivially true since, with $g = \phi_1 = 0$, the two functional equations (1.3) and (1.6) are the same. So, we may from now on assume that $g \neq 0$. Replacing y in equation (1.3) by $\sigma(y)$ and multiplying the result by $\mu(y)$ and using that $\mu(y\sigma(y)) = 1$ for all $y \in S$ we get that

$$\mu(y)f(x\sigma(y)) + f(xy) = 2f(x)\mu(y)g(\sigma(y)) + 2g(x)\mu(y)f(\sigma(y)).$$

Subtracting the last equation from (1.3) we obtain that

$$f(x)[g(y) - g^*(y)] = g(x)[f^*(y) - f(y)] \text{ for all } x, y \in S. \tag{2.3}$$

If f and g are linearly independent, then we deduce from (2.3) that f and g are even. If f and g are linearly dependent, then there exists $\lambda \in \mathbb{C}^*$ such that $f = \lambda g$. So equation (2.3) reduces to

$$\lambda g(x)[g(y) - g^*(y)] = 0. \tag{2.4}$$

Since $g \neq 0$ by hypothesis, we deduce from (2.4) that $g^* = g$ and hence $f^* = f$. Thus, in both cases f and g are even. Now, we split the discussion of (1.3) into two cases.

Case 1: σ is an involutive anti-automorphism.

By the symmetry of the right hand side of equation (1.3) we get for all $x, y \in S$ that

$$f(xy) + \mu(y)f(x\sigma(y)) = f(yx) + \mu(x)f(y\sigma(x)). \tag{2.5}$$

Since f is even and $\mu(x\sigma(x)) = 1$ for all $x \in S$ we have

$$\mu(y)f(x\sigma(y)) = \mu(y)f^*(x\sigma(y)) = \mu(y)\mu(x\sigma(y))f(y\sigma(x)) = \mu(x)f(y\sigma(x)).$$

Hence equation (2.5) becomes $f(xy) = f(yx)$ for all x, y, which means that f is central.

Now, let x, y, z be in S. By using (1.3), the centrality of f and that $\mu(x\sigma(x)) = 1$ for all $x \in S$ we have

$$\begin{aligned}
2f(z)[g(xy) + \mu(y)g(\sigma(y)x) - 2g(x)g(y)] &= [f(xyz) + \mu(z)f(xy\sigma(z)) - 2f(xy)g(z)] \\
&\quad + \mu(y)[f(\sigma(y)xz) + \mu(z)f(\sigma(y)x\sigma(z)) - 2f(\sigma(y)x)g(z)] - 4f(z)g(x)g(y) \\
&= [f(xyz) + \mu(yz)f(\sigma(y)x\sigma(z))] + \mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(\sigma(y)xz)] \\
&\quad - 2[f(xy) + \mu(y)f(x\sigma(y))]g(z) - 4f(z)g(x)g(y) \\
&= [f(xyz) + \mu(zy)f(x\sigma(z)\sigma(y))] + \mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(xz\sigma(y))] \\
&\quad - 2[2f(x)g(y) + 2f(y)g(x)]g(z) - 4f(z)g(x)g(y) \\
&= 2f(x)g(yz) + 2g(x)f(yz) + 2\mu(z)f(x)g(y\sigma(z)) + 2\mu(z)g(x)f(y\sigma(z)) \\
&\quad - 4f(x)g(y)g(z) - 4f(y)g(x)g(z) - 4f(z)g(x)g(y) \\
&= 2f(x)[g(yz) + \mu(z)g(y\sigma(z)) - 2g(y)g(z)] \\
&\quad + 2g(x)[f(yz) + \mu(z)f(y\sigma(z)) - 2f(y)g(z) - 2f(z)g(y)] \\
&= 2f(x)[g(yz) + \mu(z)g(y\sigma(z)) - 2g(y)g(z)].
\end{aligned}$$

This can be written as follows

$$f(z)\Phi(x, y) = f(x)\Psi(y, z) \text{ for all } x, y, z \in S, \tag{2.6}$$

where $\Phi, \Psi : S \times S \longrightarrow \mathbb{C}$ are defined by

$$\Phi(x, y) := g(xy) + \mu(y)g(\sigma(y)x) - 2g(x)g(y), \;\; x, y \in S,$$

$$\Psi(x, y) := g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y), \;\; x, y \in S.$$

By assumption $f \neq 0$ so there exists a $z_0 \in S$ such that $f(z_0) \neq 0$. Putting $z = z_0$ and then $x = z_0$ in (2.6) and dividing through by $f(z_0)$ we get respectively

$$\Phi(x, y) = f(x)\phi(y), \tag{2.7}$$

$$\Psi(y, z) = f(z)\psi(y), \tag{2.8}$$

where $\phi(y) = \Psi(y, z_0)/f(z_0)$ and $\psi(y) = \Phi(z_0, y)/f(z_0)$.
Substituting (2.7) and (2.8) back into (2.6) we find that $f(z)f(x)\phi(y) = f(x)f(z)\psi(y)$ for all $x, y, z \in S$, which implies that $\phi = \psi$ since $f \neq 0$.
Going back to the definitions of Φ and Ψ we get that

$$g(xy) + \mu(y)g(\sigma(y)x) - 2g(x)g(y) = f(x)\phi(y)$$

and

$$g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y) = f(y)\phi(x)$$

for all $x, y \in S$. Adding the two last equations we get that

$$2g(xy) + \mu(y)g(\sigma(y)x) + \mu(y)g(x\sigma(y)) = 4g(x)g(y) + f(x)\phi(y) + f(y)\phi(x). \tag{2.9}$$

By the symmetry of the right hand side of (2.9) we deduce that

$$2g(xy) + \mu(y)[g(\sigma(y)x) + g(x\sigma(y))] = 2g(yx) + \mu(x)[g(\sigma(x)y) + g(y\sigma(x))],$$

from which we find, using the evenness of g and that $\mu(x\sigma(x)) = 1$ for all $x \in S$, that

$$\begin{aligned}2[g(yx) - g(xy)] &= \mu(y)[g(\sigma(y)x) + g(x\sigma(y))] - \mu(x)[g(\sigma(x)y) + g(y\sigma(x))]\\ &= \mu(y)[g^*(\sigma(y)x) + g^*(x\sigma(y))] - \mu(x)[g(\sigma(x)y) + g(y\sigma(x))]\\ &= \mu(y)\mu(x\sigma(y))[g(\sigma(x)y) + g(y\sigma(x))] - \mu(x)[g(\sigma(x)y) + g(y\sigma(x))]\\ &= 0.\end{aligned}$$

This means that g is central, which implies, according to the definitions of Φ and Ψ that $\Phi = \Psi$ and hence, (2.6) reduces to

$$f(z)\Phi(x, y) = f(x)\Phi(y, z). \tag{2.10}$$

If we substitute (2.7) back into (2.10) we find that $f(z)f(x)\phi(y) = f(x)f(y)\phi(z)$. Taking $x = z = z_0$ in the last equation and then dividing through by $f^2(z_0)$ we deduce that $\phi = \beta f$, where $\beta = \phi(z_0)/f(z_0)$. Hence, $\Phi(x, y) = f(x)\phi(y) = \beta f(x)f(y)$, from which we find, using the definition of Φ and that g is central, that

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y) + \beta f(x)f(y).$$

If we choose $\alpha \in \mathbb{C}$ such that $2\alpha^2 = \beta$, the last equation becomes

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y) + 2\alpha^2 f(x)f(y) \text{ for all } x, y \in S.$$

Hence (2.1) is proved for the case of σ being an automorphism.
Case 2: σ is an involutive automorphism.
Let x, y, z be in S. Using (1.3) we get

$$\begin{aligned}2f(z)(g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y)) &= [f(xyz) + \mu(z)f(xy\sigma(z)) - 2f(xy)g(z)]\\ &+ \mu(y)[f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) - 2f(x\sigma(y))g(z)] - 4f(z)g(x)g(y)\\ &= [f(xyz) + \mu(yz)f(x\sigma(y)\sigma(z))] + \mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)z)]\\ &- 2[f(xy) + \mu(y)f(x\sigma(y))]g(z) - 4f(z)g(x)g(y)\\ &= 2f(x)g(yz) + 2g(x)f(yz) + 2\mu(z)f(x)g(y\sigma(z)) + 2\mu(z)g(x)f(y\sigma(z))\\ &- 4f(x)g(y)g(z) - 4f(y)g(x)g(z) - 4f(z)g(x)g(y)\\ &= 2f(x)[g(yz) + \mu(z)g(y\sigma(z)) - 2g(y)g(z)]\\ &+ 2g(x)[f(yz) + \mu(z)f(y\sigma(z)) - 2f(y)g(z) - 2f(z)g(y)]\\ &= 2f(x)[g(yz) + \mu(z)g(y\sigma(z)) - 2g(y)g(z)].\end{aligned}$$

This can be written as follows

$$f(z)\Phi(x,y) = f(x)\Phi(y,z), \tag{2.11}$$

where $\Phi : S \times S \longrightarrow \mathbb{C}$ is defined by

$$\Phi(x,y) := g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y), \;\; x,y \in S.$$

By assumption $f \neq 0$ so there exists a $z_0 \in S$ such that $f(z_0) \neq 0$. Putting $z = z_0$ in (2.11) and dividing through by $f(z_0)$ we get that

$$\Phi(x,y) = f(x)h(y), \tag{2.12}$$

where $h(y) = \Phi(y,z_0)/f(z_0)$. Substituting (2.12) back into (2.11) we find that

$$f(z)f(x)h(y) = f(x)f(y)h(z) \tag{2.13}$$

for all $x,y,z \in S$. Since $f \neq 0$ it follows from (2.13) that $h = \beta f$ for some $\beta \in \mathbb{C}$.
Hence, using the definition of Φ and the new form of h the identity (2.12) can be written as follows

$$g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y) = \beta f(x)f(y)$$

for all $x,y \in S$. If we choose $\alpha \in \mathbb{C}$ such that $2\alpha^2 = \beta$, then the last equation becomes

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y) + 2\alpha^2 f(x)f(y) \text{ for all } x,y \in S. \tag{2.14}$$

Hence (2.1) is proved for the case of σ being an automorphism.
Now, multiplying (1.3) by $\pm\alpha$ and adding the result to the identity (2.1) we obtain that

$$(g+\alpha f)(xy) + \mu(y)(g+\alpha f)(x\sigma(y)) = 2(g+\alpha f)(x)(g+\alpha f)(y)$$

and

$$(g-\alpha f)(xy) + \mu(y)(g-\alpha f)(x\sigma(y)) = 2(g-\alpha f)(x)(g-\alpha f)(y),$$

for all $x,y \in S$. Thus both $\phi_1 := g+\alpha f$ and $\phi_2 := g-\alpha f$ satisfy d'Alembert's μ-functional equation (1.5) and we have $g = \frac{\phi_1+\phi_2}{2}$. If $\phi_1 \neq \phi_2$, then we get that $\alpha \neq 0$ because $f \neq 0$ by hypothesis and thus $f = \frac{\phi_1-\phi_2}{2\alpha}$.
If $\phi_1 = \phi_2$, then $\alpha = 0$ and $g = \phi_1 = \phi_2$ and the functional equation (1.3) reduces to

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\phi_1(y) + 2f(y)\phi_1(x) \text{ for all } x,y \in S,$$

where ϕ_1 satisfies d'Alembert's μ-functional equation (1.5). □

The following proposition gives the solutions of the sine addition formula on a semigroup generated by its squares.

Proposition 2.2. *Let S be a semigroup. Let $\mu : S \longrightarrow \mathbb{C}$ be a multiplicative function which satisfies $\mu(x^2) = 1$ for all $x \in S$. Let $f, g : S \longrightarrow \mathbb{C}$ satisfy the functional equation* (1.3) *with $\sigma = I$. Assume furthermore that $f \neq 0$. Then, there exist two functions $\phi_1, \phi_2 : S \longrightarrow \mathbb{C}$ which are solutions of the functional equation*

$$(1+\mu(y))\varphi(xy) = 2\varphi(x)\varphi(y), \;\; x,y \in S, \tag{2.15}$$

such that

$$g = \frac{\phi_1+\phi_2}{2}. \tag{2.16}$$

Additionally we have the following.

- ***(1)*** *If $\phi_1 \neq \phi_2$, then there exists $\alpha \neq 0$ such that $f = \frac{1}{2\alpha}(\phi_1 - \phi_2)$.*
- ***(2)*** *If $\phi_1 = \phi_2$, then $g = \phi_1 = \phi_2$ and f satisfies the functional equation*

$$(1+\mu(y))f(xy) = 2f(x)\phi_1(y) + 2f(y)\phi_1(x), \;\; x,y \in S. \tag{2.17}$$

Furthermore, if S is a semigroup generated by its squares, then $\mu(x) = 1$ for all $x \in S$ and there exist two multiplicative functions $\chi_1, \chi_2 : S \longrightarrow \mathbb{C}$ such that $g = \frac{\chi_1+\chi_2}{2}$ and we have the following.
If $\chi_1 \neq \chi_2$, then there exists $\alpha \neq 0$ such that

$$f = \frac{1}{2\alpha}(\chi_1 - \chi_2). \tag{2.18}$$

If $\chi_1 = \chi_2$, then $g = \chi_1 = \chi_2$ and $f = \chi_1 a$ on $S \backslash I_{\chi_1}$ and $f = 0$ on I_{χ_1}, where a is an additive function on $S \backslash I_{\chi_1}$.

Proof. With $\sigma = I$ in (1.3), all, except the statements of S being generated by its squares, is Proposition 2.1. To complete the proof, suppose S is a semigroup which is generated by its squares, and let $x \in S$. By assumption $x = x_1^2 x_2^2 ... x_n^2$ for some $x_1, x_2, ..., x_n \in S$. Since μ is multiplicative and $\mu(x^2) = 1$ for all $x \in S$ we get that $\mu(x) = \mu(x_1^2)\mu(x_2^2)...\mu(x_n^2) = 1$ for all $x \in S$. This proves that $\mu = 1$. Thus, equation (2.15) reduces to $\varphi(xy) = \varphi(x)\varphi(y)$, $x, y \in S$ which means that the two solutions ϕ_1, ϕ_2 of (2.15) become two multiplicative functions χ_1, χ_2. The case of $\chi_1 \neq \chi_2$ is obvious. If $\chi_1 = \chi_2$ then equation (2.17) reduces to the sine addition law

$$f(xy) = f(x)\chi_1(y) + f(y)\chi_1(x), \ x, y \in S.$$

From [4] we get the desired form of f. This completes the proof. □

3. Properties of the functional equation (1.4)

In this section we study the functional equation (1.4), i.e.,

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)f(y) + 2g(x)g(y), \ x, y \in S,$$

where $\sigma : S \longrightarrow S$ is an involutive morphism. Also for this functional equation no new phenomena occur when you pass from abelian groups to non-abelian semigroups.

Proposition 3.1. *Let the pair $(f, g) : S \longrightarrow \mathbb{C}$ constitute a solution of the functional equation* (1.4) *such that f and g are linearly independent. Then there exists a constant $\alpha \in \mathbb{C}$ such that*

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)f(y) + 2f(x)g(y) + \alpha g(x)g(y) \text{ for all } x, y \in S. \tag{3.1}$$

Proof. Replacing y in equation (1.4) by $\sigma(y)$ and multiplying the resulting identity by $\mu(y)$ and using that $\mu(y\sigma(y)) = 1$ for all $y \in S$ we get that

$$\mu(y)f(x\sigma(y)) + f(xy) = 2f(x)f^*(y) + 2g(x)g^*(y).$$

Subtracting this last equation from (1.4) we find that

$$f(x)[f(y) - f^*(y)] = g(x)[g^*(y) - g(y)], \text{ for all } x, y \in S. \tag{3.2}$$

Since f and g are linearly independent we deduce from (3.2) that $f^* = f$ and $g^* = g$.
Now we split the discussion into two cases.
Case 1: σ is an involutive automorphism.
Let x, y, z be in S. Making the substitutions (xy, z) and $(x\sigma(y), z)$ in equation (1.4) and multiplying the second result by $\mu(y)$ we get respectively that

$$f(xyz) + \mu(z)f(xy\sigma(z)) = 2f(xy)f(z) + 2g(xy)g(z),$$

$$\mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2\mu(y)f(x\sigma(y))f(z) + 2\mu(y)g(x\sigma(y))g(z).$$

Adding the two last identities and using that $\mu(z\sigma(z)) = 1$ for all $z \in S$ we obtain

$$\begin{aligned}[f(xyz) + \mu(yz)f(x\sigma(y)\sigma(z))] + \mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)z)] \\ = 2[f(xy) + \mu(y)f(x\sigma(y))]f(z) + 2[g(xy) + \mu(y)g(x\sigma(y))]g(z).\end{aligned} \tag{3.3}$$

On the other hand, if we substitute (x, yz) and $(x, y\sigma(z))$ in (1.4) and multiply the second result by $\mu(z)$ we get respectively that

$$f(xyz) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2f(x)f(yz) + 2g(x)g(yz),$$
$$\mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)z)] = 2\mu(z)f(x)f(y\sigma(z)) + 2\mu(z)g(x)g(y\sigma(z)).$$

Subtracting the sum of the two last equations from (3.3) we obtain

$$f(x)[f(yz) + \mu(z)f(y\sigma(z))] + g(x)[g(yz) + \mu(z)g(y\sigma(z))] = [f(xy) + \mu(y)f(x\sigma(y))]f(z) + [g(xy) + \mu(y)g(x\sigma(y))]g(z).$$

Using (1.4) twice we get that

$$g(x)[g(yz) + \mu(z)g(y\sigma(z)) - 2g(y)f(z) - 2g(z)f(y)] = g(z)[g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)f(y) - 2g(y)f(x)],$$

which can be written as follows

$$g(x)\Phi(y, z) = g(z)\Phi(x, y), \tag{3.4}$$

where $\Phi : S \times S \longrightarrow \mathbb{C}$ is defined by

$$\Phi(x, y) := g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)f(y) - 2g(y)f(x), \ \ x, y \in S.$$

Since $g \neq 0$, there exists x_0 such that $g(x_0) \neq 0$. Putting $x = x_0$ in (3.4) we obtain that

$$\Phi(y, z) = h(y)g(z), \tag{3.5}$$

where $h(y) := \Phi(x_0, y)/g(x_0)$ for all $y \in S$. If we substitute this back into (3.4) we get that $g(x)h(y)g(z) = g(z)h(x)g(y)$, which means that $g(x)h(y) = g(y)h(x)$ for all $x, y \in S$ since $g \neq 0$. Hence $h = \alpha g$, where $\alpha = h(x_0)/g(x_0)$. Going back to the definition of Φ and using equation (3.5) we find that

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)f(y) + 2f(x)g(y) + \alpha g(x)g(y) \text{ for all } x, y \in S,$$

which is the desired formula (3.1).
Case 2: σ is an involutive anti-automorphism.
The symmetry of the right hand side of (1.4) implies that

$$f(xy) + \mu(y)f(x\sigma(y)) = f(yx) + \mu(x)f(y\sigma(x)). \tag{3.6}$$

Since f is even and $\mu(x\sigma(x)) = 1$ for all $x \in S$ we have

$$\mu(x)f(y\sigma(x)) = \mu(x)f^*(y\sigma(x)) = \mu(x)\mu(y\sigma(x))f(x\sigma(y)) = \mu(y)f(x\sigma(y)).$$

Taking the last identity into account (3.6) reduces to $f(xy) = f(yx)$ for all $x, y \in S$, which means that f is central.
Now, let x, y, z be in S. Making the substitutions (xy, z) and $(x\sigma(y), z)$ in equation (1.4) and multiplying the second result by $\mu(y)$ we get respectively that

$$f(xyz) + \mu(z)f(xy\sigma(z)) = 2f(xy)f(z) + 2g(xy)g(z),$$

$$\mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2\mu(y)f(x\sigma(y))f(z) + 2\mu(y)g(x\sigma(y))g(z).$$

The sum of the two last equations gives, by help of (1.4) and using the centrality of f, that

$$\begin{aligned} f(xyz) + \mu(z)f(xy\sigma(z)) &+ \mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) \\ &= 2[f(xy) + \mu(y)f(x\sigma(y))]f(z) + 2[g(xy) + \mu(y)g(x\sigma(y))]g(z) \\ &= 4[f(x)f(y) + g(x)g(y)]f(z) + 2[g(xy) + \mu(y)g(x\sigma(y))]g(z). \end{aligned}$$

On the other hand, using (1.4) twice and taking the centrality of f into account we get that

$$\begin{aligned} f(xyz) + \mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) \\ &= [f(zxy) + \mu(y)f(zx\sigma(y))] + \mu(z)[f(\sigma(z)xy) + \mu(y)f(\sigma(z))x\sigma(y)] \\ &= 2f(zx)f(y) + 2g(zx)g(y) + 2\mu(z)f(\sigma(z)x)f(y) + 2\mu(z)g(\sigma(z)x)g(y) \\ &= 2[f(xz) + \mu(z)f(x\sigma(z))]f(y) + 2[g(zx) + \mu(z)g(\sigma(z)x)]g(y) \\ &= 4[f(x)f(z) + g(x)g(z)]f(y) + 2[g(zx) + \mu(z)g(\sigma(z)x)]g(y). \end{aligned}$$

It follows from the computation above that

$$[g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)f(y) - 2g(y)f(x)]g(z) = [g(zx) + \mu(z)g(\sigma(z)x) - 2g(x)f(z) - 2g(z)f(x)]g(y).$$

This can be written as follows

$$g(z)\Phi(x, y) = g(y)\Psi(x, z), \tag{3.7}$$

where $\Phi, \Psi : S \times S \longrightarrow \mathbb{C}$ are defined by

$$\Phi(x, y) := g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)f(y) - 2g(y)f(x), \ x, y \in S,$$

$$\Psi(x, y) := g(yx) + \mu(y)g(\sigma(y)x) - 2g(x)f(y) - 2g(y)f(x), \ x, y \in S.$$

By assumption $g \neq 0$, so there exists a $z_0 \in S$ such that $g(z_0) \neq 0$. Putting $z = z_0$ in (3.7) and dividing through by $g(z_0)$ we get that

$$\Phi(x, y) = h(x)g(y), \tag{3.8}$$

where $h(x) = \Psi(x, z_0)/g(z_0)$. Replacing Φ by its new form in (3.7) we get that for all $x, y, z \in S$ $g(z)g(y)h(x) = g(y)\Psi(x, z)$. Since $g \neq 0$ we deduce that $\Psi(x, z) = h(x)g(z) = \Phi(x, z)$, which means, according to the definitions of Φ and Ψ, that

$$g(xy) + \mu(y)g(x\sigma(y)) = g(yx) + \mu(y)g(\sigma(y)x).$$

This can be written as follows

$$g(xy) - g(yx) = \mu(y)[g(\sigma(y)x) - g(x\sigma(y))]. \tag{3.9}$$

Interchanging x and y in (3.9) and using that g is even and that $\mu(x\sigma(x)) = 1$ for all $x \in S$ we deduce that

$$\begin{aligned} g(yx) - g(xy) &= \mu(x)[g(\sigma(x)y) - g(y\sigma(x))] \\ &= \mu(x)[g^*(\sigma(x)y) - g^*(y\sigma(x))] \\ &= \mu(x)[\mu(\sigma(x)y)g(\sigma(y)x) - \mu(y\sigma(x))g(x\sigma(y))] \\ &= \mu(y)[g(\sigma(y)x) - g(x\sigma(y))]. \end{aligned}$$

We deduce from the above computation and from (3.9) that $g(xy) - g(yx) = 0$, which means that g is central. Now, according to the definition of Φ and by help of (3.8) we have

$$g(xy) + \mu(x)g(\sigma(x)y) = 2g(y)f(x) + 2g(x)f(y) + h(y)g(x) \text{ for all } x, y \in S. \tag{3.10}$$

If we interchange x and y in equation (3.10) we obtain that

$$g(yx) + \mu(y)g(\sigma(y)x) = 2g(x)f(y) + 2g(y)f(x) + h(x)g(y) \text{ for all } x, y \in S. \tag{3.11}$$

Subtracting (3.11) from (3.10) and taking into account that g is both central and even and that $\mu(x\sigma(x)) = 1$ for all $x \in S$ we obtain that

$$\begin{aligned} h(y)g(x) - h(x)g(y) &= \mu(x)g(\sigma(x)y) - \mu(y)g(\sigma(x)y) \\ &= \mu(x)g(\sigma(x)y) - \mu(y)g^*(\sigma(x)y) \\ &= \mu(x)g(\sigma(x)y) - \mu(y)\mu(\sigma(y)x)g(\sigma(x)y) = 0. \end{aligned}$$

Hence $g(y)h(x) = g(x)h(y)$ for all $x, y \in S$, which implies, since $g \neq 0$, that $h = \alpha g$, where $\alpha \in \mathbb{C}$. If we replace h by its new form in equation (3.10) we get that

$$g(xy) + \mu(x)g(\sigma(x)y) = 2g(x)f(y) + 2f(x)g(y) + \alpha g(x)g(y), \text{ for all } x, y \in S.$$

This completes the proof. □

The following proposition describes how the solutions of the functional equations (1.4) are built up by solutions of d'Alembert's-μ functional equation (1.5) and solutions of (1.6). The proof can be compared to the one by Stetkær [6].

Proposition 3.2. *The solutions $f, g \colon S \longrightarrow \mathbb{C}$ of the functional equation* (1.4) *can be listed as follows:*

(1) $f = g = 0$.

(2) There exist $\phi : S \longrightarrow \mathbb{C}$ which is a solution of d'Alembert's μ-functional equation (1.5)*, and $c \in \mathbb{C}\backslash\{\pm i\}$ such that*

$$f = \frac{1}{1+c^2}\phi \text{ and } g = \frac{c}{1+c^2}\phi.$$

(3) There exist two different solutions $\phi_1, \phi_2 : S \longrightarrow \mathbb{C}$ of d'Alembert's μ-functional equation (1.5)*, and a constant $\beta \in \mathbb{C}\backslash\{0, i, -i\}$ such that*

$$f = \frac{\beta\phi_2 + \frac{1}{\beta}\phi_1}{\beta + \frac{1}{\beta}} \quad \text{and} \quad g = \frac{\phi_2 - \phi_1}{\beta + \frac{1}{\beta}}.$$

(4) There exists a solution $\phi : S \longrightarrow \mathbb{C}$ of d'Alembert's μ-functional equation (1.5) *such that (g, ϕ) satisfies the functional equation* (1.6) *and $f = \phi + ig$ or $f = \phi - ig$.*

Proof. It is easy to see that each of the pairs (f, g) described in Proposition 3.2 is a solution of (1.4). So it is left to prove that any solution of (1.4) falls into one of the cases listed above.
If $f = 0$ then we get from (1.4) that $g = 0$, which is case (1), so we may assume from now on that $f \neq 0$. If f and g are linearly dependent, then there exists a constant $c \in \mathbb{C}$ such that $g = cf$. Substituting this into (1.4) we find that

$$f(xy) + \mu(y)f(x\sigma(y)) = 2(1 + c^2)f(x)f(y).$$

It follows from the last equation that $(1 + c^2)f$ is a solution of d'Alembert's μ-functional equation (1.5). If $1 + c^2 \neq 0$, then since $g = cf$ we are in case (2). If $1 + c^2 = 0$, then $c = \pm i$. Hence we are in case (4) for $\phi = 0$.
From now on we may assume that f and g are linearly independent. So, according to Proposition 3.1, g satisfies the formula (3.1). Now, combining (1.4) and (3.1), simple computation shows that for any $\beta \in \mathbb{C}$ we get

$$\phi(xy) + \mu(y)\phi(x\sigma(y)) = 2\phi(x)\phi(y) - (2\beta^2 + \alpha\beta - 2)g(x)g(y), \tag{3.12}$$

where $\phi := f - \beta g$. Let β_1 and β_2 be the two roots of the polynomial $2z^2 + \alpha z - 2$. Now, we get from (3.12) that $\phi_1 := f - \beta_1 g$ and $\phi_2 := f - \beta_2 g$ are solutions of d'Alembert's μ-functional equation (1.5).
If $\beta_1 \neq \beta_2$ then $\phi_1 \neq \phi_2$. In that case we obtain

$$f = \frac{\beta_1\phi_2 - \beta_2\phi_1}{\beta_1 - \beta_2} \text{ and } g = \frac{\phi_2 - \phi_1}{\beta_1 - \beta_2}$$

and since $\beta_1\beta_2 = -1$, this is case (3).
If $\beta_1 = \beta_2$ then $\phi_1 = \phi_2$ and so we see from $\beta_1\beta_2 = -1$ that either $\beta_1 = i$ or $\beta_1 = -i$. Let us consider the first case: $\beta_1 = i$ is the root of the polynomial $2z^2 + \alpha z - 2$ we get that $\alpha = -4i$. Noting that $\phi_1 = f - \beta_1 g = f - ig$ we get from (1.4) that

$$(\phi_1 + ig)(xy) + \mu(y)(\phi_1 + ig)(x\sigma(y)) = 2(\phi_1 + ig)(x)(\phi_1 + ig)(y) + 2g(x)g(y).$$

Using that ϕ_1 is a solution of d'Alembert's μ-functional equation (1.5) we get from the last equation after some reduction that g satisfies the functional equation

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)\phi_1(y) + 2g(y)\phi_1(x),$$

and this is case (4).
If $\beta_1 = -i$ then similar arguments show that we are in case (4). □

4. Solutions of the functional equation (1.8)

Throughout this section $\sigma : S \longrightarrow \mathbb{C}$ is an involutive anti-automorphism. The aim of this section is to determine the solutions of the functional equation (1.8), i.e.,

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\phi(y) + 2\psi(x)f(y), \ x, y \in S,$$

where ϕ and ψ are two non-zero different solutions of the d'Alembert's μ-functional equation (1.5). First, we treat the functional equation (4.1) which contains (1.8).

Lemma 4.1. *Let $f, m, g : S \longrightarrow \mathbb{C}$ be a solution of the functional equation*

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)m(y) + 2g(x)f(y), \ x, y \in S. \tag{4.1}$$

If f and g are linearly independent and $m \neq g^e$, where g^e is the even part of g, then there exists $\alpha \in \mathbb{C}^*$ such that $f = \alpha(m - g^e)$.

Proof. If we replace y in equation (4.1) by $\sigma(y)$ and multiply the resulting identity by $\mu(y)$ and use that $\mu(y\sigma(y)) = 1$ for all $y \in S$ we find that

$$f(x)[m^*(y) - m(y)] = g(x)[f(y) - f^*(y)]. \tag{4.2}$$

Since f and g are linearly independent we deduce from (4.2) that $f^* = f$ and $m^* = m$, which means that f and m are even.
Now, interchanging x and y in equation (4.1) we get

$$f(yx) + \mu(x)f(y\sigma(x)) = 2f(y)m(x) + 2g(y)f(x). \tag{4.3}$$

On the other hand, since f is even and $\mu(x\sigma(x)) = 1$ for all $x \in S$ we have

$$\mu(x)f(y\sigma(x)) = \mu(x)\mu(y\sigma(x))f(x\sigma(y)) = \mu(y)f(x\sigma(y)).$$

Subtracting equation (4.3) from (4.1) and taking the last equation into account we obtain that

$$f(xy) - f(yx) = 2f(x)[m(y) - g(y)] - 2f(y)[m(x) - g(x)]. \tag{4.4}$$

Replacing x by $\sigma(x)$ and y by $\sigma(y)$ in the last equation and multiplying the result by $\mu(xy)$ we find

$$f^*(yx) - f^*(xy) = 2f^*(x)[m^*(y) - g^*(y)] - 2f^*(y)[m^*(x) - g^*(x)]. \tag{4.5}$$

Since f and m are even equation (4.5) becomes

$$f(yx) - f(xy) = 2f(x)[m(y) - g^*(y)] - 2f(y)[m(x) - g^*(x)]. \tag{4.6}$$

The sum of (4.6) and (4.4) implies that

$$f(x)[2m(y) - g(y) - g^*(y)] = f(y)[2m(x) - g(x) - g^*(x)],$$

which can be written as follows

$$f(x)[m(y) - g^e(y)] = f(y)[m(x) - g^e(x)]. \tag{4.7}$$

By assumption $m \neq g^e$ so there exists $y_0 \in S$ such that $m(y_0) \neq g^e(y_0)$. Replacing y by y_0 in equation (4.7) we deduce that $f = \alpha(m - g^e)$, where $\alpha = f(y_0)/(m(y_0) - g^e(y_0))$. This completes the proof. □

The following lemma proves a kind of symmetry between the two functions ϕ and ψ in the functional equation (1.8).

Lemma 4.2. *Let the functions $f, m, g : S \longrightarrow \mathbb{C}$, with g even, satisfy the functional equation* (4.1). *If f and g are linearly independent then m satisfies d'Alembert's μ-functional equation* (1.5) *if, and only if g does.*

Proof. Let f be a non-zero solution of (4.1). The case of $m = g$ is obvious, so we can from now on assume that $m \neq g$. Since f and g are linearly independent and $g^e = g \neq m$, then according to Lemma 4.1 there exists $\alpha \in \mathbb{C}^*$ such that $f = \alpha(m - g)$. Substituting this back into (4.1) and simplifying by α we get

$$(m-g)(xy) + \mu(y)(m-g)(x\sigma(y)) = 2(m-g)(x)m(y) + 2g(x)(m-g)(y),$$

or equivalently

$$m(xy) + \mu(y)m(x\sigma(y)) - 2m(x)m(y) = g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y),$$

which proves obviously the desired equivalence. □

The following proposition gives the solutions of the functional equation (1.8).

Proposition 4.3. *If $\phi, \psi : S \longrightarrow \mathbb{C}$ are two non-zero different solutions of d'Alembert's μ-functional equation* (1.5)*, then the solutions $f : S \longrightarrow \mathbb{C}$ of the functional equation* (1.8) *are of the form $f = \alpha(\phi - \psi)$, where $\alpha \in \mathbb{C}$.*

Proof. We start by proving that any solution of (1.8) can be written as $f = \alpha(\phi - \psi)$ for some $\alpha \in \mathbb{C}$. Assume that f and ψ are linearly dependent. In that instance $f = c\psi$ for some $c \in \mathbb{C}$. Now ψ is as a solution of (1.5) both central and even (see [8, Proposition 9.17(a) and (b)]), and hence so is f. That implies that

$$\begin{aligned} f(xy) + \mu(y)f(x\sigma(y)) &= f(yx) + \mu(y)f^*(x\sigma(y)) \\ &= f(yx) + \mu(y)\mu(x\sigma(y))f(y\sigma(x)) = f(yx) + \mu(x)f(y\sigma(x)), \end{aligned}$$

which says that the left hand side of (1.8) is symmetric in x and y. Hence so is the right hand side, meaning that

$$f(x)\phi(y) + \psi(x)f(y) = f(y)\phi(x) + \psi(y)f(x),$$

so that

$$f(x)[\phi(y) - \psi(y)] = f(y)[\phi(x) - \psi(x)].$$

Since $\phi \neq \psi$ we see that $f = \alpha(\phi - \psi)$ for some $\alpha \in \mathbb{C}$.
Assume next that f and ψ are linearly independent. In this case the statement follows from Lemma 4.1. Now, we shall prove the converse. Let f be a function such that $f = \alpha(\phi - \psi)$ with $\alpha \in \mathbb{C}$. We have

$$\begin{aligned} f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)\phi(y) - 2\psi(x)f(y) &= \alpha(\phi(xy) - \psi(xy)) + \alpha\mu(y)(\phi(x\sigma(y)) - \psi(x\sigma(y))) \\ &\quad - 2\alpha(\phi(x) - \psi(x))\phi(y)] - 2\alpha\psi(x)(\phi(y) - \psi(y)) \\ &= \alpha(\phi(xy) + \mu(y)\phi(x\sigma(y)) - 2\phi(x)\phi(y)) \\ &\quad - \alpha(\psi(xy) + \mu(y)\psi(x\sigma(y)) - 2\psi(x)\psi(y)). \end{aligned}$$

Since both ϕ and ψ satisfy (1.5) we deduce that

$$f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)\phi(y) - 2\psi(x)f(y) = 0,$$

and thus, the converse is proved. □

Stetkær [6, Lemma 5] solved the functional equation

$$\int_K f(x + k \cdot y)dk = f(x) + \phi(x)f(y), \;\; x, y \in G, \tag{4.8}$$

where K is a compact transformation group of a topological abelian group $(G, +)$, acting by automorphisms on G. In the following Lemma we give the solutions of the functional equation (4.8) for $K = \mathbb{Z}_2$, in any semigroup which is not necessary abelian.

Lemma 4.4. *Let $f, \psi : S \longrightarrow \mathbb{C}$ constitute a solution of the functional equation*

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2\psi(x)f(y), \; x, y \in S, \tag{4.9}$$

such that $f \neq 0$ and ψ is even. Then ψ satisfies d'Alembert's functional equation

$$\psi(xy) + \psi(x\sigma(y)) = 2\psi(x)\psi(y), \; x, y \in S. \tag{4.10}$$

Furthermore, if $\psi \neq 0$ and $\psi \neq 1$, then $f = \alpha(\psi - 1)$, where $\alpha \in \mathbb{C}^*$.

Proof. $\psi = 0$ satisfies (4.10), so we may in the derivation of (4.10) assume that $\psi \neq 0$. If f and ψ are linearly dependent then there exists a constant $\lambda \in \mathbb{C}$ such that $\psi = \lambda f$ and equation (4.9) becomes

$$f(xy) + f(x\sigma(y)) = 2f(x)[1 + \lambda f(y)].$$

$\psi \neq 0$ gives that $\lambda \neq 0$. Multiplying the last equation by λ we get that

$$\psi(xy) + \psi(x\sigma(y)) = 2\psi(x)[1 + \psi(y)]. \tag{4.11}$$

If we interchange x and y in (4.11) we get

$$\psi(yx) + \psi(y\sigma(x)) = 2\psi(y)[1 + \psi(x)]. \tag{4.12}$$

Subtracting (4.12) from (4.11) and using that ψ is even we get

$$\psi(xy) - \psi(yx) = 2(\psi(x) - \psi(y)). \tag{4.13}$$

Replacing x by $\sigma(x)$ and y by $\sigma(y)$ in the last equation and using that ψ is even we obtain that

$$\psi(yx) - \psi(xy) = 2(\psi(x) - \psi(y)). \tag{4.14}$$

The sum of the two equations (4.14) and (4.13) gives that $\psi(x) = \psi(y)$ for all $x, y \in S$, which means that $\psi = c$, where $c \in \mathbb{C}^*$ is a constant. If we replace ψ by c in equation (4.11) we get that $2c = 2c[1 + c]$ which implies that $c = 0$, and hence $\psi = 0$. This contradicts the hypothesis that $\psi \neq 0$. Consequently, f and ψ are linearly independent. Since ψ is even and $m := 1$ satisfies d'Alembert's functional equation (4.10), we deduce, according to Lemma 4.2 that ψ satisfies d'Alembert's functional equation (4.10). Now, using that $\psi \neq 1$, we can deduce from Proposition 4.3 that $f = \alpha(\psi - 1)$ where $\alpha \in \mathbb{C}^*$. □

In Proposition 4.5 we give the solutions of (4.9) and those of a similar functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x)\phi(y) + f(y), \; x, y \in S, \tag{4.15}$$

where ϕ is a non-zero solution of (4.10).

Proposition 4.5. *If $\phi : S \longrightarrow \mathbb{C}$ is a non-zero solutions of d'Alembert's functional equation* (4.10) *such that $\phi \neq 1$, then the solutions $f : S \longrightarrow \mathbb{C}$ of the functional equations* (4.9) *and* (4.15) *are of the form $f = \alpha(\phi - 1)$, where $\alpha \in \mathbb{C}$.*

Proof. Since ϕ and $\psi := 1$ satisfy both d'Alembert's functional equation (4.10) and $\phi \neq 1$ then, we can apply Proposition 4.3 for $\mu = 1$ and then get the desired formula for the solutions of (4.9) and (4.15). □

It is well known that a multiplicative function, which is even, is a solution of d'Alembert's μ-functional equation (1.5). So we may choose, in equation (1.8), $\phi = \chi$ and $\psi = \nu$ where χ and ν are two non zero different multiplicative functions which are both even. If $\sigma = I$, equation (4.16) is analogous to [9, eq. (6)], although Corollary 4.6 treats just the case of σ being an anti-automorphism.

Corollary 4.6. *Let σ be an involutive anti-automorphism of S, and let $f : S \longrightarrow \mathbb{C}$ be a non-zero solution of the functional equation*

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)\chi(y) + 2\nu(x)f(y), \; x, y \in S, \tag{4.16}$$

where $\chi, \nu : S \longrightarrow \mathbb{C}$ are two non-zero multiplicative functions of S which are different and both even. Then the solutions of (4.16) *are of the $f = \alpha(\chi - \nu)$, where $\alpha \in \mathbb{C}^*$.*

5. Pexider functional equation

Stetkær [9] treated the Pexider functional equation

$$f(xy) = 2g_1(x)h_1(y) + 2\mu(x)h_2(y),\ \ x, y \in G,$$

where G is a group and $\mu : G \longrightarrow \mathbb{C}$ is a given character of the group G by using the solutions of the functional equation

$$f(xy) = 2f(x)\chi(y) + 2\mu(x)f(y),\ \ x, y \in G,$$

where $\chi, \mu : G \longrightarrow \mathbb{C}$ are two characters of G.
In this section we solve an analogous case: the functional equation (1.7), i.e.,

$$f(xy) + \mu(y)f(x\sigma(y)) = 2g_1(x)h_1(y) + 2\psi(x)h_2(y),\ \ x, y \in M,$$

where M is a monoid, $\psi : M \longrightarrow \mathbb{C}$ is a non-zero given solution of d'Alembert's μ-functional equation (1.5). We split the study into two cases: whether f and ψ are linearly independent or not. The proofs can be compared to the one by [9].

Proposition 5.1. *The solutions $f, g_1, h_1, h_2 : M \longrightarrow \mathbb{C}$ of equation* (1.7) *such that $f = c\psi$, where $c \in \mathbb{C}$ are*

- ***(1)*** $(f, g_1, h_1, h_2) = (c\psi, \delta, 0, c\psi)$ *with δ arbitrary.*
- ***(2)*** $(f, g_1, h_1, h_2) = (c\psi, c_1\psi, \gamma, c\psi - c_1h_1)$, *with $\gamma \neq 0$ arbitrary, $c_1 \in \mathbb{C}$.*

Proof. Using the hypothesis that $f = c\psi$ equation (1.7) becomes

$$c(\psi(xy) + \mu(y)\psi(x\sigma(y))) = 2g_1(x)h_1(y) + 2\psi(x)h_2(y).$$

Since ψ is a solution of (1.5) we find that

$$c\psi(x)\psi(y) = g_1(x)h_1(y) + \psi(x)h_2(y).$$

Hence

$$\psi(x)(c\psi(y) - h_2(y)) = g_1(x)h_1(y). \tag{5.1}$$

If $h_1 = 0$ we get obviously case (1). Assume that $h_1 \neq 0$. So there exists $y_0 \in M$ such that $h_1(y_0) \neq 0$. Taking $y = y_0$ in equation (5.1) we find that

$$g_1 = c_1\psi,\ \text{where } c_1 = \frac{c\psi(y_0) - h_2(y)}{h_1(y_0)}.$$

Substituting this back into equation (5.1) we find that $h_2 = c\psi - c_1h_1$ which is case (2). □

Proposition 5.2. *The solutions $f, g_1, h_1, h_2 : M \longrightarrow \mathbb{C}$ of equation* (1.7) *such that f is not proportional to ψ, and g_1 is even have, one of the following forms.*

(1)

$$f = \alpha\phi + (c_1 - \alpha)\psi,\ \ g_1 = \frac{1}{c}[\alpha\phi + (c_2 - \alpha)\psi],$$
$$h_1 = c\phi \text{ and } h_2 = (\alpha - c_2)\phi + (c_1 - \alpha)\psi,$$

where $\phi : M \longrightarrow \mathbb{C}$ is a solution of (1.5)*, and $c, c_1, c_2 \in \mathbb{C}$, with $c \neq 0$ and $\alpha \neq 0$, are constants.*

(2)

$$f = F + c_1\psi,\ \ g_1 = \frac{1}{c}[F + c_2\psi],$$
$$h_1 = c\psi \text{ and } h_2 = F + (c_1 - c_2)\psi,$$

where $F : M \longrightarrow \mathbb{C}$ is a solution of the functional equation

$$F(xy) + \mu(y)F(x\sigma(y)) = 2F(x)\psi(y) + F(y)\psi(x),\ \ x, y \in M,$$

and $c, c_1, c_2 \in \mathbb{C}$, with $c \neq 0$ are constants.

Proof. Since f and ψ are linearly independent we deduce that $F := f - f(e)\psi \neq 0$.
Let $x, y \in M$. According to the definition of F and using that ψ satisfies (1.5) we obtain that

$$\begin{aligned} F(xy) + \mu(y)F(x\sigma(y)) &= [f(xy) - f(e)\psi(xy)] + \mu(y)[f(x\sigma(y)) - f(e)\psi(x\sigma(y))] \\ &= f(xy) + \mu(y)f(x\sigma(y)) - 2f(e)\psi(x)\psi(y). \end{aligned}$$

By help of (1.7) we find that

$$F(xy) + \mu(y)F(x\sigma(y)) = 2g_1(x)h_1(y) + 2\psi(x)h_2(y) - 2f(e)\psi(x)\psi(y). \tag{5.2}$$

Now, if we put $y = e$ in equation (1.7) we get the formula

$$f = h_1(e)g_1 + h_2(e)\psi. \tag{5.3}$$

Since f and ψ are linearly independent we deduce from equation (5.3) that $h_1(e) \neq 0$. Hence

$$g_1 = \frac{1}{h_1(e)}f - \frac{h_2(e)}{h_1(e)}\psi. \tag{5.4}$$

Taking $x = e$ in equation (1.7) we get that

$$f(y) + \mu(y)f(\sigma(y)) = 2g_1(e)h_1(y) + 2\psi(e)h_2(y). \tag{5.5}$$

By help of (5.3) we can see easily that f is even since g_1 and ψ are both even by assumption. Thus equation (5.5) can be written as follows

$$h_2 = f - g_1(e)h_1. \tag{5.6}$$

Substituting g_1 and h_2 by their formulas (5.4) and (5.6) in (5.2) we obtain

$$\begin{aligned} F(xy) + \mu(y)F(x\sigma(y)) &= 2[\frac{f(x)}{h_1(e)} - \frac{h_2(e)}{h_1(e)}\psi(x)]h_1(y) + 2\psi(x)[f(y) - g_1(e)h_1(y)] - 2f(e)\psi(x)\psi(y) \\ &= 2[f(x) - f(e)\psi(x)]\frac{h_1(y)}{h_1(e)} + 2\psi(x)[f(y) - f(e)\psi(y)] - 2\frac{h_1(y)}{h_1(e)}\psi(x)[g_1(e)h_1(e) + h_2(e) - f(e)]. \end{aligned}$$

Thus

$$F(xy) + \mu(y)F(x\sigma(y)) = 2F(x)\frac{h_1(y)}{h_1(e)} + 2\psi(x)F(y), \tag{5.7}$$

since $f(e) = g_1(e)h_1(e) + h_2(e)$ (take $x = y = e$ in (1.7)). Defining $\phi := h_1/h_1(e)$ equation (5.7) reduces to

$$F(xy) + \mu(y)F(x\sigma(y)) = 2F(x)\phi(y) + 2\psi(x)F(y). \tag{5.8}$$

Since $F \neq 0$, ψ is a non-zero solution of (1.5) and f and ψ are linearly independent then, we deduce from Lemma 4.2 that ϕ in equation (5.8) is also a solution of (1.5). Hence F satisfies the functional equation (1.8). So we have two cases:
Case 1: If $\phi \neq \psi$ then, according to Proposition 4.3 $F = \alpha(\phi - \psi)$ where $\alpha \in \mathbb{C}^*$, so that

$$f = F + f(e)\psi = \alpha(\phi - \psi) + f(e)\psi.$$

Now, we put $c := h_1(e)$, $c_1 := f(e)$, $c_2 = g_1(e)h_1(e)$ and we get from (1.7) that $h_2(e) = c_1 - c_2$.
Using those constants we have

$$f = F + c_1\psi = \alpha(\phi - \psi) + c_1\psi = \alpha\phi + (c_1 - \alpha)\psi,$$

$$g_1 = (f - h_2(e)\psi)/h_1(e) = [(\alpha\phi + (c_1 - \alpha)\psi) - (c_1 - c_2)\psi]/c = [\alpha\phi + (c_2 - \alpha)\psi]/c$$

and

$$h_2 = (\alpha - c_2)\phi + (c_1 - \alpha)\psi.$$

Case 2: If $\phi = \psi$ then $f = F + f(e)\psi$, where F satisfies the functional equation (1.6).
Using the same constants in this case we have $f = F + c_1\psi$. We derive the expression of g_1 from (5.4)

$$g_1 = (f - h_2(e)\psi)/h_1(e) = [(F + c_1\psi - (c_1 - c_2)\psi)]/c = [F + c_2\psi]/c.$$

By help of equation (5.6) we can deduce the form of h_2

$$h_2 = f - g_1(e)h_1 = F + (c_1 - c_2)\psi.$$

This completes the proof. □

6. Solutions of equation (6.1) on a nilpotent group generated by its squares

In this section we study the solutions of the functional equation

$$f(xy) + \mu(y)f(xy^{-1}) = 2f(x)\phi(y) + 2f(y)\phi(x),\ \ x, y \in G, \tag{6.1}$$

where G is a nilpotent group which is generated by its squares and ϕ is a non-zero solution of d'Alembert's μ-functional equation

$$\phi(xy) + \mu(y)\phi(xy^{-1}) = 2\phi(x)\phi(y),\ \ x, y \in G. \tag{6.2}$$

Definition 6.1. The commutator $[x, y]$ between $x \in G$ and $y \in G$ is $[x, y] := xyx^{-1}y^{-1} \in G$, and let $[G, G]$ denote the subgroup of G generated by $\{[x, y]\ x, y \in G\}$.

For simplification we use the notation $\check{\chi}(x) := \chi(x^{-1})$ for $x \in G$.

Proposition 6.2. *Let G be a nilpotent group which is generated by its squares. Let $f, \phi : G \longrightarrow \mathbb{C}$ be a solution of the functional equation* (6.1)*. Then there exists a non-zero character $\chi : G \longrightarrow \mathbb{C}$ such that $\phi = \frac{\chi+\mu\check{\chi}}{2}$. Furthermore,*

(1) if $\chi \neq \mu\check{\chi}$ then f is abelian.

(2) if $\chi = \mu\check{\chi}$, then $f = \chi q$, where $q : G \longrightarrow \mathbb{C}$ satisfies the quadratic functional equation

$$q(xy) + q(xy^{-1}) = 2q(x) + 2q(y),\ \ x, y \in G.$$

Proof. Let $a, x, y \in G$, and define $F_a := f(ax) - f(a)\phi(x) - f(x)\phi(a)$. Using (6.1) we have

$$F_a(x) + \mu(x)F_a(x^{-1}) = f(ax) + \mu(x)f(ax^{-1}) - 2f(a)\phi(x) - 2f(x)\phi(a) = 0,$$

which means that F_a is odd.
By help of (6.1) and (6.2) we obtain that

$$\begin{aligned}F_a(xy) + \mu(y)F_a(xy^{-1}) &= f(axy) + \mu(y)f(axy^{-1}) - \phi(a)[f(xy) + \mu(y)f(xy^{-1})] - f(a)[\phi(xy) + \mu(y)\phi(xy^{-1})]\\ &= 2f(ax)\phi(y) + 2\phi(ax)f(y) - 2f(x)\phi(y)\phi(a) - 2f(y)\phi(x)\phi(a) - 2f(a)\phi(x)\phi(y)\\ &= 2[f(ax) - f(x)\phi(a) - f(a)\phi(x)]\phi(y) + 2f(y)[\phi(ax) - \phi(x)\phi(a)]\\ &= 2F_a(x)\phi(y) + 2f(y)[\phi(ax) - \phi(x)g(a)].\end{aligned}$$

Hence

$$F_a(xy) + \mu(y)F_a(xy^{-1}) = 2F_a(x)\phi(y) + 2f(y)\phi_a(x), \tag{6.3}$$

where $\phi_a(x) := \phi(ax) - \phi(a)\phi(x)$.
On the other hand, since ϕ satisfies d'Alembert's functional equation (6.2) and G is a nilpotent group which is generated by its squares then, we deduce from [10, Theorem 7.1] that ϕ is abelian. So there exists a character $\chi : G \longrightarrow \mathbb{C}$

such that $\phi = (\chi + \mu\check{\chi})/2$. Here we split the discussion in two cases.
The case of $\chi = \mu\check{\chi}$ is simple. Assume from now on that $\chi \neq \mu\check{\chi}$. Hence for any $a, x \in G$ we have

$$\begin{aligned}\phi_a(x) &= \phi(ax) - \phi(a)\phi(x)\\ &= \frac{\chi(ax) + \mu(ax)\check{\chi}(ax)}{2} - \frac{(\chi(a) + \mu(a)\check{\chi}(a))(\chi(x) + \mu(x)\check{\chi}(x))}{4}\\ &= \frac{(\chi(a) - \mu(a)\check{\chi}(a))}{2}\frac{(\chi(x) - \mu(x)\check{\chi}(x))}{2}\\ &= N(a)N(x),\end{aligned}$$

where $N := (\chi - \mu\check{\chi})/2$. Now, let c be an element of $[G, G]$. We have $\phi(c) = 1$ and $N(c) = 0$ because $\chi(c) = \chi(xyx^{-1}y^{-1}) = \chi(e) = 1$ for some $x, y \in G$. Hence

$$\phi_c(x) = N(c)N(x) = 0 \cdot N(x) = 0 \text{ for all } x \in G.$$

Thus equation (6.3) becomes

$$F_c(xy) + \mu(y)F_c(xy^{-1}) = 2F_c(x)\phi(y). \tag{6.4}$$

Since $f \neq 0$ and ϕ is abelian and not a character we deduce from [10, Theorem 8.1] that

$$F_c(x) = \alpha(c)\frac{\chi(x) - \mu(x)\check{\chi}(x)}{2} + \beta(c)\frac{\chi(x) + \mu(x)\check{\chi}(x)}{2} = \alpha(c)N(x) + \beta(c)\phi(x),$$

where $\alpha(c)$ and $\beta(c)$ are depending on c. Since F_c is odd we deduce that $\beta(c) = 0$, which implies that

$$F_c(x) = \alpha(c)N(x), \tag{6.5}$$

for all $x \in G$ and all $c \in [G, G]$ such that $F_c \neq 0$. Here we can extend the expression (6.5) of F_c to all $c \in [G, G]$ by taking $\alpha(c) = 0$ if $F_c = 0$. So by help of the definition of F_c and using that $\phi = 1$ on $[G, G]$, equation (6.5) becomes

$$f(cx) = f(c)\phi(x) + f(x) + \alpha(c)N(x) \tag{6.6}$$

for all $x \in G$ and for all $c \in [G, G]$. Let $b \in [G, G]$ and $x \in G$. Since $\chi(bx) = \chi(b)\chi(x) = \chi(x)$ and $\mu(bx) = \mu(b)\mu(x) = 1$ we have $\phi(bx) = \phi(x)$ and $N(bx) = N(x)$. Hence, replacing x in (6.6) by bx we obtain

$$f(cbx) = f(c)\phi(bx) + f(bx) + \alpha(c)N(bx) = f(c)\phi(x) + f(bx) + \alpha(c)N(x).$$

If we replace c in (6.6) by b we find that

$$f(bx) = f(b)\phi(x) + f(x) + \alpha(b)N(x).$$

Combining the two last equations we find that

$$f(cbx) = [f(c) + f(b)]\phi(x) + f(x) + [\alpha(b) + \alpha(c)]N(x). \tag{6.7}$$

On the other hand, if we replace c in (6.6) by cb we find

$$f(cbx) = f(cb)\phi(x) + f(x) + \alpha(cb)N(x). \tag{6.8}$$

Hence we derive from (6.7) and (6.8) that

$$[f(cb) - f(b) - f(c)]\phi(x) = [\alpha(b) + \alpha(c) - \alpha(cb)]N(x). \tag{6.9}$$

Since $\chi \neq 0$ and $\chi \neq \mu\check{\chi}$ we deduce that ϕ and N are linearly independent. So equation (6.9) implies that

$$f(cb) = f(c) + f(b), \quad \text{and } \alpha(bc) = \alpha(b) + \alpha(c)$$

for all $c, b \in [G, G]$. We see from those two last equations that f and α are additive on $[G, G]$. So we may use the notation A in stead of α. Hence we have

$$f(cx) = f(c)g(x) + A(c)N(x) + f(x), \text{ for all } x \in G, \text{ and all } c \in [G, G]. \tag{6.10}$$

Now, since f is additive on $[G, G]$ we have $f(c^2) = 2f(c)$. On the other hand, we find that $f(c^2) = 4f(c)g(c) = 4f(c)$ by taking $x = y = c$ in equation (6.1). Thus $f(c) = 0$ for all $c \in [G, G]$. Consequently equation (6.10) reduces to

$$f(cx) = f(x) + A(c)N(x),\ x \in G,\ c \in [G, G]. \tag{6.11}$$

Replacing x by x^{-1} and c by c^{-1} in equation (6.11) and multiplying the resulting equation by $\mu(x)$ we obtain

$$\mu(x)f(c^{-1}x^{-1}) = \mu(x)f(x^{-1}) + A(c^{-1})\mu(x)N(x^{-1}).$$

Since f is even and N is odd we deduce that

$$f(xc) = f(x) - A(c^{-1})N(x).$$

Substituting the last equation from (6.11) and using that f is central we find that

$$[A(c) + A(c^{-1})]N(x) = 0,$$

which implies that $A(c^{-1}) = -A(c)$ for all $c \in [G, G]$ since $N \neq 0$.
Replacing x in (6.11) by yxy^{-1} and using that f and N are central we get that

$$f(cyxy^{-1}) = f(yxy^{-1}) + A(c)N(yxy^{-1}) = f(x) + A(c)N(x). \tag{6.12}$$

On the other hand, using that f is central and that $y^{-1}cy \in [G, G]$ and applying (6.11) we have

$$f(cyxy^{-1}) = f(y^{-1}cyx) = f(x) + A(y^{-1}cy)N(x). \tag{6.13}$$

Since $N \neq 0$ we deduce from the substraction of (6.12) from (6.13) that

$$A(y^{-1}cy) = A(c) \text{ for all } y \in G \text{ and } c \in [G, G].$$

Now, let $x, y \in G$. Replacing c in equation (6.11) by $[x, y]$ and x by y we find that

$$f([x, y]y) = f(y) + A([x, y])N(y). \tag{6.14}$$

Using that f is central we get that $f([x, y]y) = f(xyx^{-1}) = f(y)$. Thus equation (6.14) reduces to

$$f(y) = f(y) + A([x, y])N(y),$$

which implies that $A[x, y]N(y) = 0$. Here we discuss two cases: If $N(y) \neq 0$ then $A([x, y]) = 0$. If $N(y) = 0$ then $\chi(y) = \mu\check{\chi}(y)$ which means that $y \in [G, G]$. Since A is additive on $[G, G]$ we get

$$A([x, y]) = A(xyx^{-1}y^{-1}) = A(xyx^{-1}) + A(y^{-1}) = A(y) + A(y^{-1}) = 0$$

because $A(y^{-1}) = -A(y)$ for all $y \in [G, G]$. So $A([x, y]) = 0$ for all $x, y \in G$. Hence equation (6.11) reduces to

$$f(cx) = f(x) \text{ for all } c \in [G, G] \text{ and } x \in G.$$

Now let $x, y, z \in G$. If we replace c in the last equation by $[x, y]$ and x by yxz we get

$$f([x, y]yxz) = f(yxz).$$

Since $[x, y]yxz = xyz$ we deduce that

$$f(xyz) = f(yxz) \text{ for all } x, y, z \in G.$$

Consequently f is abelian. This completes the proof. □

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