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Some examples of application of the operator ${}_zO^{\alpha}_{\beta}$ to special functions, in particular to the Christoffel-Darboux identity for orthogonal polynomials

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Abstract

In the field of special functions, the theory relating to sequences of orthogonal polynomials functions and multiple orthogonal polynomials is fundamental. There are several formulas and many useful applications in mathematical physics, numerical analysis, statistics and probability and in many other disciplines. For example, the well-known identity of Christoffel-Darboux has generated a large number of research articles. Recently, Tygert (*Analogues for Bessel functions of Christoffel-Darboux identity*, Research Report Yaleu/Dcs/Rr-1351, 1–8, 2006) obtained two similar new identities for Bessel functions according to the well-known Christoffel-Darboux formula. This identity has even been generalized for orthogonal matrix polynomials (E. Daems and A. B. J. Kuijlaars, *A Christoffel–Darboux formula for multiple orthogonal polynomials*, J. Approx. Theory 130, 188–200, 2004). In this article, we obtain several summation formulas involving the classical orthogonal polynomials (Sections 7 and 8) using the well-balanced fractional operator defined in terms of fractional derivative, ${}_zO^{\alpha}_{\beta}f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}z^{1-\beta}D^{\alpha-\beta}_z z^{\alpha-1}f(z)$ (Section 2). This operator has several operational properties (Section 3) and it has already been used in several papers involving special functions, for example obtaining several new higher-order transformations of the Gaussian hypergeometric function (R. Tremblay, *New quadratic transformations of hypergeometric functions and associated summation formulas obtained with the well-poised fractional calculus operator*, Montes Taurus J. Pure Appl. Math. 2 (1), 36–62, 2020 and R. Tremblay, S. Gaboury, *Well-posed fractional calculus: obtaining new transformations formulas involving Gauss hypergeometric functions with rational quadratic, cubic and higher degree arguments*, Math. Methods Appl. Sci. 41 (13), 4967–4985, 2018). Firstly, we demonstrate unequivocally using examples the efficiency of the fractional operator ${}_zO^{\alpha}_{\beta}f(z)$ to generate new relations involving the special functions of one or more variables. As for the fractional derivative, these functions can be represented in several forms using this operator (Section 4 and Tables A.1, A.2, A.3, A.4). Second, this operator also offers the possibility of discovering new avenues of research. We prove this in Section 5 and Section 6 where we obtain new formulas involving the generalized hypergeometric function and an extension of the generalized Bernoulli polynomials. Thirdly, we apply the operator ${}_zO^{\alpha}_{\beta}f(z)$ to the classical Christoffel-Darboux identity and we obtain two general summation formulas for orthogonal polynomials (Theorem 7.1 and Corollary 7.2). Section 7 explicitly explores these formulas for the main orthogonal polynomials. Finally, a generalization of the formulas for any family of functions satisfying a three-term symmetric recurrence formula is given in Section 8.

Keywords: Fractional derivatives, Well-Poised fractional calculus operator, special functions, generalized hypergeometric functions, orthogonal polynomials, Christoffel-Darboux identity, Bessel functions, generalized Bernoulli polynomials, summation formulas

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1. Introduction

The fractional derivative of arbitrary order α ($\alpha \in \mathbb{C}$), is an extension of the familiar nth derivative $D^n_{g(z)}F(z) = d^nF(z)/d(g(z))^n$ of the function $F(z)$ with respect to $g(z)$ to non-integral values of n and is denoted by $D^\alpha_{g(z)}F(z)$. We can find many surveys and discussions on several of these approaches in texts on the fractional calculus [20, 22, 35, 36]. The interest of this operator lies in the fact that most of the fundamental results of differential calculus have been generalized to fractional order, such as Taylor's series [23, 25, 26], Lagrange's expansion [25], the Leibniz rule [10, 23, 25, 28, 31, 32], the chain rule and applications [24, 37, 38] , the $D^aD^b = D^{a+b}$ law of composition and others [27, 29, 42, 44]. Other new formulas have been added. For example, the expansion of an analytic function in terms of powers of the quadratic function $(z-a)(z-b)$ generalizing the Taylor's series (see [43, Theorem 3.1] for definitions and conditions on the parameters)

$$\sum_{k\in K} a^{-1}\omega^{-\gamma k}\Big[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^\alpha\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^\beta - e^{i\pi(\alpha-\beta)}\frac{sin(\alpha+a-\gamma)\pi}{sin(\beta+a-\gamma)\pi} \tag{1.1}$$
$$f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^\alpha\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^\beta\Big]$$
$$= \sum_{-\infty}^{\infty}\frac{sin(\beta-an-\gamma)\pi}{sin(\beta+a-\gamma)\pi}e^{-i\pi a(n+1)}\theta(z)^{an+\gamma}\frac{D^{-\alpha+an+\gamma}_{z-z_2}}{\Gamma(1-\alpha+an+\gamma)}\Big[(z-z_2)^{\beta-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)f(z)\Big]\Big|_{z=z_1},$$

where $\theta(z) = zq(z)$ be a given function such that $q(z)$ is regular and univalent function,

$$\Delta_k = (z_1-z_2)^2 + 4V\left(\theta(z)\omega^k\right) \tag{1.2}$$

and

$$V(z) = \sum_{r=1}^{\infty} D^{r-1}_z\left(q(z)^{-r}\right)\Big|_{z=0} z^r/r!\,; \tag{1.3}$$

and the transformation formula [45]

$$D^\alpha_{g(z)}g(z)^pf(z) = \frac{\Gamma(1+p)}{\Gamma(-\alpha)}D^{-p-1}_{g(z)}g(z)^{-\alpha-1}f(g^{-1}(g(w)-g(z)))\Big|_{w=z} \tag{1.4}$$

which has found several applications [11, 13].

Several particular cases of (1.1) are discussed in [43], as for example, if $z_1 \neq z_2$, $\gamma = 0$ and $q(z) = 1$, we obtain the following interesting formula

$$(z-z_1)^\alpha(z-z_2)^\beta f(z) = \sum_{-\infty}^{\infty} a_n(z-z_1)^n(z-z_2)^n, \tag{1.5}$$

where

$$\begin{aligned} a_n &= \frac{D^{-\alpha+n}_{z_1-z_2}}{\Gamma(1-\alpha+n)}\Big[f(z_1)(z_1-z_2)^{\beta-n-1}(z_1-z_2+z-w)\Big]\Big|_{w=z_1} \\ &= \frac{D^{-\alpha+n}_{z_1-z_2}}{\Gamma(1-\alpha+n)}\Big[f(z_1)(z_1-z_2)^{\beta-n}\Big] + (z-z_1)\frac{D^{-\alpha+n}_{z_1-z_2}}{\Gamma(1-\alpha+n)}\Big[f(z_1)(z_1-z_2)^{\beta-n-1}\Big]. \end{aligned} \tag{1.6}$$

1.1. Preliminaries

Several representations of this operator have been proposed, each of which is valid in a specific domain and has appropriate restrictions on the parameters. However the relatively less restrictive representation of the fractional derivative according to parameters appears to be the one based on the Pochhammer's contour integral introduced by Tremblay [16, 17].

Definition 1.1. Let $f(z)$ be analytic in a simply connected region of $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Let $F(a) = f(a)g(a)^p(g(a) - g(z))^{-\alpha-1}$ denote the principal value. Then if α is not a negative integer, p is not an integer, and $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ to be

$$D^{\alpha}_{g(z)} \{[g(z)]^p f(z)\} = \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}\, d\xi. \tag{1.7}$$

For non-integers α and p, the functions $g(\xi)^p$ and $[g(\xi) - g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both branches pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = C_1 \cup C_2 \cup C_3 \cup C_4$ at any other point as shown in Figure 1. Here $F(a)$ denotes the principal value of the integrand in (1.7) at the beginning and the ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

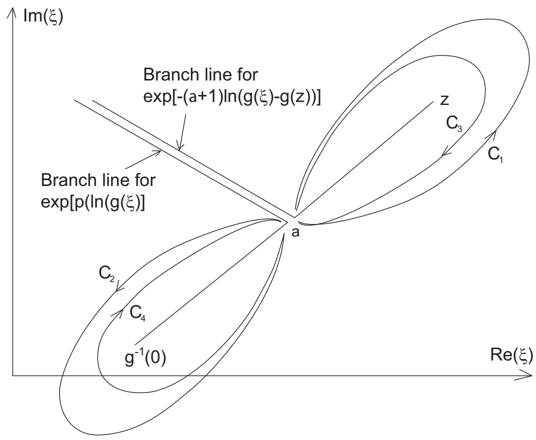

Figure 1. The contour used in integral (1.7)

2. The well poised fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$

Now let us recall the well-posed fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$. The operator ${}_{g(z)}O^{\alpha}_{\beta}$ has been introduced by Tremblay [39] and is defined in terms of fractional calculus operator $D^{\alpha}_{g(z)}$ as

$${}_{g(z)}O^{\alpha}_{\beta}\, f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, g(z)^{1-\beta}\, D^{\alpha-\beta}_{g(z)}\, g(z)^{\alpha-1}\, f(z). \tag{2.1}$$

It seems to be simply a rewrite of the fractional derivative. However, it is motivated because of it has an important list of easy-to-demonstrate properties that simplify calculations while having simpler analytical properties than the fractional derivative itself. It is more appropriate to explore special functions as it has already been demonstrated in [11, 12, 39, 44]. These are the reasons why we called it 'well poised fractional operator'.

The operator ${}_{g(z)}O^{\alpha}_{\beta}$ possesses the following integral representation with regard of Pochhammer's integral contour.

Definition 2.1. Let $f(z)$ be analytic in a simply connected region of $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Let $F(a) = f(a)g(a)^{\alpha-1}(g(a) - g(z))^{\beta-\alpha-1}$ denote the principal value. Then if $\alpha \neq 1, 2, \ldots, \beta - \alpha \neq 1, 2, \ldots, \beta \neq 0, -1, -2 \ldots$ and $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$ with parameters α and β on $f(z)$ with respect to $g(z)$ to be

$${}_{g(z)}O^{\alpha}_{\beta} f(z) = -\frac{e^{-i\pi\beta}\Gamma(\beta)}{4\pi\, \sin(\pi\alpha)\sin(\pi(\beta-\alpha))} \int_{C(z^+,g^{-1}(0)^+,z^-,g^{-1}(0)^-;F(a),F(a))} \frac{f(\xi)\,[g(\xi)]^{\alpha-1}\, g'(\xi)}{[g(\xi)-g(z)]^{\alpha-\beta+1}}\, d\xi. \tag{2.2}$$

Remark 2.2. The restriction on the parameters in Definition 2.1 derive essentially from the product of the three gamma functions $\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1+\alpha-\beta)$ in (2.2). However, as mentioned in Definition 2.1 of the fractional derivative, integrating by parts N times the integral in (2.2) two different ways, we can show that $\alpha = 1, 2, \ldots$, and $\beta - \alpha = 1, 2, \ldots$ are removable singularities [17]. Moreover, the operator $\frac{{}_{g(z)}O^{\alpha}_{\beta}}{\Gamma(\beta)}$ no longer has the restriction on the parameter β and becomes well defined for all values of α and β. Furthermore, the function $f(z)$ can have an essential singularity at $z = g^{-1}(0)$. The cases where α or/and β are negative integers or nulls have been studied in detail in [39].

An important feature of Definition 2.1 is the symmetry of the contour of Pochhammer used around points of singularities $g^{(-1)}(0)$ and z. By a simple change of variables $\zeta = z - \xi$ in (2.2), the author ([39], see also [45]) deduces the following transformation formula for the fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$.

Theorem 2.3. *Let $f(z)$ be a function that satisfies the conditions for the existence of the fractional derivative ${}_{g(z)}O^{\alpha}_{\beta}f(z)$ listed in Definition* 2.1 *and using a Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ laid out around the points $\xi = g^{-1}(0)$ and z (see Figure* 1*). If $f(g^{-1}(0)) \neq 0$ and $\beta \neq 0, -1, -2, \ldots$ then we have*

$$ {}_{g(z)}O^{\alpha}_{\beta}f(z) = {}_{g(z)}O^{\beta-\alpha}_{\beta} f(g^{-1}(g(w) - g(z)))\Big|_{w=z} \tag{2.3}$$

for $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$. Note that we must have $w \to z$ in the right side of (2.3) *after evaluation of the fractional derivative, the point w must be near the point z inside the loop C_3.*

In terms of fractional derivative, the transformation (2.3) becomes (1.4).

3. Some properties of the well-posed fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$

In this section, we give some of the important properties of the fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$ introduced at the first time by the author [39]. We chose to simply list them since the proofs are readily obtainable.

1) Linearity

$$ {}_{g(z)}O^{\alpha}_{\beta}\{\lambda_1 f(z) + \lambda_2 h(z)\} = \lambda_1\, {}_{g(z)}O^{\alpha}_{\beta}f(z) + \lambda_2\, {}_{g(z)}O^{\alpha}_{\beta}h(z), \tag{3.1}$$

where λ_1 and λ_2 are arbitrary complex numbers.

2) Invariability

$$ {}_{\lambda g(z)}O^{\alpha}_{\beta}f(z) = {}_{g(z)}O^{\alpha}_{\beta}f(z), \tag{3.2}$$

where $\lambda \neq 0$ is an arbitrary complex number.

3) Identity

$$ {}_{g(z)}O^{\alpha}_{\alpha} = I. \tag{3.3}$$

4) Reductions

$$ {}_{g(z)}O^{\alpha}_{\beta}\ {}_{g(z)}O^{\beta}_{\gamma} = {}_{g(z)}O^{\alpha}_{\gamma}, \tag{3.4}$$

$$ {}_{g(z)}O^{\alpha}_{\beta}\ {}_{g(z)}O^{\gamma}_{\alpha} = {}_{g(z)}O^{\gamma}_{\beta}. \tag{3.5}$$

5) Elementary cases

$$ {}_{g(z)}O_{\beta}^{\alpha}\, 1 = 1, \tag{3.6} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^n = \frac{(\alpha)_n}{(\beta)_n}\, [g(z)]^n, \tag{3.7} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(w) - g(z)]^n \Big|_{w=z} = \frac{(\beta-\alpha)_n}{(\beta)_n}\, [g(z)]^n, \tag{3.8} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^n [g(w) - g(z)]^m \Big|_{w=z} = \frac{(\alpha)_n(\beta-\alpha)_m}{(\beta)_{n+m}}\, [g(z)]^{n+m}, \tag{3.9} $$

where m and n are integers and $(\lambda)_k$ holds for the Pochhammer symbol defined, in terms of the Gamma function, by

$$ (\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}; \lambda \in \mathbb{C}) \\ 1 & (k = 0; \lambda \in \mathbb{C} \setminus \{0\}). \end{cases} $$

6) Useful cases

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^{\lambda} f(z) = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)}\, [g(z)]^{\lambda}\, {}_{g(z)}O_{\beta+\lambda}^{\alpha+\lambda}\, f(z), \tag{3.10} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(w) - g(z)]^{\theta} \Big|_{w=z} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha+\theta)}{\Gamma(\beta-\alpha)\Gamma(\beta+\theta)}\, [g(z)]^{\theta}\, {}_{g(z)}O_{\beta+\theta}^{\alpha}\, f(z). \tag{3.11} $$

7) Combined case

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^{\lambda} [g(w) - g(z)]^{\theta} f(z) \Big|_{w=z} = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)\Gamma(\beta-\alpha+\theta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\beta+\lambda+\theta)}\, [g(z)]^{\lambda+\theta}\, {}_{g(z)}O_{\beta+\lambda+\theta}^{\alpha+\lambda}\, f(z). \tag{3.12} $$

8) Commutativity

$$ {}_{g(z)}O_{\beta}^{\alpha}\, {}_{g(z)}O_{\theta}^{\delta} = {}_{g(z)}O_{\theta}^{\delta}\, {}_{g(z)}O_{\beta}^{\alpha}, \tag{3.13} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\; g(z)^{\gamma}\; {}_{g(z)}O_{\theta}^{\delta} = \frac{\Gamma(\beta)\Gamma(\theta)\Gamma(\alpha+\gamma)\Gamma(\delta-\gamma)}{\Gamma(\alpha)\Gamma(\delta)\Gamma(\beta+\gamma)\Gamma(\theta-\gamma)}\, {}_{g(z)}O_{\theta-\gamma}^{\delta-\gamma}\; g(z)^{\gamma}\; {}_{g(z)}O_{\beta+\gamma}^{\alpha+\gamma}. \tag{3.14} $$

9) Effect on a hypergeometric function

$$ {}_{g(z)}O_{\beta}^{\alpha}\; {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| g(z)\right] = {}_{p+1}F_{q+1}\left[\begin{matrix} \alpha, a_1, a_2, \cdots, a_p \\ \beta, b_1, b_2, \cdots, b_q \end{matrix}\middle| g(z)\right], \tag{3.15} $$

where

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| g(z)\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdot (b_2)_n \cdots (b_q)_n} \cdot \frac{[g(z)]^n}{n!}. $$

10) Special cases

$$ {}_zO_{\beta}^{-n}\, f(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{(k)}(0) \frac{z^k}{(\beta)_k}, \tag{3.16} $$

$$ {}_zO_{\alpha}^{\alpha+n}\, f(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0) \frac{z^k}{(\alpha)_k}, \tag{3.17} $$

$$ {}_zO_{-m}^{-n}\, f(z) = \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{m}{k}} f^{(k)}(0) \frac{z^k}{k!} + \lim_{\epsilon\to 0} \sum_{k=m+1}^{\infty} \frac{(-n+\epsilon)_k}{(-m+\epsilon)_k} f^{(k)}(0) \frac{z^k}{k!} \qquad m \geq n. \tag{3.18} $$

11) Particular transformations

$$
\begin{aligned}
{}_{\frac{z}{1-z}}O^{\alpha}_{\beta}\, f(z) &= (1-z)^{\alpha}\, {}_zO^{\alpha}_{\beta}\,(1-z)^{-\beta} f(z), && (3.19)\\
{}_{\frac{az+b}{cz+d}}O^{\alpha}_{\beta}\, f(z) &= (\frac{a}{ad-bc})^{\alpha-\beta}\,(cz+d)^{\alpha}\ {}_{az+b}O^{\alpha}_{\beta}\,(cz+d)^{-\beta} f(z), && (3.20)\\
{}_zO^{\alpha}_{\beta}\,(1-z)^{\theta} f(z) &= (1-z)^{\beta-\alpha+\theta}\, {}_zO^{\beta+\theta}_{\beta}\,(1-z)^{\alpha-\beta}\, {}_zO^{\alpha}_{\beta+\theta}\, f(z). && (3.21)
\end{aligned}
$$

12) Logarithmic functions:

$$
{}_zO^{\alpha}_{\beta}\,(w-z)^{-\gamma}(ln(z))^{\delta}(ln(w-z))^{\theta}\Big|^{*}_{w=z} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha-\gamma)}{\Gamma(\beta-\alpha)\Gamma(\beta-\gamma)} \tag{3.22}
$$
$$
\Big\{\big[\psi(\alpha)-\psi(\beta-\gamma)+\ln(z)\big]^{\delta}\big[\psi(\beta-\alpha-\gamma)-\psi(\beta-\gamma)+\ln(z)\big]^{\theta}-\delta\theta\psi'(\beta-\gamma)\Big\}
$$

with δ, $\theta = 0$ or 1 and $\psi(z) = \Gamma(z)' / \Gamma(z)$ is the Psi (or Digamma) function.

Let's mention that ${}_{g(z)}O^{\alpha}_{\beta}\left((g(z))^{p}((g(z)-g(w))^{\theta}(ln(g(z))^{\delta} f(z)\right)\big|_{w=z}$ (with δ, $\theta = 0$ or 1) has many more interesting properties and applications. The analytical properties with respect to parameters α, β, p and z were studied in depth in [16, 39] where several kinds of applications to special functions have been explored. An article about the more general operator ${}_{g(z)}O^{\alpha}_{\beta}F(w,z)\big|_{w=z}$ where $F(w,z) = ((g(z))^{p}(g(w)-g(z))^{q}\{\ln(g(z))\}^{\theta}\{\ln(g(w))-g(z)\}^{\delta} f(z)$ is in preparation. The effectiveness of this operator to study special functions and to facilitate the obtaining of new hypergeometric transformations is illustrated in the next sections.

4. Representation of special functions with the well poised fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$

One finds in the literature a large number of articles developing the concept of fractional derivative, let us quote for example [4]-[7], [15, 19], [21]-[31]. This operator has been widely used to study the properties of special functions in mathematical physics. Most of these functions can be written in terms of a fractional derivative acting on a more elementary function to facilitate calculations. Due to its easy-to-use properties, the ${}_{g(z)}O^{\alpha}_{\beta}$ operator is a powerful tool for studying special functions because it further simplifies calculations while offering new avenues of research.

Let's give some examples. Using the property (3.7) with $g(z) = z$, we have

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha}{2}+\frac{1}{2},\beta\\ \frac{\gamma}{2},\frac{\gamma}{2}+\frac{1}{2}\end{matrix}\middle|1\right] &= {}_3F_2\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha}{2}+\frac{1}{2},\beta\\ \frac{\gamma}{2},\frac{\gamma}{2}+\frac{1}{2}\end{matrix}\middle|\frac{z^2}{w^2}\right]\Bigg|_{w=z} && (4.1)\\
&= {}_zO^{\alpha}_{\gamma}\,{}_1F_0\left[\begin{matrix}\beta\\ -\end{matrix}\middle|\frac{z^2}{w^2}\right]\Bigg|_{w=z} = {}_z\,O^{\alpha}_{\gamma}(1-\frac{z^2}{w^2})^{-\beta}\Big|_{w=z}\\
&= z^{\beta}\,{}_zO^{\alpha}_{\gamma}(w-z)^{-\beta}(1+\frac{z}{w})^{-\beta}\Big|_{w=z} = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma-\beta\end{matrix}\middle|-1\right]
\end{aligned}
$$

by using the properties (3.8), (3.15) and the representation of the Gauss hypergeometric function in Table A.1.

Putting $\gamma - 1 + 2\beta - \alpha$ in (4.7), with the Kummer's theorem [33, Eq. (2), p. 68], we obtain a special case of the Dixon's theorem [33, p. 92]

$$
{}_3F_2\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha}{2}+\frac{1}{2},\beta\\ \frac{1}{2}+\beta-\frac{\alpha}{2},1+\beta-\frac{\alpha}{2}\end{matrix}\middle|1\right] = \frac{\Gamma(1+2\beta-\alpha)\Gamma(1+\beta-2\alpha)\Gamma(1+\frac{\beta}{2})}{\Gamma(1+2\beta-2\alpha)\Gamma(1+\frac{\beta}{2}-\alpha)\Gamma(1+\beta)}. \tag{4.2}
$$

As another example, we can write

$$
\begin{aligned}
{}_4F_3\left[\left.\begin{matrix}\frac{\alpha}{3},\frac{\alpha}{3}+\frac{1}{3},\frac{\alpha}{3}+\frac{2}{3},\beta\\ \frac{\gamma}{3},\frac{\gamma}{3}+\frac{1}{3},\frac{\gamma}{3}+\frac{2}{3}\end{matrix}\right|1\right] &= {}_4F_3\left[\left.\begin{matrix}\frac{\alpha}{3},\frac{\alpha}{3}+\frac{1}{3},\frac{\alpha}{3}+\frac{2}{3},\beta\\ \frac{\gamma}{3},\frac{\gamma}{3}+\frac{1}{3},\frac{\gamma}{3}+\frac{2}{3}\end{matrix}\right|\frac{z^3}{w^3}\right]\Bigg|_{w=z} \\
&= {}_zO_\gamma^\alpha {}_1F_0\left[\left.\begin{matrix}\beta\\ -\end{matrix}\right|\frac{z^3}{w^3}\right]\Bigg|_{w=z} = {}_z\,O_\gamma^\alpha w^{3\beta}(w-z)^{-\beta}(w^2+wz+z^2)^{-\beta}\Big|_{w=z} \\
&= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} z^{2\beta}\ {}_zO_{\gamma-\beta}^\alpha (w^2+wz+z^2)^{-\beta}\Big|_{w=z}.
\end{aligned}
\tag{4.3}
$$

With $\beta=-n$, we obtain

$$
\begin{aligned}
{}_4F_3\left[\left.\begin{matrix}\frac{\alpha}{3},\frac{\alpha}{3}+\frac{1}{3},\frac{\alpha}{3}+\frac{2}{3},-n\\ \frac{\gamma}{3},\frac{\gamma}{3}+\frac{1}{3},\frac{\gamma}{3}+\frac{2}{3}\end{matrix}\right|1\right] &= z^{-2n}\frac{(\gamma-\alpha)_n}{(\gamma)_n}\sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3=n}}\binom{n}{k_1,k_2,k_3} z^{2k_1+k_2}\ {}_zO_\gamma^{\alpha+n} z^{k_2+2k_3} \\
&= (\gamma-\alpha)_n\sum_{k_3=0}^{n}\sum_{k_2=0}^{k_3}\frac{n!(\alpha)_{2k_3-k_2}}{k_2!(k_3-k_2)!(n-k_3)!(\gamma)_{n+2k_3-k_2}} \\
&= (\gamma-\alpha)_n\sum_{k_3=0}^{n}\binom{n}{k_3}\frac{(\alpha)_{2k_3}}{(\gamma)_{n+2k_3}}\ {}_2F_1\left[\left.\begin{matrix}-k_3,1-\gamma-n-2k_3\\ 1-\alpha-2k_3\end{matrix}\right|-1\right].
\end{aligned}
\tag{4.4}
$$

If $\alpha=\gamma+n$, then

$$
{}_2F_1\left[\left.\begin{matrix}-k_3,1-\gamma-n-2k_3\\ 1-\alpha-2k_3\end{matrix}\right|-1\right]=2^{k_3}
$$

and we finally find

$$
{}_4F_3\left[\left.\begin{matrix}\frac{\gamma}{3}+\frac{n}{3},\frac{\gamma}{3}+\frac{n}{3}+\frac{1}{3},\frac{\gamma}{3}+\frac{n}{3}+\frac{2}{3},-n\\ \frac{\gamma}{3},\frac{\gamma}{3}+\frac{1}{3},\frac{\gamma}{3}+\frac{2}{3}\end{matrix}\right|1\right]=\frac{1}{(\gamma)_n}\lim_{\epsilon\to 0}\frac{\Gamma(\epsilon)}{\Gamma(-n+\epsilon)}\sum_{k_3=0}^{n}\binom{n}{k_3}2^{k_2}=\frac{(-1)^n n!}{(\gamma)_n}3^n. \tag{4.5}
$$

By a similar way, we can generalize this result as

$$
{}_{p+1}F_p\left[\left.\begin{matrix}\frac{\gamma}{p}+\frac{n}{p},\frac{\gamma}{p}+\frac{n}{p}+\frac{1}{p},\ldots,\frac{\gamma}{p}+\frac{n}{p}+\frac{(p-1)}{p},-n\\ \frac{\gamma}{p},\frac{\gamma}{p}+\frac{1}{p},\ldots,\frac{\gamma}{p}+\frac{(p-1)}{p}\end{matrix}\right|1\right]=\frac{(-1)^n n!}{(\gamma)_n}p^n. \tag{4.6}
$$

We could of course multiply the examples of application of the operator ${}_zO_\gamma^\alpha$.

Just like for the fractional derivative, we can represent a special function in the form

$$
\mathfrak{I}(z)=K(z)_{g(z)}O_\beta^\alpha F(z), \tag{4.7}
$$

where $K(z)$, $g(z)$ and $F(z)$ are functions of a more elementary nature than $\mathfrak{I}(z)$. Moreover, the operator ${}_{g(z)}O_\beta^\alpha$ offers several possibilities of representation.

Table A.1 shows a few representations of the form (4.7). It is also possible to extend fractional calculus to several variables and to represent special functions of two or more variables by means of fractional partial differentiation introduced in [3, 34] and used in [17, 28, 31] which is quite easily explained by the formal expression

$$
{}_{g(z),h(w)}O_{\beta,\beta'}^{\alpha,\alpha'} = {}_{g(z)}\ O_\beta^\alpha\ {}_{h(w)}O_{\beta'}^{\alpha'}.
$$

Also in Table A.2 are found some representations of Appell and Humbert's hypergeometric functions of two variables using partial fractional differential operators. The notations used for the special functions are those of Erdelyi et al. [9].

5. First examples of use the well poised fractional operator ${}_{g(z)}O_{\beta}^{\alpha}$

Theorem 5.1. *For any analytic function $f(z)$ and for arbitrary complex numbers α and β, we have*

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(z)(-1)^n\frac{(\beta-\alpha)_n}{(\beta)_n}\frac{z^n}{n!}. \tag{5.1}$$

Proof. Using the elementary series

$$f(w-z) = \sum_{n=0}^{\infty} f^{(n)}(w)(-1)^n\frac{z^n}{n!} \tag{5.2}$$

valid for analytic function $f(z)$, applying the operator ${}_zO_{\beta}^{\alpha}$ on each side, we have

$${}_zO_{\beta}^{\alpha} f(z) = \sum_{n=0}^{\infty} f^{(n)}(0)\, {}_zO_{\beta}^{\alpha}\frac{z^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(w)(-1)^n\, {}_zO_{\beta}^{\beta-\alpha}\frac{z^n}{n!}\bigg|_{w=z}.$$

Using (3.7) with $g(z) = z$, we obtain (5.1). □

5.1. Some special cases deduced from Theorem 5.1

We can deduce from (5.1) several interesting special cases.

Case 1. Setting $f(z) = (1-z)^{-\gamma}$ and $f(z) = e^z$ we easily obtain the well-known transformations of Euler ([33, Eq. (4), p. 60]) and Kummer ([33, Eq. (2), p. 125])

$${}_2F_1\left[\begin{matrix}\alpha,\gamma\\ \beta\end{matrix}\bigg| z\right] = (1-z)^{-\gamma}\, {}_2F_1\left[\begin{matrix}\beta-\alpha,\gamma\\ \beta\end{matrix}\bigg| \frac{-z}{1-z}\right] \tag{5.3}$$

and

$${}_1F_1\left[\begin{matrix}\alpha\\ \beta\end{matrix}\bigg| z\right] = e^z\, {}_1F_1\left[\begin{matrix}\beta-\alpha\\ \beta\end{matrix}\bigg| -z\right]. \tag{5.4}$$

As other examples, if we put $f(z) = \ln(1+z)$ in (5.1), we obtain

$$\frac{\alpha}{\beta} z\, {}_3F_2\left[\begin{matrix}\alpha+1,1,1\\ \beta+1,2\end{matrix}\bigg| -z\right] = \ln(1+z) - \frac{(\beta-\alpha)}{\beta}\frac{z}{1+z}\, {}_3F_2\left[\begin{matrix}\beta-\alpha+1,1,1\\ \beta+1,2\end{matrix}\bigg| \frac{z}{1+z}\right]. \tag{5.5}$$

Case 2. Consider any addition formula more general than (5.2)

$$f(w-z) = \sum_i a_i g_i(w) h_i(z). \tag{5.6}$$

Applying the operator ${}_zO_{\beta}^{\alpha}$ we have

$${}_zO_{\beta}^{\alpha} f(z) = {}_zO_{\beta}^{\beta-\alpha} f(w-z)\bigg|_{w=z} = \sum_i a_i g_i(z)\, {}_zO_{\beta}^{\beta-\alpha} h_i(z). \tag{5.7}$$

Setting $f(z) = \cos z$ and $f(z) = \sin z$ in (5.6), we obtain the two following equalities:

$${}_2F_3\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha}{2}+\frac{1}{2}\\ \frac{\beta}{2},\frac{\beta}{2}+\frac{1}{2},\frac{1}{2}\end{matrix}\bigg| -\frac{z^2}{4}\right] = \cos z\, {}_2F_3\left[\begin{matrix}\frac{\beta}{2}-\frac{\alpha}{2},\frac{\beta}{2}-\frac{\alpha}{2}+\frac{1}{2}\\ \frac{\beta}{2},\frac{\beta}{2}+\frac{1}{2},\frac{1}{2}\end{matrix}\bigg| -\frac{z^2}{4}\right] + \frac{(\beta-\alpha)}{\beta} z \sin z\, {}_2F_3\left[\begin{matrix}\frac{\beta}{2}-\frac{\alpha}{2}+\frac{1}{2},\frac{\beta}{2}-\frac{\alpha}{2}+1\\ \frac{\beta}{2}+\frac{1}{2},\frac{\beta}{2}+1,\frac{3}{2}\end{matrix}\bigg| -\frac{z^2}{4}\right]$$

and

$$\frac{\alpha}{\beta}\, z\, {}_2F_3\left[\begin{matrix} \frac{\alpha}{2}+\frac{1}{2}, \frac{\alpha}{2}+1 \\ \frac{\beta}{2}+\frac{1}{2}, \frac{\beta}{2}+1, \frac{3}{2} \end{matrix}\middle| -\frac{z^2}{4}\right] = \sin z\, {}_2F_3\left[\begin{matrix} \frac{\beta}{2}-\frac{\alpha}{2}, \frac{\beta}{2}-\frac{\alpha}{2}+\frac{1}{2} \\ \frac{\beta}{2}, \frac{\beta}{2}+\frac{1}{2}, \frac{1}{2} \end{matrix}\middle| -\frac{z^2}{4}\right] - \frac{(\beta-\alpha)}{\beta}\, z\, \cos z\, {}_2F_3\left[\begin{matrix} \frac{\beta}{2}-\frac{\alpha}{2}+\frac{1}{2}, \frac{\beta}{2}-\frac{\alpha}{2}+1 \\ \frac{\beta}{2}+\frac{1}{2}, \frac{\beta}{2}+1, \frac{3}{2} \end{matrix}\middle| -\frac{z^2}{4}\right]. \tag{5.8}$$

Many addition formulas of the type (5.6) can be found. Table A.5 gives some of them [1]. Computations of the corresponding right side of (5.7) are shown in Table A.6.

Case 3. If we put $\beta = 2\alpha$ in (5.1), we get

$$\sum_{n=0}^{\infty} f^{(n)}(0)\, \frac{(\alpha)_n}{(2\alpha)_n}\, \frac{z^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(z)\, \frac{(\alpha)_n}{(2\alpha)_n}\, (-1)^n\, \frac{z^n}{n!} \tag{5.9}$$

and we can deduce the identity

$${}_{p+1}F_{q+1}\left[\begin{matrix} a_p, \alpha \\ b_q, 2\alpha \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} {}_pF_q\left[\begin{matrix} a_p+n \\ b_q+n \end{matrix}\middle| z\right] \frac{(\alpha)_n \prod_{i=1}^{p}(a_i)_n}{(2\alpha)_n \prod_{j=1}^{q}(b_j)_n}\, (-1)^n \frac{z^n}{n!}.$$

Now, using the fact that

$$\lim_{\alpha\to\infty} \frac{(\alpha)_n}{(2\alpha)_n} = \frac{1}{2^n} \tag{5.10}$$

and

$$\lim_{\alpha\to 0} \frac{(\alpha)_n}{(2\alpha)_n} = \begin{cases} 1 & (n=0) \\ \frac{1}{2} & (n \geq 1) \end{cases} \tag{5.11}$$

we get the obvious formula

$$f(0) = \sum_{n=0}^{\infty} f^{(n)}(z)\, (-1)^n\, \frac{z^n}{n!} \tag{5.12}$$

and the new formula

$$\sum_{n=0}^{\infty} f^{(n)}(0)\, \frac{(\frac{z}{2})^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(z)\, (-1)^n\, \frac{(\frac{z}{2})^n}{n!}, \tag{5.13}$$

and we can deduce

$${}_pF_q\left[\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} {}_pF_q\left[\begin{matrix} a_p+n \\ b_q+n \end{matrix}\middle| 2z\right] \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\, (-1)^n \frac{z^n}{n!} \tag{5.14}$$

and the obvious formula

$$\sum_{n=0}^{\infty} {}_pF_q\left[\begin{matrix} a_p+n \\ b_q+n \end{matrix}\middle| z\right] \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\, (-1)^n \frac{z^n}{n!} = 1. \tag{5.15}$$

6. Some formulas involving the generalized Bernoulli polynomials

Let's define the polynomials associated with the Bernoulli polynomials

$$\begin{aligned} Q_n^{(\alpha,a,b)}(z) = {}_zO_b^a\, B_n^{(\alpha)}(z) &= \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} B_{n,k}^{(\alpha)}\, z^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{(b)_k}\, B_{n-k}^{(\alpha)}(0)\, z^k. \end{aligned} \tag{6.1}$$

By using ([18, Eq. (14), p. 21]),

$$\frac{d^k}{dz^k} B_n^{(\alpha)}(z) = \frac{n!}{(n-k)!}\, B_{n-k}^{(\alpha)}(z). \tag{6.2}$$

We have $Q_n^{(\alpha,a,a)}(z) = B_n^{(\alpha)}(z)$ and $Q_n^{(\alpha,0,b)}(z) = B_n^{(\alpha)}(0)$.

Case 4. Generating functions of associated polynomials $Q_n^{(\alpha,a,b)}(z)$.

From generating functions of generalized Bernoulli polynomials ([18, Eq. (1), p. 18])

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{zt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(z)\, \frac{t^n}{n!} \tag{6.3}$$

and

$$(1+t)^z \left(\frac{\ln(1+t)}{t}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(1+\alpha+n)}(1+z)\, \frac{t^n}{n!}, \tag{6.4}$$

we can write

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\middle| zt\right] = \sum_{n=0}^{\infty} Q_n^{(\alpha,a,b)}(z) \frac{t^n}{n!} \tag{6.5}$$

and

$$\frac{\left(\frac{\ln(1+t)}{t}\right)^{\alpha}}{1+t}\, {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\middle| z\, \ln(1+t)\right] = \sum_{n=0}^{\infty} Q_n^{(1+\alpha+n,a,b)}(z) \frac{t^n}{n!}. \tag{6.6}$$

Case 5.

$$\begin{aligned} Q_n^{(\alpha,a,b)}(z) &= {}_zO_b^a\, B_n^{(\alpha)}(z) = (-1)^n\, {}_zO_b^a\, B_n^{(\alpha)}(\alpha - z) \\ &= (-1)^n \sum_{k=0}^{n} B_{n,k}^{(\alpha)}\, {}_zO_b^a\, (\alpha - z)^k = (-1)^n \sum_{k=0}^{n} B_{n,k}^{(\alpha)}\, \alpha^k\, {}_zO_b^a \left(1 - \frac{z}{\alpha}\right)^k \\ &= (-1)^n \sum_{k=0}^{n} B_{n,k}^{(\alpha)}\, \alpha^k\, {}_2F_1\left[\begin{matrix} -k, a \\ b \end{matrix}\middle| \frac{z}{\alpha}\right]. \end{aligned} \tag{6.7}$$

If $z = \alpha$, using the Gauss summation theorem ([33, Theorem 18, p. 49])

$${}_2F_1\left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}\middle| 1\right] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{6.8}$$

we obtain

$$Q_n^{(\alpha,a,b)}(\alpha) = (-1)^n\, Q_n^{(\alpha,b-a,b)}(\alpha). \tag{6.9}$$

We can directly obtain (6.9) using (2.3)

$$B_n^{(\alpha)}(z) = (-1)^n \, B_n^{(\alpha)}(\alpha - z) \tag{6.10}$$

and

$$Q_n^{(\alpha,a,b)}(\alpha) = \; {}_zO_b^a \, B_n^{(\alpha)}(z)\Big|_{z=\alpha} = (-1)^n \; {}_zO_b^a \, B_n^{(\alpha)}(\alpha - z)\Big|_{z=\alpha} = (-1)^n \; {}_zO_b^{b-a} \, B_n^{(\alpha)}(z)\Big|_{z=\alpha} = (-1)^n \; Q_n^{(\alpha,b-a,b)}(\alpha) \tag{6.11}$$

which shows the efficiency of the operator ${}_zO_b^a$.

If $b = 2a$ in (6.9), we deduce from (6.11) that

$$Q_{2n+1}^{(\alpha,a,2a)}(\alpha) = \; {}_zO_{2a}^a \, B_{2n+1}^{(\alpha)}(z)\Big|_{z=\alpha} = \sum_{k=0}^{n} \frac{(a)_k}{(2a)_k} B_{2n+1,k}^{(\alpha)} \, \alpha^k = 0. \tag{6.12}$$

Moreover, starting directly from (6.1) and (5.9), we have

$$\begin{aligned} Q_n^{(\alpha,a,2a)}(z) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{(2a)_k} \, B_{n-k}^{(\alpha)}(z) \, z^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k}{(2a)_k} \, B_{n-k}^{(\alpha)}(0) \, z^k. \end{aligned} \tag{6.13}$$

If $z = 2\alpha$, we have

$${}_2F_1\left[\begin{matrix} -2k, a \\ 2a \end{matrix}\middle| 2\right] = \frac{(\frac{1}{2})_k}{(a+\frac{1}{2})_k} \tag{6.14}$$

and

$${}_2F_1\left[\begin{matrix} -2k-1, a \\ 2a \end{matrix}\middle| 2\right] = 0, \tag{6.15}$$

then we get

$$Q_n^{(\alpha,a,2a)}(2\alpha) = (-1)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(\frac{1}{2})_k}{(a+\frac{1}{2})_k} B_{n,2k}^{(\alpha)} \, \alpha^{2k}. \tag{6.16}$$

Case 6. From the generating function (6.4),

$$(1+t)^z \left(\frac{\ln(1+t)}{t}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(1+\alpha+n)}(1+z) \, \frac{t^n}{n!} \tag{6.17}$$

applying on the both side the operator ${}_zO_b^a$ and using (3.15) we obtain

$$\left(\frac{\ln(1+t)}{t}\right)^{\alpha} \; {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\middle| z \, \ln(1+t)\right] = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} B_{n-i}^{(1+\alpha+n)}(0) \; {}_2F_1\left[\begin{matrix} -i, a \\ b \end{matrix}\middle| -z\right] t^n. \tag{6.18}$$

If $z = -1$, using the Gauss summation theorem (6.8), we find (6.6) with $z = 1$.

Case 7 (Addition formulas). We start with the addition formula for the generalized Bernoulli polynomials ([18, Eq. (12), p. 21])

$$B_n^{(\alpha+\beta)}(w - z) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(\alpha)}(-z) \, B_r^{(\beta)}(w). \tag{6.19}$$

Applying the operator ${}_zO_b^a|_{w=z}$ on the both side, we easily get

$$Q_n^{(\alpha+\beta,a,b)}(z) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(\beta)}(z)\, Q_r^{(\alpha,b-a,b)}(-z)$$

which it reduces with $\beta = 0$

$$Q_n^{(\alpha,a,b)}(z) = \sum_{r=0}^{n} \binom{n}{r} Q_{n-r}^{(\alpha,b-a,b)}(-z)\, z^r.$$

Case 8. Pochhammer symbol $(z)_n$ and power term z^n. From ([18, Eq. (15)-(16), p. 21])

$$(-1)^n (z)_n = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(n+1)}(0)\ (1-z)^r \tag{6.20}$$

and

$$(z)_n = \sum_{r=0}^{n-1} \binom{n-1}{r} (-1)^r\, B_r^{(n)}(0)\ z^{n-r}, \tag{6.21}$$

applying the operator ${}_zO_b^a$ on the both sides of each equation, we easily obtain

$$(-1)^n\, {}_zO_b^a\,(z)_n = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(n+1)}(0)\ \ {}_2F_1\left[\begin{matrix}-r, a\\ b\end{matrix}\middle| z\right] \tag{6.22}$$

and

$$\begin{aligned} {}_zO_b^a\,(z)_n &= \sum_{r=0}^{n-1} \binom{n-1}{r} (-1)^r \frac{(a)_{n-r}}{(b)_{n-r}}\, B_r^{(n)}(0)\, z^{n-r} \\ &= (-1)^{n+1} \frac{a}{b} z\, Q_{n-1}^{(n,a+1,b+1)}(-z). \end{aligned} \tag{6.23}$$

If $z = 1$ in (6.22) and (6.27), we find

$$\begin{aligned} (-1)^n\, {}_zO_b^a\,(z)_n\Big|_{z=1} &= \sum_{r=0}^{n} \binom{n}{r} \frac{(b-a)_r}{(b)_r}\, B_{n-r}^{(n+1)}(0) = Q_n^{(n+1,b-a,b)}(1) \\ &= -\frac{a}{b}\, Q_{n-1}^{(n,a+1,b+1)}(-1). \end{aligned} \tag{6.24}$$

Similarly, from the formula ([18, Eq. (19), p. 21])

$$z^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r\, B_{n-r}^{(-r)}(0)\ (-z)_r, \tag{6.25}$$

we obtain by applying the operator

$$\frac{(a)_n}{(b)_n} z^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r\, B_{n-r}^{(-r)}(0)\, {}_zO_b^a\,(-z)_r. \tag{6.26}$$

Using (6.19) and (6.26) in (6.1), we deduce the formula

$$Q_n^{(\alpha,a,b)}(z) = \sum_{k=0}^{n} \Big(\sum_{r=0}^{k} \binom{k}{r} (-1)^r\, B_{n,k}^{(\alpha)}\, B_{k-r}^{(-r)}(0) \Big)\, {}_zO_b^a\,(-z)_r = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(\alpha-r)}(0)(-1)^r\ {}_zO_b^a\,(-z)_r\,. \tag{6.27}$$

7. The Christoffel-Darboux formula

In this section, we give some ways to apply the operator ${}_zO^{\alpha}_{\beta}$ to the Chistoffel-Darboux formula associated with the families of orthogonal polynomials $f_m(z)$ [9, Vol. 2, Eq. (10), p. 159]

$$\sum_{m=0}^{n} \frac{1}{h_m} f_m(w) f_m(z) = \frac{k_n}{k_{n+1} h_n} \frac{f_{n+1}(w) f_n(z) - f_n(w) f_{n+1}(z)}{w - z}, \tag{7.1}$$

where

$$h_n = \int_a^b w(z)[f_n(x)]^2 dx \text{ and } f_n(x) = k_n x^n + k_n x^{n-1} + \dots .$$

The squared norm h_n and the slope coefficient k_n of the orthogonal polynomials considered in this article (obtained from [1, Tables 22.2 and 22.3, p. 774–775]) are listed in Table A.7.

Theorem 7.1. *If $f_n(z)$ is the n-th term of a set of orthogonal polynomials, then*

$$\frac{\beta - \alpha}{\beta} z \sum_{m=0}^{n} \frac{f_m(z)}{h_m} \, {}_zO^{\alpha}_{\beta+1} f_m(z) = \frac{k_n}{k_{n+1} h_n} \Big[f_{n+1}(z) \, {}_zO^{\alpha}_{\beta} f_n(z) - f_n(z) \, {}_zO^{\alpha}_{\beta} f_{n+1}(z) \Big] \tag{7.2}$$

and

$$\frac{z}{\alpha} \sum_{m=0}^{n} \frac{f_m(z)}{h_m} \, {}_zO^{\alpha}_{\alpha+1} f_m(z) = \frac{k_n}{k_{n+1} h_n} \lim_{\beta \to \alpha} \Big[f_{n+1}(z) \frac{\partial}{\partial \beta} \, {}_zO^{\alpha}_{\beta} f_n(z) - f_n(z) \frac{\partial}{\partial \beta} \, {}_zO^{\alpha}_{\beta} f_{n+1}(z) \Big]. \tag{7.3}$$

Proof. Modifying (7.1) by replacing x by w, y by z and putting $z_0 = 0$, multiplying both sides by $z^p (w - z)^{q+1}$, we obtain

$$\sum_{m=0}^{n} \frac{1}{h_m} \frac{f_m(w) z^p f_m(z)}{(w - z)^{-q-1}} = \frac{k_n}{k_{n+1} h_n} \frac{f_{n+1}(w) z^p f_n(z) - f_n(w) z^p f_{n+1}(z)}{(w - z)^{-q}}. \tag{7.4}$$

Now applying the operator ${}_zO^{\alpha}_{\beta}$ on each side of (7.4) where $w = z$ before the operation, we have

$$\sum_{m=0}^{n} \frac{f_m(z)}{h_m} \, {}_zO^{\alpha}_{\beta} z^p (w - z)^{q+1} f_m(z) \Big|^{*}_{w=z} = \frac{k_n}{k_{n+1} h_n} \left[f_{n+1}(z) \, {}_zO^{\alpha}_{\beta} z^p (w - z)^q f_n(z) \Big|^{*}_{w=z} - f_n(z) \, {}_zO^{\alpha}_{\beta} z^p (w - z)^q f_{n+1}(z) \Big|^{*}_{w=z} \right]. \tag{7.5}$$

Now using (3.12) with $g(z) = z$, we get

$${}_zO^{\alpha}_{\beta} z^r (w - z)^s F(z) \Big|^{*}_{w=z} = \frac{\Gamma(\beta)\Gamma(\alpha + r)\Gamma(\beta - \alpha + s)}{\Gamma(\alpha)\Gamma(\beta - \alpha)\Gamma(\beta + r + s)} z^{r+s} \, {}_zO^{\alpha+r}_{\beta+r+s} F(z). \tag{7.6}$$

Using (7.6) in (7.5) and simplifying we can write

$$\frac{\beta - \alpha + q}{\beta + p + q} z \sum_{m=0}^{n} \frac{f_m(z)}{h_m} \, {}_zO^{\alpha+p}_{\beta+p+q+1} f_n(z) = \frac{k_n}{k_{n+1} h_n} \left[f_{n+1}(z) \, {}_zO^{\alpha+p}_{\beta+p+q} f_n(z) - f_n(z) \, {}_zO^{\alpha+p}_{\beta+p+q} f_{n+1}(z) \right]. \tag{7.7}$$

Note that the parameters p and q are redundant. If $\alpha \to \alpha + p$ and putting $p = q = 0$ in (7.7), we obtain (7.2). Now, with $\beta \to \alpha$, we get the particular form (7.3)

$$\begin{aligned} \frac{z}{\alpha} \sum_{m=0}^{n} \frac{f_m(z)}{h_m} \, {}_zO^{\alpha}_{\alpha+1} f_m(z) &= \frac{k_n}{k_{n+1} h_n} \lim_{\beta \to \alpha} \frac{\Big[f_{n+1}(z) \, {}_zO^{\alpha}_{\beta} f_n(z) - f_n(z) \, {}_zO^{\alpha}_{\beta} f_{n+1}(z) \Big]}{\beta - \alpha} \\ &= \frac{k_n}{k_{n+1} h_n} \lim_{\beta \to \alpha} \left[f_{n+1}(z) \frac{\partial}{\partial \beta} \, {}_zO^{\alpha}_{\beta} f_n(z) - f_n(z) \frac{\partial}{\partial \beta} \, {}_zO^{\alpha}_{\beta} f_{n+1}(z) \right]. \end{aligned} \tag{7.8}$$

This completes the proof. □

Corollary 7.2. *For all sets of orthogonal polynomials, we have*

$$\sum_{m=0}^{n} \frac{1}{h_m}\, f_m(z)\; {}_zO_\beta^\alpha f_m(z) = \frac{k_n}{k_{n+1}h_n}\frac{(1-\beta)}{(1+\alpha-\beta)z} \times \left\{ f_{n+1}(z) \sum_{k=1}^{n} \frac{f_n^{(k)}(z)}{k!}(-1)^k \frac{(\beta-\alpha-1)_k}{(\beta-1)_k} z^k f_n(z) \sum_{k=1}^{n+1} \frac{f_{n+1}^{(k)}(z)}{k!}(-1)^k \frac{(\beta-\alpha-1)_k}{(\beta-1)_k} z^k \right\}. \tag{7.9}$$

Proof. Using the transformation (2.3) and the property (3.7) with $g(z) = z$, one easily obtains the additional

$$\begin{aligned} \sum_{m=0}^{n} \frac{1}{h_m}\, f_m(z)\; {}_zO_\beta^\alpha f_m(z) &= \frac{k_n}{k_{n+1}h_n}\; {}_zO_\beta^{\beta-\alpha} \left\{ \frac{f_{n+1}(w) f_n(w-z) - f_n(w) f_{n+1}(w-z)}{z} \right\} \Big|_{w=z} \\ &= \frac{k_n}{k_{n+1}h_n z}\frac{(1-\beta)}{(1+\alpha-\beta)} \left\{ f_{n+1}(w)\; {}_zO_{\beta-1}^{\beta-\alpha-1} f_n(w-z)\Big|_{w=z} - f_n(w)\; {}_zO_{\beta-1}^{\beta-\alpha-1} f_{n+1}(w-z)\Big|_{w=z} \right\}. \end{aligned} \tag{7.10}$$

With (5.2) and using the property (3.7), we easily obtain (7.9).

If $\alpha = 0$ in (7.9), we get the particular formula

$$z\sum_{m=0}^{n} \frac{1}{h_m}\, f_m(z) f_m(0) = \frac{k_n}{k_{n+1}h_n} \left\{ f_{n+1}(z)\, f_n(0) - f_n(z) f_{n+1}(0) \right\}, \tag{7.11}$$

an obvious case of (7.2) with $\alpha = p = 0$ because ${}_zO_\beta^0 F(z) = F(0)$ if $F(z)$ is analytic at $z = 0$. □

7.1. Special cases

In this section, we give some examples to demonstrate the effectiveness of the O operator in finding and discovering formulas associated with orthogonal polynomials. As in the previous section, we could multiply the examples in other areas of special functions.

Example 7.3 (**Laguerre polynomials**)**.** Consider the Laguerre polynomials

$$L_n^{(a)}(z) = \frac{(1+a)_m}{m!}\; {}_1F_1\left[\begin{matrix} -m \\ 1+a \end{matrix}\middle| z\right]$$

with (see Table A.1)

$$h_n = \frac{\Gamma(1+a+n)}{n!}, \quad k_n = \frac{(-1)^n}{n!}.$$

Using the property (3.15), we have

$${}_zO_\mu^\nu L_n^{(a)}(z) = \frac{(1+a)_n}{n!}\; {}_2F_2\left[\begin{matrix} -n, \mu \\ 1+a, \nu \end{matrix}\middle| z\right]. \tag{7.12}$$

From the formula (7.2), we obtain after some calculations the following summation formula

$$\begin{aligned} z\frac{(\beta-\alpha)}{\beta} \sum_{m=0}^{n} Ł_m^{(a)}(z)\; {}_2F_2\left[\begin{matrix} -m, \alpha \\ 1+a, \beta+1 \end{matrix}\middle| z\right] &= (1+a+n)\, L_n^{(a)}(z)\; {}_2F_2\left[\begin{matrix} -n-1, \alpha \\ 1+a, \beta \end{matrix}\middle| z\right] \\ &\quad - (n+1)\, L_{n+1}^{(a)}(z)\; {}_2F_2\left[\begin{matrix} -n, \alpha \\ 1+a, \beta \end{matrix}\middle| z\right]. \end{aligned} \tag{7.13}$$

If $\alpha = 0$, (7.13) gives

$$z\sum_{m=0}^{n} Ł_m^{(a)}(z) = (1+a+n)\, L_n^{(a)}(z) - (n+1)\, L_{n+1}^{(a)}(z), \tag{7.14}$$

a formula that is similar to the known formula [9, Vol. 2, Eq. (38), p. 192]

$$z\sum_{m=0}^{n}Ł_m^{(a)}(z)=L_n^{(a+1)}(z)=(z-n)\,L_n^{(a)}(z)+(a+n)\,L_{n-1}^{(a)}(z). \tag{7.15}$$

If $a=\alpha-1$ and $\alpha\to\alpha+1$, it is easy to obtain

$$z\,\frac{(\beta-\alpha-1)}{(n+1)!}\sum_{m=0}^{n}\frac{m!}{(1+\beta)_m}Ł_m^{(\alpha)}(z)L_m^{(\beta)}(z)=\frac{(1+\alpha+n)}{(1+\beta)_n}Ł_n^{(\alpha)}(z)L_{n+1}^{(\beta-1)}(z)-\frac{1}{(1+\beta)_{n-1}}Ł_{n+1}^{(\alpha)}(z)L_n^{(\beta-1)}(z). \tag{7.16}$$

If $\beta=\alpha$ in (7.16), we obtain

$$z\sum_{m=0}^{n}\frac{m!}{(1+\alpha)_m}(Ł_m^{(\alpha)}(z))^2=\frac{(n+1)!}{(1+\alpha)_n}\left[(\alpha+n)Ł_{n+1}^{(\alpha)}(z)L_n^{(\alpha-1)}(z)-(1+\alpha+n)Ł_n^{(\alpha)}(z)L_{n+1}^{(\alpha-1)}(z)\right]. \tag{7.17}$$

Using the property (3.15) and the fact that

$$\frac{d^k}{dz^k}L_n^{(a)}(z)=(-1)^kL_{n-k}^{(a+k)}(z),$$

from (7.8) and (7.9), we easily obtain after some calculations the following formulas

$$\begin{aligned}\frac{z}{1+a}\sum_{m=0}^{n}\frac{m!}{(2+a)_m}L_m^{(a)}(z)Ł_m^{(a+1)}(z)&=(1+a+n)L_n^{(a)}(z)\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-1)^kz^k}{(1+a)_k}(\psi(1+a)-\psi(1+a+k))\\&\quad-(n+1)L_{n+1}^{(a)}(z)\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^kz^k}{(1+a)_k}\,(\psi(1+a)-\psi(1+a+k))\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}\sum_{m=0}^{n}L_m^{(a)}(z)\,{}_2F_2\left[\begin{matrix}-m,\alpha\\1+a,\beta\end{matrix}\middle|z\right]&=\frac{(n+1)!}{(1+a)_n}\frac{(1-\beta)}{(1+\alpha-\beta)z}\left\{L_n^{(a)}(z)\sum_{k=1}^{n+1}\frac{L_{n+1-k}^{(a+k)}(z)}{k!}\frac{(\beta-\alpha-1)_k}{(\beta-1)_k}z^k\right.\\&\quad\left.-L_{n+1}^{(a)}(z)\sum_{k=1}^{n}\frac{L_{n-k}^{(a+k)}(z)}{k!}\frac{(\beta-\alpha-1)_k}{(\beta-1)_k}z^k\right\}.\end{aligned} \tag{7.19}$$

If $\beta=\alpha$ in (7.19), we obtain after simplification

$$\sum_{m=0}^{n}\frac{m!}{(1+a)_m}(L_m^{(a)}(z))^2=\frac{(n+1)!}{(1+a)_n}\left[L_n^{(a)}(z)L_n^{(a+1)}(z)-L_{n+1}^{(a)}(z)L_{n-1}^{(a+1)}(z)\right] \tag{7.20}$$

a probably new formula.

One can find in the literature several similar formulas involving the product of two Laguerre polynomials such as for example

$$\sum_{m=0}^{n}\frac{(xy)^m}{m!(1+a)_m}L_{n-m}^{(a+2m)}(x+y)=\frac{n!}{(1+a)_n}Ł_n^{(a)}(x)L_n^{(a)}(y) \tag{7.21}$$

obtained by W. N. Bailey in 1939 [2] and

$$\sum_{m=0}^{n}\frac{(2m)!(2n-2m)!}{m!((n-m)!)^2(1+a)_m}L_{2m}^{(2a)}(2z)=\frac{2^{2n}n!}{(1+a)_n}\left(L_n^{(a)}(z)\right)^2 \tag{7.22}$$

obtained by W. T. Howell in 1937 [14].

Example 7.4 (Jacobi polynomials). Consider the Jacobi polynomials

$$P_n^{(a,b)}(z) = \frac{(1+a)_n}{n!} {}_2F_1\left[\begin{matrix}-n, 1+a+b+n\\ 1+a\end{matrix}\middle|\frac{1-z}{2}\right],$$

then [1, Tables 22.2 and 22.3, p. 774–775],

$$h_n = \frac{2^{a+b+1}\Gamma(1+a+n)\Gamma(1+b+n)}{(2n+a+b+1)n!\Gamma(1+a+b+n)}, \quad k_n = \frac{1}{2^n}\binom{2n+a+b}{n}.$$

For this case, we must use the following modified formula

$$\frac{\beta-\alpha}{\beta}\,(1-z)\sum_{m=0}^{n}\frac{f_m(z)}{h_m}\ {}_{1-z}O_{\beta+1}^{\alpha+p} f_n(z) = \frac{k_n}{k_{n+1}h_n}\Big[f_{n+1}(z)\ {}_{1-z}O_\beta^\alpha f_n(z) - f_n(z)\ {}_{1-z}O_\beta^\alpha f_{n+1}(z)\Big]. \tag{7.23}$$

After some calculations we get the following explicit summation formula

$$\begin{aligned}&\frac{(\beta-\alpha)}{\beta}\left(\frac{1-z}{2}\right)\sum_{m=0}^{n}\frac{(2m+a+b+1)(1+a+b)_m}{(1+b)_m}P_m^{(a,b)}(z)\ {}_3F_2\left[\begin{matrix}-m, 1+a+b+m, \alpha\\ 1+a, \beta+1\end{matrix}\middle|\frac{1-z}{2}\right] \\ &= \frac{(1+a+b)(2+a+b)_n}{(2+a+b+2n)(1+b)_n}\Bigg\{(1+a+n)P_n^{(a,b)}(z)\ {}_3F_2\left[\begin{matrix}-n-1, 2+a+b+n, \alpha\\ 1+a, \beta\end{matrix}\middle|\frac{1-z}{2}\right] \\ &\qquad - (n+1)P_{n+1}^{(a,b)}(z)\ {}_3F_2\left[\begin{matrix}-n, 1+a+b+n, \alpha\\ 1+a, \beta\end{matrix}\middle|\frac{1-z}{2}\right]\Bigg\}.\end{aligned} \tag{7.24}$$

If $\alpha = 0$ and if $\alpha = 1+a$ in (7.24), we obtain

$$\begin{aligned}&\left(\frac{1-z}{2}\right)\sum_{m=0}^{n}\frac{(2m+a+b+1)(1+a+b)_m}{(1+b)_m}P_m^{(a,b)}(z) \\ &= \frac{(1+a+b)(2+a+b)_n}{(2+a+b+2n)(1+b)_n}\left\{(1+a+n)P_n^{(a,b)}(z) - (n+1)P_{n+1}^{(a,b)}(z)\right\}\end{aligned} \tag{7.25}$$

and

$$\begin{aligned}&\frac{(\beta-1-a)}{(\beta)}\left(\frac{1-z}{2}\right)\sum_{m=0}^{n}\frac{(2m+a+b+1)(1+a+b)_m m!}{(1+b)_m(1+\beta)_m}P_m^{(a,b)}(z)\ P_m^{(\beta,a+b-\beta)}(z) \\ &= \frac{(1+a+b)(2+a+b)_n}{(2+a+b+2n)(1+b)_n}\left\{\frac{(n+1)!(1+a+n)}{(\beta)_{n+1}}P_n^{(a,b)}(z)\ P_{n+1}^{(\beta-1,1+a+b-\beta)}(z) - \frac{(n+1)!}{(\beta)_n}P_{n+1}^{(a,b)}(z)\ P_n^{(\beta-1,1+a+b-\beta)}(z)\right\}.\end{aligned} \tag{7.26}$$

From a formula equivalent to (7.3) for the operator ${}_{1-z}O_\beta^\alpha$, we easily obtain

$$\begin{aligned}&\frac{(1-z)}{2\alpha}\sum_{m=0}^{n}\frac{(2m+a+b+1)(1+a+b)_m}{(1+b)_m}P_m^{(a,b)}(z)\ {}_3F_2\left[\begin{matrix}-m, 1+a+b+m, \alpha\\ 1+a, 1+\alpha\end{matrix}\middle|\frac{1-z}{2}\right] \\ &= \frac{(1+a+b+n)(1+a+b)_n}{(2+2n+a+b)(1+b)_n} \\ &\times\Bigg((1+a+n)P_n^{(a,b)}(z)\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(2+a+b+n)_k}{(1+a)_k}(-1)^k\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k)) \\ &- (n+1)P_{n+1}^{(a,b)}(z)\sum_{k=0}^{n}\binom{n}{k}\frac{(2+a+b+n)_k}{(1+a)_k}(-1)^k\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k))\Bigg).\end{aligned} \tag{7.27}$$

Example 7.5 (**Legendre polynomials**). With $a = b = 0$ in (7.27), we obtain the equivalent formulas for the Legendre polynomials

$$\frac{(\beta-\alpha)}{\beta}\left(\frac{1-z}{2}\right)\sum_{m=0}^{n}(2m+1)P_m(z)\ {}_3F_2\left[\begin{matrix}-m,1+m,\alpha\\1,\beta+1\end{matrix}\middle|\frac{1-z}{2}\right] = \frac{(n+1)}{2}\left\{(P_n(z)\ {}_3F_2\left[\begin{matrix}-n-1,2+n,\alpha\\1,\beta\end{matrix}\middle|\frac{1-z}{2}\right]\right. \tag{7.28}$$
$$\left. - P_{n+1}(z)\ {}_3F_2\left[\begin{matrix}-n,1+n,\alpha\\1,\beta\end{matrix}\middle|\frac{1-z}{2}\right]\right\},$$

$$(1-z)\sum_{m=0}^{n}(2m+1)P_m(z) = (n+1)\Big[P_n(z) - P_{n+1}(z)\Big] \tag{7.29}$$

and

$$\frac{(\beta-1)}{\beta}\left(\frac{1-z}{2}\right)\sum_{m=0}^{n}\frac{(2m+1)m!}{(1+\beta)_m}P_m(z)P_m^{(\beta,-\beta)}(z) = \frac{(n+1)!}{2(\beta)_{n+1}}\Big[P_n(z)P_{n+1}^{(\beta-1,1-\beta)}(z) - (\beta+n)P_{n+1}(z)P_n^{(\beta-1,1-\beta)}(z)\Big], \tag{7.30}$$

$$2(1-z)\sum_{m=0}^{n}(m+\frac{1}{2})P_m(z) = (n+1)[P_n(z) - P_{n+1}(z)]. \tag{7.31}$$

In addition, with the following hypergeometric form for the Legendre polynomial

$$P_n(z) = \frac{(\frac{1}{2})_n(2z)^n}{n!}\ {}_2F_1\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\\\frac{1}{2}-n\end{matrix}\middle|\frac{1}{z^2}\right], \tag{7.32}$$

from (7.2) and (7.3), we obtain

$$\frac{(\beta-\alpha)}{\beta}z\sum_{m=0}^{n}\frac{(2m+1)(\frac{1}{2})_m(\alpha)_m}{m!(\beta)_m}2^mz^m\ P_m(z)\ {}_4F_3\left[\begin{matrix}-\frac{\beta}{2}-\frac{m}{2},\frac{1}{2}-\frac{\beta}{2}-\frac{m}{2},-\frac{m}{2},-\frac{m}{2}+\frac{1}{2}\\\frac{1}{2}-\frac{\alpha}{2}-\frac{m}{2},1-\frac{\alpha}{2}-\frac{m}{2},\frac{1}{2}-m\end{matrix}\middle|\frac{1}{z^2}\right] \tag{7.33}$$
$$= \frac{(n+1)(\frac{1}{2})_n(\alpha)_n}{n!(\beta)_n}2^nz^n\left\{P_{n+1}(z){}_4F_3\left[\begin{matrix}\frac{1}{2}-\frac{\beta}{2}-\frac{n}{2},1-\frac{\beta}{2}-\frac{n}{2},-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\\\frac{1}{2}-\frac{\alpha}{2}-\frac{n}{2},1-\frac{\alpha}{2}-\frac{n}{2},\frac{1}{2}-n\end{matrix}\middle|\frac{1}{z^2}\right]\right.$$
$$\left. - P_n(z)\frac{(2n+1)(\alpha+n)}{(n+1)(\beta+n)}z{}_4F_3\left[\begin{matrix}-\frac{\beta}{2}-\frac{n}{2},\frac{1}{2}-\frac{\beta}{2}-\frac{n}{2},-\frac{n}{2}-\frac{1}{2},-\frac{n}{2}\\-\frac{\alpha}{2}-\frac{n}{2},\frac{1}{2}-\frac{\alpha}{2}-\frac{n}{2},-\frac{1}{2}-n\end{matrix}\middle|\frac{1}{z^2}\right]\right\}$$

and

$$z\sum_{m=0}^{n}\frac{(2m+1)(\frac{1}{2})_m}{m!(\alpha+m)}2^mz^m\ P_m(z)\ {}_3F_2\left[\begin{matrix}-\frac{\alpha}{2}-\frac{m}{2},-\frac{m}{2},-\frac{m}{2}+\frac{1}{2}\\1-\frac{\alpha}{2}-\frac{m}{2},\frac{1}{2}-n\end{matrix}\middle|\frac{1}{z^2}\right] \tag{7.34}$$
$$= \frac{(n+1)}{n!}(\frac{1}{2})_n2^nz^n\left\{P_{n+1}(z)\sum_{k=0}^{n}\frac{(-\frac{n}{2})_k(-\frac{n}{2}+\frac{1}{2})_k}{(\frac{1}{2}-n)_kk!z^{2k}}(\psi(\alpha)-\psi(\alpha-2k))\right.$$
$$\left. - P_n(z)\frac{(2n+1)}{(n+1)}z\sum_{k=0}^{n}\frac{(-\frac{n}{2}-\frac{1}{2})_k(-\frac{n}{2})_k}{(-\frac{1}{2}-n)_kk!z^{2k}}(\psi(\alpha)-\psi(1+\alpha-2k))\right\}.$$

As the latest example for Legendre polynomials, using the fact that [33, Eq. (7), p. 158]

$$P_{2m}(0) = \frac{(-1)^m}{n!}\left(\frac{1}{2}\right)_m \quad\text{and}\quad P_{2m+1}(0) = 0,$$

we have

$$z\sum_{m=0}^{n}(-1)^m\frac{(4m+1)}{m!}\left(\frac{1}{2}\right)_m P_{2m}(z)=(-1)^n\frac{(2n+1)}{n!}\left(\frac{1}{2}\right)_n P_{2n+1}(z). \tag{7.35}$$

Example 7.6 (Hermite polynomials). Consider the Hermite polynomials

$$H_n(z)=(2z)^n{}_2F_0\left[-\tfrac{n}{2},-\tfrac{n}{2}+\tfrac{1}{2}\,\middle|\,\frac{-1}{z^2}\right],$$

we have $h_n=\sqrt{\pi}2^n n!$ and $k_n=2^n$.

By using the same properties of the operator ${}_zO^{\alpha}_{\beta}$, with (7.2), we can obtain similar formulas

$$\begin{aligned}\frac{(\beta-\alpha)}{\beta}z\sum_{m=0}^{n}\frac{z^m}{m!}\frac{(\alpha)_m}{(\beta+1)_m}H_m(z){}_4F_2&\left[\begin{matrix}-\frac{m}{2},-\frac{m}{2}+\frac{1}{2},-\frac{\beta}{2}-\frac{m}{2},\frac{1}{2}-\frac{\beta}{2}-\frac{m}{2}\\ \frac{1}{2}-\frac{\alpha}{2}-\frac{m}{2},1-\frac{\alpha}{2}-\frac{m}{2}\end{matrix}\,\middle|\,-\frac{1}{z^2}\right]\\ =\frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!}\Bigg[\frac{1}{2}H_{n+1}(z)\,{}_4F_2&\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\frac{1}{2}-\frac{\beta}{2}-\frac{n}{2},1-\frac{\beta}{2}-\frac{n}{2}\\ \frac{1}{2}-\frac{\alpha}{2}-\frac{n}{2},1-\frac{\alpha}{2}-\frac{n}{2}\end{matrix}\,\middle|\,-\frac{1}{z^2}\right]\\ -\frac{(\alpha+n)}{(\beta+n)}z\,H_n(z)\,{}_4F_2&\left[\begin{matrix}-\frac{n}{2}-\frac{1}{2},-\frac{n}{2},-\frac{\beta}{2}-\frac{n}{2},\frac{1}{2}-\frac{\beta}{2}-\frac{n}{2}\\ -\frac{\alpha}{2}-\frac{n}{2},\frac{1}{2}-\frac{\alpha}{2}-\frac{n}{2}\end{matrix}\,\middle|\,-\frac{1}{z^2}\right]\Bigg].\end{aligned} \tag{7.36}$$

If $\alpha=\beta+1$, the formula (7.36) reduces to

$$\begin{aligned}z\sum_{m=0}^{n}\frac{z^m}{2^m m!}H_m^2(z)=\frac{z^n}{n!}\Bigg[&-\frac{1}{2}(\beta+n)H_{n+1}(z)\,{}_3F_1\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-\frac{\beta}{2}-\frac{n}{2}\\ -\frac{\beta}{2}-\frac{n}{2}\end{matrix}\,\middle|\,-\frac{1}{z^2}\right]\\ &+z\,(\beta+n+1)H_n(z)\,{}_3F_1\left[\begin{matrix}-\frac{n}{2}-\frac{1}{2},-\frac{n}{2},\frac{1}{2}-\frac{\beta}{2}-\frac{n}{2}\\ -\frac{1}{2}-\frac{\beta}{2}-\frac{n}{2}\end{matrix}\,\middle|\,-\frac{1}{z^2}\right]\Bigg].\end{aligned} \tag{7.37}$$

In addition, with (7.3), after some calculations and simplifications, we obtain

$$\begin{aligned}&2z\sum_{m=0}^{n}\frac{z^m}{(\alpha+m)m!}H_m(z){}_3F_1\left[\begin{matrix}-\frac{m}{2},-\frac{m}{2}+\frac{1}{2},-\frac{\alpha}{2}-\frac{m}{2}\\ 1-\frac{\alpha}{2}-\frac{m}{2}\end{matrix}\,\middle|\,-\frac{1}{z^2}\right]\\ &=\frac{z^n}{n!}\Bigg\{H_{n+1}(z)\sum_{k=0}^{[\frac{n}{2}]}\frac{(-\frac{n}{2})_k(-\frac{n}{2}+\frac{1}{2})_k(-1)^k}{k!z^{2k}}(\psi(\alpha)-\psi(\alpha+n-2k))\\ &-2z\,H_n(z)\sum_{k=0}^{[\frac{n+1}{2}]}\frac{(-\frac{n}{2}-\frac{1}{2})_k(-\frac{n}{2})_k(-1)^k}{k!z^{2k}}(\psi(\alpha)-\psi(1+\alpha+n-2k))\Bigg\}.\end{aligned} \tag{7.38}$$

Note that using the fact that [33, Eq. (4), p. 188]

$$H^{(a)}_{2m}(0)=\frac{(-1)^m}{n!}2^{2m}\left(\frac{1}{2}\right)_m \quad\text{and}\quad H^{(a)}_{2m+1}(0)=0,$$

we have

$$z\sum_{m=0}^{n}\frac{(-1)^m}{2^{2m}m!}H_{2m}(z)=\frac{(-1)^n}{2^{2n+1}n!}H_{2n+1}(z) \tag{7.39}$$

and

$$z\sum_{m=0}^{n-1}\frac{(-1)^m}{2^{2m}(2m!}H_{2m}(z)=\frac{(-1)^{n+1}}{2^{2n-1}(n-1)!}H_{2n-1}(z). \tag{7.40}$$

We can also deduce other results from a following formula associated with the operator ${}_{z^2}O^{\alpha}_{\beta}$, a formula similar to (7.2):

$$\frac{\beta-\alpha}{\beta}\,z\sum_{m=0}^{n}\frac{f_m(z)}{h_m}\,{}_{z^2}O^{\alpha}_{\beta+1}f_m(z)=\frac{k_n}{k_{n+1}h_n}\Big[f_{n+1}(z)\,{}_{z^2}O^{\alpha}_{\beta}f_n(z)-f_n(z)\,{}_{z^2}O^{\alpha}_{\beta}f_{n+1}(z)\Big]$$
$$+\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta+\frac{1}{2})}\frac{k_n}{k_{n+1}h_n}\Big[f_{n+1}(z)\,{}_{z^2}O^{\alpha+\frac{1}{2}}_{\beta+\frac{1}{2}}f_n(z)-f_n(z)\,{}_{z^2}O^{\alpha+\frac{1}{2}}_{\beta+\frac{1}{2}}f_{n+1}(z)\Big]. \tag{7.41}$$

Replacing $f_n(z)$ by $H_n(z)=(2z)^n{}_2F_0\left[\left.{-\frac{n}{2},-\frac{n}{2}+\frac{1}{2} \atop -}\right|-\frac{1}{z^2}\right]$, we have

$$(\beta-\alpha)\,z\sum_{m=0}^{n}\frac{\Gamma(\alpha+\frac{m}{2})}{\Gamma(\beta+1+\frac{m}{2})}\frac{z^m}{m!}H_m(z)\,{}_3F_1\left[\left.{-\frac{m}{2},-\frac{m}{2}+\frac{1}{2},-\beta-\frac{m}{2} \atop 1-\alpha-\frac{m}{2}}\right|-\frac{1}{z^2}\right] \tag{7.42}$$
$$=\frac{z^n}{n!}\Bigg[\frac{1}{2}H_{n+1}(z)\left\{\frac{\Gamma(\alpha+\frac{n}{2})}{\Gamma(\beta+\frac{n}{2})}\,{}_3F_1\left[\left.{-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-\beta-\frac{n}{2} \atop 1-\alpha-\frac{m}{2}}\right|-\frac{1}{z^2}\right]+\frac{\Gamma(\alpha+\frac{1}{2}+\frac{n}{2})}{\Gamma(\beta+\frac{1}{2}+\frac{n}{2})}\,{}_3F_1\left[\left.{-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\frac{1}{2}-\beta-\frac{n}{2} \atop \frac{1}{2}-\alpha-\frac{n}{2}}\right|-\frac{1}{z^2}\right]\right\}$$
$$-zH_n(z)\left\{\frac{\Gamma(\alpha+\frac{1}{2}+\frac{n}{2})}{\Gamma(\beta+\frac{1}{2}+\frac{n}{2})}\,{}_3F_1\left[\left.{-\frac{n}{2}-\frac{1}{2},-\frac{n}{2},\frac{1}{2}-\beta-\frac{n}{2} \atop \frac{1}{2}-\alpha-\frac{n}{2}}\right|-\frac{1}{z^2}\right]+\frac{\Gamma(\alpha+1+\frac{n}{2})}{\Gamma(\beta+1+\frac{n}{2})}\,{}_3F_1\left[\left.{-\frac{n}{2}-\frac{1}{2},-\frac{n}{2},-\beta-\frac{n}{2} \atop -\alpha-\frac{n}{2}}\right|-\frac{1}{z^2}\right]\right\}\Bigg].$$

Example 7.7 (Gegenbauer polynomials). We can add an other example with the Gegenbauer polynomials

$$C^{(a)}_n(z)=\frac{(2a)_n}{n!}{}_2F_1\left[\left.{-n,2a+n \atop a+\frac{1}{2}}\right|\frac{1-z}{2}\right]$$

with $h_n=\frac{\pi 2^{1-2a}\Gamma(a+2n)}{n(a+n)(\Gamma(a))^2}$ and $k_n=\frac{2^n\Gamma(a+n)}{n!\Gamma(a)}$. With (7.2), we obtain

$$-2\frac{(\beta-\alpha)}{\beta}(1-z)\sum_{m=0}^{n}(a+m)\,C^{(a)}_m(z)\,{}_3F_2\left[\left.{-m,2a+m,\alpha \atop a+\frac{1}{2},\beta+1}\right|\frac{1-z}{2}\right]=(n+1)\,C^{(a)}_{n+1}(z)\,{}_3F_2\left[\left.{-n,2a+n,\alpha \atop a+\frac{1}{2},\beta}\right|\frac{1-z}{2}\right] \tag{7.43}$$
$$-(2a+n)\,C^{(a)}_n(z)\,{}_3F_2\left[\left.{-n-1,2a+1+n,\alpha \atop a+\frac{1}{2},\beta}\right|\frac{1-z}{2}\right].$$

If $\alpha=0$, (7.43) becomes

$$-2(1-z)\sum_{m=0}^{n}(a+m)\,C^{(a)}_m(z)=(n+1)\,C^{(a)}_{n+1}(z)-(2a+n)\,C^{(a)}_n(z) \tag{7.44}$$

and if $\alpha=a+\frac{1}{2}$ and $\beta=a-\frac{1}{2}$, we obtain

$$\frac{4(1-z)}{(2a-1)}\sum_{m=0}^{n}\frac{(a+m)m!}{(2a)_m}\left\{C^{(a)}_m(z)\right\}^2 \tag{7.45}$$
$$=\frac{(n+1)!}{2(a-\frac{1}{2})_{n+1}}\left\{(2a-1+2n)\,C^{(a)}_{n+1}(z)\,P^{(a-\frac{3}{2},a+\frac{1}{2})}_n(z)-2(2a+n)\,C^{(a)}_n(z)\,P^{(a-\frac{3}{2},a+\frac{1}{2})}_{n+1}(z)\right\}.$$

In addition, with (7.3), after some calculations, we get

$$\frac{2(1-z)}{\alpha}\sum_{m=0}^{n}(a+m)\,C^{(a)}_m(z)\,{}_3F_2\left[\left.{-m,2a+m,\alpha \atop a+\frac{1}{2},\alpha+1}\right|\frac{1-z}{2}\right] \tag{7.46}$$
$$=(2a+n)\,C^{(a)}_n(z)\sum_{k=0}^{n+1}\frac{(2a+n+1)_k(-1)^k}{(a+\frac{1}{2})_k}\binom{n+1}{k}\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k))$$
$$-(n+1)\,C^{(a)}_{n+1}(z)\sum_{k=0}^{n}\frac{(2a+n)_k(-1)^k}{(a+\frac{1}{2})_k}\binom{n}{k}\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k)).$$

Also, using the fact that [33, Eq. (5), p. 278]

$$C_{2m}^{(a)}(0) = \frac{(-1)^m (a)_m}{m!} \text{ and } C_{2m+1}^{(a)}(0) = 0,$$

we have

$$z\sum_{m=0}^{n}(-1)^m(a+2m)\frac{(\frac{1}{2})_m}{(\frac{1}{2}+a)_m}\,C_{2m}^{(a)}(z) = (-1)^n\frac{(2n+1)}{2}\frac{(\frac{1}{2})_n}{(\frac{1}{2}+a)_n}C_{2n+1}^{(a)}(z). \tag{7.47}$$

Example 7.8 (**Tchebicheff polynomials**). By the same technique, it is possible to deduce similar summation formulas for the other classical orthogonal Tchebicheff polynomials T_n and U_n. We have:

$$\begin{aligned}\frac{(\beta-\alpha)}{\beta}(1-z)\left[1+2\sum_{m=1}^{n}T_m(z){}_3F_2\left[\begin{matrix}-m,m,\alpha\\1/2,1+\beta\end{matrix}\middle|\frac{1-z}{2}\right]\right]\\ = T_n(z)\,{}_3F_2\left[\begin{matrix}-n-1,n+1,\alpha\\1/2,\beta\end{matrix}\middle|\frac{1-z}{2}\right] - T_{n+1}(z){}_3F_2\left[\begin{matrix}-n,n,\alpha\\1/2,\beta\end{matrix}\middle|\frac{1-z}{2}\right],\end{aligned} \tag{7.48}$$

$$\begin{aligned}\frac{(1-z)}{\alpha}\left[1+2\sum_{m=1}^{n}T_m(z){}_3F_2\left[\begin{matrix}-m,m,\alpha\\1/2,1+\alpha\end{matrix}\middle|\frac{1-z}{2}\right]\right] = T_n(z)\sum_{k=1}^{n+1}\frac{(1+n)_k(-1)^k}{(\frac{1}{2})_k}\binom{n+1}{k}\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k))\\ - T_{n+1}(z)\sum_{k=1}^{n}\frac{(n)_k(-1)^k}{(\frac{1}{2})_k}\binom{n}{k}\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k)),\end{aligned} \tag{7.49}$$

$$\begin{aligned}2\frac{(\beta-\alpha)}{\beta}(1-z)\left\{\sum_{m=0}^{n}(m+1)U_m(z){}_3F_2\left[\begin{matrix}-m,m+2,\alpha\\3/2,1+\beta\end{matrix}\middle|\frac{1-z}{2}\right]\right\}\\ = (n+2)U_n(z){}_3F_2\left[\begin{matrix}-n-1,n+3,\alpha\\3/2,\beta\end{matrix}\middle|\frac{1-z}{2}\right] - (n+1)U_{n+1}(z){}_3F_2\left[\begin{matrix}-n,n+2,\alpha\\3/2,\beta\end{matrix}\middle|\frac{1-z}{2}\right],\end{aligned} \tag{7.50}$$

$$\begin{aligned}2\frac{(1-z)}{\alpha}\left\{\sum_{m=0}^{n}(m+1)U_m(z){}_3F_2\left[\begin{matrix}-m,m+2,\alpha\\3/2,1+\alpha\end{matrix}\middle|\frac{1-z}{2}\right]\right\}\\ = (n+2)U_n(z)\sum_{k=1}^{n+1}\frac{(3+n)_k(-1)^k}{(\frac{3}{2})_k}\binom{n+1}{k}\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k))\\ - (n+1)U_{n+1}(z)\sum_{k=1}^{n}\frac{(2+n)_k(-1)^k}{(\frac{3}{2})_k}\binom{n}{k}\left(\frac{1-z}{2}\right)^k(\psi(\alpha)-\psi(\alpha+k)),\end{aligned} \tag{7.51}$$

$$(1-z)\left[1+2\sum_{m=1}^{n}T_m(z)\right] = T_n(z) - T_{n+1}(z) \tag{7.52}$$

and

$$2(1-z)\sum_{m=0}^{n}(m+1)U_m(z) = (n+2)U_n(z) - (n+1)U_{n+1}(z). \tag{7.53}$$

From the summation formula (7.11) and using the fact that

$$T_{2m}(0) = (-1)^m \text{ and } T_{2m+1}(0) = 0$$

and

$$U_{2m}(0)=(-1)^m 2 \text{ and } U_{2m+1}(0)=0,$$

we have

$$z+2z\sum_{m=0}^{n}(-1)^m\,T_{2m}(z)=(-1)^n\,T_{2n+1}(z) \tag{7.54}$$

and

$$2z\sum_{m=0}^{n}(-1)^m\,U_{2m}(z)=(-1)^n\,U_{2n+1}(z). \tag{7.55}$$

8. Bessel functions of the Christoffel-Darboux identity

In [48], Tygert obtains two new analogues formulas for Bessel functions of the Christoffel-Darboux identity

$$\sum_{m=1}^{\infty}2(\nu+m)J_{\nu+m}(w)J_{\nu+m}(z)=\frac{wz}{w-z}\{J_{\nu+1}(w)J_\nu(z)-J_\nu(w)J_{\nu+1}(z)\} \tag{8.1}$$

and

$$2(\nu+1)\sum_{m=1}^{\infty}2(\nu+2m)J_{\nu+2m}(w)J_{\nu+2m}(z)=\frac{w^2z^2}{w^2-z^2}\{J_{\nu+2}(w)J_\nu(z)-J_\nu(w)J_{\nu+2}(z)\}. \tag{8.2}$$

Applying the operator ${}_zO_\beta^{\alpha-1}\big|_{w=z}$ on the both sides of (8.1)

$$\sum_{m=1}^{\infty}2(\nu+m)J_{\nu+m}(z)_zO_\beta^{\alpha-1}J_{\nu+m}(z)=z\left\{J_{\nu+1}(w)\;{}_zO_\beta^{\alpha-1}\Big(\frac{z}{w-z}J_\nu(z)\Big)\Big|_{w=z}-J_\nu(w)\;{}_zO_\beta^{\alpha-1}\Big(\frac{z}{w-z}J_{\nu+1}(z)\Big)\Big|_{w=z}\right\}, \tag{8.3}$$

and using the property (3.9), we have

$${}_zO_\beta^{\alpha-1}J_{\nu+m}(z)=\frac{\Gamma(\beta)\Gamma(\alpha-1+\nu+m)}{\Gamma(\alpha-1)\Gamma(\beta+\nu+m)\Gamma(1+\nu+m)}\Big(\frac{z}{2}\Big)^{\nu+m}{}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+\frac{m}{2}-\frac{1}{2},\frac{\alpha}{2}+\frac{\nu}{2}+\frac{m}{2}\\1+\nu+m,\frac{\beta}{2}+\frac{\nu}{2}+\frac{m}{2},\frac{\beta}{2}+\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}\end{matrix}\middle|-\frac{z^2}{4}\right] \tag{8.4}$$

and

$${}_zO_\beta^{\alpha-1}\Big(\frac{z}{w-z}J_\nu(z)\Big)\Big|_{w=z}=\frac{\Gamma(\beta)\Gamma(\alpha+\nu)}{(\beta-\alpha)\Gamma(\alpha-1)\Gamma(\beta+\nu)\Gamma(1+\nu)}\Big(\frac{z}{2}\Big)^{\nu}{}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2},\frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}\\1+\nu,\frac{\beta}{2}+\frac{\nu}{2},\frac{\beta}{2}+\frac{\nu}{2}+\frac{1}{2}\end{matrix}\middle|-\frac{z^2}{4}\right]. \tag{8.5}$$

Substituting (8.4) and (8.5) in (8.3), we obtain after simplifications

$$\begin{aligned}&\frac{(\beta-\alpha)}{(\beta+\nu)}z\sum_{m=0}^{\infty}\frac{(\alpha+\nu)_m}{(\beta+\nu+1)_m(1+\nu)_m}\Big(\frac{z}{2}\Big)^m J_{\nu+m+1}(z){}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+\frac{m}{2},\frac{\alpha}{2}+\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}\\2+\nu+m,\frac{\beta}{2}+\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2},\frac{\beta}{2}+\frac{\nu}{2}+\frac{m}{2}+1\end{matrix}\middle|-\frac{z^2}{4}\right]\\&=z\,J_{\nu+1}(z){}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2},\frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}\\1+\nu,\frac{\beta}{2}+\frac{\nu}{2},\frac{\beta}{2}+\frac{\nu}{2}+\frac{1}{2}\end{matrix}\middle|-\frac{z^2}{4}\right]-z\,J_\nu(z)\frac{(\alpha+\nu)}{(\beta+\nu)(1+\nu)}\Big(\frac{z}{2}\Big){}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\alpha}{2}+\frac{\nu}{2}+1\\2+\nu,\frac{\beta}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\beta}{2}+\frac{\nu}{2}+1\end{matrix}\middle|-\frac{z^2}{4}\right].\end{aligned} \tag{8.6}$$

By dividing by $\beta-\alpha$ the both sides of (8.6) and taking the limit $\beta\to\alpha$, we obtain by simplifying

$$\begin{aligned}&2\sum_{m=1}^{\infty}J_{\nu+m}(z)\frac{(\frac{z}{2})^{\nu+m}}{\Gamma(\nu+m)(\alpha-1+\nu+m)}{}_1F_2\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+\frac{m}{2}-\frac{1}{2}\\1+\nu+m,\frac{\alpha}{2}+\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}\end{matrix}\middle|-\frac{z^2}{4}\right]\\&=\frac{z}{\alpha+\nu}J_\nu(z)J_{\nu+1}(z)+zJ_{\nu+1}(z)\frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\sum_{m=0}^{\infty}\frac{\{\psi(\alpha+\nu)-\psi(\alpha+\nu+2m)\}}{(1+\nu)_m m!}\left(-\frac{z^2}{4}\right)^m\\&-zJ_\nu(z)\frac{(\frac{z}{2})^{\nu+1}}{\Gamma(2+\nu)}\sum_{m=0}^{\infty}\frac{\{\psi(\alpha+1+\nu)-\psi(\alpha+1+\nu+2m)\}}{(2+\nu)_m m!}\left(-\frac{z^2}{4}\right)^m.\end{aligned} \tag{8.7}$$

The application of operators ${}_{z}O^{\alpha}_{\beta}\Big|_{w=z}$ and ${}_{z^2}O^{\alpha}_{\beta}\Big|_{w=z}$ on the both sides of (8.2) gives after simplifications

$$2\frac{\alpha(\beta+\nu+1)}{(\alpha+\nu)}\sum_{m=0}^{\infty}(\nu+2m)J_{\nu+2m}(z)\frac{(\alpha+\nu)_{2m}}{(\beta+\nu+1)_{2m}(1+\nu)_{2m}}\left(\frac{z}{2}\right)^{2m} {}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+m,\frac{\alpha}{2}+\frac{\nu}{2}+m+\frac{1}{2}\\ 1+\nu+2m,\frac{\beta}{2}+\frac{\nu}{2}+m+\frac{1}{2},\frac{\beta}{2}+\frac{\nu}{2}+m+1\end{matrix}\middle|-\frac{z^2}{4}\right]$$
$$+2\alpha\sum_{m=0}^{\infty}(\nu+2m)J_{\nu+2m}(z)\frac{(\alpha+\nu+1)_{2m}}{(\beta+\nu+2)_{2m}(1+\nu)_{2m}}\left(\frac{z}{2}\right)^{2m}{}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+m+\frac{1}{2},\frac{\alpha}{2}+\frac{\nu}{2}+m+1\\ 1+\nu+2m,\frac{\beta}{2}+\frac{\nu}{2}+m+1,\frac{\beta}{2}+\frac{\nu}{2}+m+\frac{3}{2}\end{matrix}\middle|-\frac{z^2}{4}\right]$$
$$=\frac{\alpha(\alpha+\nu+1)}{2(\nu+1)(\beta-\alpha)}z^2\ J_{\nu+2}(z)\ {}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+1,\frac{\alpha}{2}+\frac{\nu}{2}+\frac{3}{2}\\ 1+\nu,\frac{\beta}{2}+\frac{\nu}{2}+1,\frac{\beta}{2}+\frac{\nu}{2}+\frac{3}{2}\end{matrix}\middle|-\frac{z^2}{4}\right]$$
$$-\frac{\alpha(\alpha+\nu+1)(\alpha+\nu+2)(\alpha+\nu+3)}{8(\beta-\alpha)(\beta+\nu+2)(\beta+\nu+3)(1+\nu)^2(2+\nu)}z^4\ J_\nu(z)\ {}_2F_3\left[\begin{matrix}\frac{\alpha}{2}+\frac{\nu}{2}+2,\frac{\alpha}{2}+\frac{\nu}{2}+\frac{5}{2}\\ 3+\nu,\frac{\beta}{2}+\frac{\nu}{2}+2,\frac{\beta}{2}+\frac{\nu}{2}+\frac{5}{2}\end{matrix}\middle|-\frac{z^2}{4}\right] \tag{8.8}$$

and

$$\frac{2(\beta-\alpha+1)}{(2\beta+\nu)(2+\nu)}\sum_{m=0}^{\infty}(\nu+2m+2)J_{\nu+2m+2}(z)\frac{(\alpha+\frac{\nu}{2}+1)_m}{(\beta+\frac{\nu}{2}+1)_m(3+\nu)_{2m}}\left(\frac{z}{2}\right)^{2m}{}_1F_2\left[\begin{matrix}\alpha+\frac{\nu}{2}+m+1\\ 3+\nu+2m,\beta+\frac{\nu}{2}+m+1\end{matrix}\middle|-\frac{z^2}{4}\right]$$
$$=J_{\nu+2}(z)\ {}_1F_2\left[\begin{matrix}\alpha+\frac{\nu}{2}+1\\ 1+\nu,\beta+\frac{\nu}{2}\end{matrix}\middle|-\frac{z^2}{4}\right]-\frac{z^2}{4}\ J_\nu(z)\frac{(2\alpha+\nu+2)}{(2\beta+\nu)(1+\nu)(2+\nu)}\ {}_1F_2\left[\begin{matrix}\alpha+\frac{\nu}{2}+2\\ 3+\nu,\beta+\frac{\nu}{2}+1\end{matrix}\middle|-\frac{z^2}{4}\right]. \tag{8.9}$$

8.1. Generalizations

In [48], Tygert derives an analogue, for any family of functions satisfying a symmetric "three-term" recurrence relation, of what is known as the Christoffel-Darboux identity for orthonormal polynomials.

Theorem 8.1. *Suppose that $g(z)$ and $\ldots$ $f_{-2}(z), f_{-1}(z), f_0(z), f_1(z), f_2(z), \ldots$ are complex-valued functions on a set S, and $\ldots$ $c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ and $\ldots$ $d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots$ are complex numbers, such that $g(x)f_k(x) = c_{k-1}f_{k-1}(x) + d_kf_k(x) + c_kf_{k+1}(x)$ for any $z \in S$, and any integer k, then*

$$\sum_{k=m+1}^{n} f_k(z)f_k(w) = \frac{c_m}{g(z)-g(w)}\Big\{f_m(z)f_{m+1}(w)-f_{m+1}(z)f_m(w)\Big\}+\frac{c_n}{g(z)-g(w)}\Big\{f_{n+1}(z)f_n(w)-f_n(z)f_{n+1}(w)\Big\} \tag{8.10}$$

for any $z \in S$ and $w \in S$ such that $g(z) \neq g(w)$ with $m < n$.

Using the operator ${}_{g(z)}O^{\alpha}_{\beta}\Big|_{w=z}$ and the property (3.9), we can deduce the following corollary.

Corollary 8.2. *With the same hypothesis on the set S of complex-valued functions $f_n(z)$ in Theorem (8.1), if*

$$\frac{\alpha}{\beta}\ g(z)\ {}_{g(z)}O^{\alpha+1}_{\beta+1}f_k(z) = c_{k-1}\ {}_{g(z)}O^{\alpha}_{\beta}f_{k-1}(z) + d_k\ {}_{g(z)}O^{\alpha}_{\beta}f_k(z) + c_k\ {}_{g(z)}O^{\alpha}_{\beta}f_{k+1}(z) \tag{8.11}$$

then

$$\frac{(\beta-\alpha-1)}{(\beta-1)}g(z)\sum_{k=m+1}^{n} f_k(z)\ {}_{g(z)}O^{\alpha}_{\beta}f_k(z) = c_m\left\{f_m(z)\ {}_{g(z)}O^{\alpha}_{\beta-1}f_{m+1}(z)-f_{m+1}(z)\ {}_{g(z)}O^{\alpha}_{\beta-1}f_m(z)\right\}$$
$$+c_n\left\{f_{n+1}(z)\ {}_{g(z)}O^{\alpha}_{\beta-1}f_n(z)-f_n(z)\ {}_{g(z)}O^{\alpha}_{\beta-1}f_{n+1}(z)\right\} \tag{8.12}$$

and

$$\frac{1}{(\alpha-1)}g(z)\sum_{k=m+1}^{n} f_k(z)\ {}_{g(z)}O^{\alpha-1}_{\alpha}f_k(z) = c_m\left\{f_m(z)\lim_{\beta\to\alpha}\frac{\partial}{\partial\beta}\ {}_{g(z)}O^{\alpha}_{\beta-1}f_{m+1}(z)-f_{m+1}(z)\lim_{\beta\to\alpha}\frac{\partial}{\partial\beta}\ {}_{g(z)}O^{\alpha}_{\beta-1}f_m(z)\right\}$$
$$+c_n\left\{f_{n+1}(z)\lim_{\beta\to\alpha}\frac{\partial}{\partial\beta}\ {}_{g(z)}O^{\alpha}_{\beta-1}f_n(z)-f_n(z)\lim_{\beta\to\alpha}\frac{\partial}{\partial\beta}\ {}_{g(z)}O^{\alpha}_{\beta-1}f_{n+1}(z)\right\}. \tag{8.13}$$

9. Conclusion

The main objective of this paper is to demonstrate the usefulness and efficiency of the well-posed operator ${}_{g(z)}O_{\beta}^{\alpha}$ introduced by the author several years ago [39]. This two-parameter ${}_{g(z)}O_{\beta}^{\alpha}$ operator is a variant of the fractional derivative D_z^{α} and can be represented using the integral Pochhammer contour to minimize restrictions on these settings [16]. This fractional operator gives more flexibility to represent special functions of one or more variables (see Tables A.1 to A.4) and it opens up new avenues for studying special functions in mathematical physics. Several examples of applications of the well-posed operator ${}_{g(z)}O_{\beta}^{\alpha}$ are discussed to demonstrate the ability to discover new results. Motivated by Tygert's article [48], the focus is on summation formulas derived from the classical Christoffel-Darboux formula for orthogonal polynomials. The formulas obtained clearly demonstrate the usefulness and efficiency of this operator. It is certainly a powerful tool for systematically discovering new formulas involving the generalized hypergeometric function as well as the special functions of mathematical physics. In future work, we will explore other applications of this fractional operator, in particular the search for new summation theorems for hypergeometric functions.

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Appendix A. Tables

Name	Representation with the Well-Posed Fractional Calculus Operator ${}_{g(z)}O_{\beta}^{\alpha}$
Gauss hypergeometric function	${}_2F_1(\alpha,\beta;\gamma;z) = {}_zO_{\gamma}^{\beta}(1-z)^{-\alpha} = {}_zO_{\gamma}^{\alpha}(1-z)^{-\beta}$ $= {}_zO_{\gamma}^{\frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2}}\ {}_2F_1(\frac{\alpha}{2},\frac{\beta}{2};\frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2};4z(1-z))$
Degenerate hypergeometric function	${}_1F_1(\alpha;\beta;z) = {}_zO_{\beta}^{\alpha}\ e^z = e^z\ {}_zO_{\beta}^{\beta-\alpha}\ e^{-z}$
Generalized hypergeometric function	${}_{p+1}F_{q+1}(\alpha,a_1,\ldots,a_p;\beta,b_1,\ldots,b_q;z) = {}_zO_{\gamma}^{\beta}\ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$
Bessel function	$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ {}_{z^2}O_{1+\nu}^{\frac{1}{2}}\cos z = \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ {}_{z^2}O_{1+\nu}^{\frac{3}{2}}\frac{\sin z}{z}$ $= \frac{(\frac{z}{2})^{\nu-\mu}\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})}\ {}_{z^2}O_{1+\nu-\frac{\mu}{2}}^{1+\frac{\mu}{2}}\ J_\mu(z)$ $= \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ e^{-iz}\ {}_zO_{1+2\nu}^{\frac{1}{2}+\nu}\ e^{i2z}$ $= \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)\cos z}\ {}_zO_{1+2\nu}^{\frac{1}{2}+\nu}\ \cos 2z$ $= \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)\sin z}\ {}_zO_{1+2\nu}^{\frac{1}{2}+\nu}\ \sin 2z$
Modified Bessel function	$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ {}_{z^2}O_{1+\nu}^{\frac{1}{2}}\cosh z = \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ {}_{z^2}O_{1+\nu}^{\frac{3}{2}}\frac{\sinh z}{z}$ $= \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ e^{-z}\ {}_zO_{1+\nu}^{\frac{3}{2}}\ e^{2z}$
Struve function	$H_\nu(z) = \frac{2}{\sqrt{\pi}}\ \frac{(\frac{z}{2})^{\nu+1}}{\Gamma(\frac{3}{2}+\nu)}\ {}_{z^2}O_{\frac{3}{2}+\nu}^{1}\ \frac{\sin z}{z}$ $= \frac{2}{\sqrt{\pi}}\ (\frac{z}{2})^{\nu+1}\ {}_{z^2}O_{\frac{3}{2}}^{1}\ (\frac{z}{2})^{-\nu-\frac{1}{2}} J_{\nu+\frac{1}{2}}(z)$ $= \frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})}\ (\frac{z}{2})^{\nu-\mu}\ {}_{z^2}O_{1+\nu-\frac{\mu}{2}}^{1+\frac{\mu}{2}}\ H_\mu(z)$
Modified Struve function	$L_\nu(z) = \frac{2}{\sqrt{\pi}}\ \frac{(\frac{z}{2})^{\nu+1}}{\Gamma(\frac{3}{2}+\nu)}\ {}_{z^2}O_{\frac{3}{2}+\nu}^{1}\ \frac{\sinh z}{z}$ $= \frac{2}{\sqrt{\pi}}(\frac{z}{2})^{\nu+1}\ {}_{z^2}O_{\frac{3}{2}}^{1}\ (\frac{z}{2})^{-\nu-\frac{1}{2}} I_{\nu+\frac{1}{2}}(z)$ $= \frac{\Gamma(1+\frac{\mu}{2})}{\Gamma(1+\nu-\frac{\mu}{2})}\ (\frac{z}{2})^{\nu-\mu}\ {}_{z^2}O_{1+\nu-\frac{\mu}{2}}^{1+\frac{\mu}{2}}\ L_\nu(z)$
Legendre function of the first kind	$P_\nu(z) = 2^{-\nu}\ {}_{1-z}O_1^{1+\nu}\ (1+z)^\nu$ $= 2^{\nu+1}\ {}_{1-z}O_1^{-\nu}\ (1+z)^{-\nu-1}$ $= 2^{-\nu}\ {}_{1-z}O_1^{-\nu}\ (1+z+t(1-z))^\nu\|_{t=1}$
Associated Legendre function of the first kind	$P_\nu^\mu(z) = \frac{2^{-\nu}}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}}\ {}_{1-z}O_{1-\mu}^{1+\nu}\ (1+z)^\nu$ $= \frac{2^{\nu+1}}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}}\ {}_{1-z}O_{1-\mu}^{-\nu}\ (1+z)^{-\nu-1}$ $= \frac{2^{-\nu}}{\Gamma(1-\mu)}\ (z^2-1)^{\frac{-\mu}{2}}\ {}_{1-z}O_{1-\mu}^{-\nu}\ (1+z+t(1-z))^{\mu+\nu}\|_{t=1}$
Jacobi Polynomial	$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{2^n n!}\ {}_{1-z}O_{1+\alpha}^{1+\alpha+\beta+n}\ (1+z)^n$ $= \frac{(1+\alpha)_n 2^{\alpha+\beta+n-1}}{n!}\ {}_{1-z}O_{1+\alpha}^{-n}(1+z)^{-\alpha-\beta-n-1}$ $= (-1)^n\frac{(1+\beta)_n}{2^n n!}\ {}_{1+z}O_{1+\beta}^{1+\alpha+\beta+n}\ (1-z)^n$
Laguerre function	$L_\nu^{(\alpha)}(z) = \frac{\Gamma(1+\alpha+\nu)}{\Gamma(1+\nu)\Gamma(1+\alpha)}\ {}_zO_{1+\alpha}^{-\nu}\ e^z$ $= \frac{\Gamma(1+\alpha+\nu)}{\Gamma(1+\nu)\Gamma(1+\alpha)}e^z\ {}_zO_{1+\alpha}^{1+\alpha+\nu}\ e^{-z}$ $= \frac{\Gamma(1+\alpha+\nu)\Gamma(1+\beta)}{\Gamma(1+\nu)\Gamma(1+\alpha)\Gamma(1+\beta+\nu)}\ {}_zO_{1+\alpha}^{1+\beta}\ L_\nu^{(\beta)}(z)$ $= \frac{\Gamma(1+\alpha+\nu)\Gamma(1+\mu)}{\Gamma(1+\nu)\Gamma(1+\alpha)\Gamma(1+\alpha+\nu)}\ {}_zO_{-\mu}^{-\nu}\ L_\nu^{(\alpha)}(z)$
Whittaker function	$M_{\mu;\nu}(z) = z^{\nu+1/2}\ e^{-z/2}\ {}_zO_{1+2\nu}^{1/2+\nu-\nu}e^z$ $= z^{\nu+1/2}\ e^{z/2}\ {}_zO_{1+2\nu}^{1/2+\nu+\nu}e^{-z}$

Table A.1. Special functions expressed with operator ${}_zO_{\beta}^{\alpha}$

Name	Representation with the Well-Posed Fractional Calculus Operator ${}_{g(z)}O^{\alpha}_{\beta}$
Hermite polynomial	$H_{2n}(z) = \frac{(-1)^n(2n)!}{2^n n!}\ {}_{z^2}O^{-n}_{\frac{1}{2}}\ e^{\frac{z^2}{2}}$ $= \frac{(-1)^n(2n)!\Gamma(\frac{5}{4})}{2^n n!}\ {}_{z^2}O^{-n}_{\frac{1}{2}}\ e^{\frac{z^2}{4}} \left(\frac{4}{z^2}\right)^{\frac{3}{4}} J_{-\frac{1}{4}}(\frac{z^2}{4})$ $H_{2n+1}(z) = \frac{(-1)^n(2n+1)!}{2^n n!}\ {}_{z^2}O^{-n}_{\frac{3}{2}}\ e^{\frac{z^2}{2}}$ $= \frac{(-1)^n(2n+1)!\Gamma(\frac{7}{4})}{2^n n!}\ {}_{z^2}O^{-n}_{\frac{3}{4}}\ e^{\frac{z^2}{4}} \left(\frac{4}{z^2}\right)^{\frac{5}{4}} J_{\frac{1}{4}}(\frac{z^2}{4})$ $H_n(z) = (-1)^n e^{z^2}\ z^{-n}\ \frac{{}_zO^1_{1-n}}{\Gamma(1-n)}\ e^{-z^2}$ $H_n(z) = \frac{(2z)^{n-m} n!}{m!(n-m)!}\ {}_zO^1_{1+n-m}\ H_m(z)$
Tchebicheff polynomial of the first kind	$T_n(z) = 2^n_{1-z}O^{-n}_{\frac{1}{2}}\ (1+z)^{-n}$ $= 2^{-n}\ {}_{1-z}O^n_{\frac{1}{2}}\ (1+z)^n$ $= (-1)^n\ 2^{-n}\ {}_{1+z}O^n_{\frac{1}{2}}\ (1-z)^n$
Tchebicheff polynomial of the second kind	$U_n(z) = (n+1)2^{n+2}\ {}_{1-z}O^{-n}_{\frac{3}{2}}\ (1+z)^{-n-2}$ $= (n+1)2^{-n}\ {}_{1-z}O^{2+n}_{\frac{3}{2}}\ (1+z)^n$ $= (n+1)2^{n+2}\ {}_{1+z}O^{-n}_{\frac{3}{2}}\ (1-z)^{-n-2}$
Modified Bessel function	$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ {}_{z^2}O^{\frac{1}{2}}_{1+\nu} \cosh z = \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ {}_{z^2}O^{\frac{3}{2}}_{1+\nu} \frac{\sinh z}{z}$ $= \frac{(\frac{z}{2})^\nu}{\Gamma(1+\nu)}\ e^{-z}\ {}_zO^{\frac{3}{2}}_{1+\nu}\ e^{2z}$
Bessel polynomial	$y_n(z) = \left(\frac{-z}{2}\right)^n\ {}_zO^{1+2n}_{\frac{1}{2}}\ H_{2n}\left(\sqrt{\frac{2}{z}}\right)$ $y_n(z) = \left(\frac{z}{2}\right)^n\ \frac{(2n)!}{n!}\ {}_tO^{-n}_{-2n}\ e_n^{\frac{2t}{z}}\vert_{t=1}$
Rice polynomial	$H_n(\varsigma, p; z) = {}_zO^{\varsigma}_p\ P_n(1-2z)$ $= \frac{n!}{(p)_n}\ {}_zO^{\varsigma}_1\ P_n^{(p-1,1-p)}(1-2z)$ $= {}_zO^{-n}_p\ {}_2F_1(n+1, \varsigma; 1; z)$
Incomplete gamma function	$\gamma(\alpha, z) = \frac{z^\alpha}{\alpha}\ {}_zO^1_{1+\alpha} e^{-z}$ $= \frac{z^\alpha}{\alpha}\ e^{-z}\ {}_zO^1_{1+\alpha} e^z$
Error function	$erf(z) = \frac{2z}{\sqrt{\pi}}\ {}_{z^2}O^{\frac{1}{2}}_{\frac{3}{2}}\ e^{-z^2}$ $= \frac{2z}{\sqrt{\pi}}\ e^{-z^2}\ {}_{z^2}O^1_{\frac{3}{2}}\ e^{z^2}$ $= \frac{2z}{\sqrt{\pi}}\ {}_zO^1_2\ e^{-z^2}$
Psi function	$\Psi(\xi) = -\gamma + \ln(z) -{}_z\ O^1_\xi \ln(z)$

Table A.2. Special functions expressed with operator ${}_zO^{\alpha}_{\beta}$

Name	Representation with the Well-Posed Fractional Calculus Operators ${}_{g(z)}O^{\alpha}_{\beta}$ and ${}_{g(z),h(w)}O^{\alpha,\alpha'}_{\beta,\beta'}$
First Appell function	$F_1(\alpha,\beta,\beta';\gamma;xt,yt) = {}_tO^{\beta}_{\gamma}\,(1-xt)^{-\beta}(1-yt)^{-\beta'}$ $= {}_{x,y,t}O^{\beta,\beta',\frac{\beta}{2}}_{\frac{\beta}{2},\frac{\beta}{2},\beta}\,{}_2F_1(\alpha,\beta;\gamma;xt-yt+yw)\|_{w=t}$ $={}_t\,O^{\frac{\gamma}{2}}_{\gamma}\,F_2(\alpha,\beta,\beta';\tfrac{\gamma}{2},\tfrac{\gamma}{2};xt,y(w-t))\|_{w=t}$ $F_1(\alpha,\beta,\beta';\gamma;x,y) = {}_{x,t}O^{\beta',\beta}_{\gamma,\gamma-\beta'}\,(1-x+xt-yt)^{-\alpha}\|_{t=1}$
Second Appell function	$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) ={}_{x,y}\,O^{\beta,\beta'}_{\gamma,\gamma'}\,(1-x-y)^{-\alpha}$ $= {}_{x,y}O^{\alpha,\alpha}_{\gamma,\gamma'}\,(1-x)^{-\beta}\,(1-y)^{-\beta'}\,{}_2F_1(\beta,\beta';\alpha;xy/(1-x)(1-y))$ $= {}_{y,t}O^{1+\alpha-\gamma',\alpha}_{\gamma,\gamma'}\,(1-yt)^{-\beta}\left(1+\frac{xt}{1-t}\right)^{-\beta}\Big\|_{t=1}$ $F_2(\alpha,\beta,\beta';\gamma,\gamma';xt,yt) = {}_{y,t}O^{\beta,\beta'}_{\beta,2\beta}\,F_4(\alpha,2\beta;\gamma,\gamma';xt,y(w-t))\|_{w=t}$
Third Appell function	$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y) = {}_xO^{\alpha}_{\gamma-\alpha'}\,{}_xO^{\beta}_{\gamma-\beta'}(1-x)^{\alpha'+\beta'-\gamma}\,{}_2F_1(\alpha',\beta';\gamma;x+y-xy)$ $= {}_xO^{\alpha}_{\gamma-\alpha'}\,{}_xO^{\beta}_{\gamma-\beta'}(1-y)^{\alpha+\beta-\gamma}\,{}_2F_1(\alpha,\beta;\gamma;x+y-xy)$ $F_3(\alpha,\alpha',\beta,\beta';\gamma;xt,yt) = {}_{y,t}O^{\beta',\beta}_{\beta,2\beta}F_1(2\beta,\alpha,\alpha';\gamma;xt,y(w-t))\|_{w=t}$ $= {}_tO^{\frac{\gamma}{2}}_{\gamma}\,{}_2F_1(\alpha,\beta;\tfrac{\gamma}{2};xt)\,{}_2F_1(\alpha',\beta';\tfrac{\gamma}{2};y(w-t))\|_{t=1}$
Fourth Appell function	$F_4(\alpha,\beta;\gamma,\gamma';x,y) = {}_{x,y}O^{\alpha,\alpha'}_{\gamma,\gamma}\,(1-x-y)^{-\beta}$ ${}_2F_1(\beta/2,\beta/2+1/2;\alpha;-4xy/(1-x-y)^2)$ $= {}_{x,y}O^{\beta,\beta'}_{\gamma,\gamma}\,(1-x-y)^{-\alpha}_2F_1(\alpha/2,\alpha/2+1/2;\beta;-4xy/(1-x-y)^2)$ $= {}_{x,t}O^{\beta,1+\alpha-\gamma'}_{\gamma,\gamma}\left(1+\frac{t(x-y+yt)}{1-t}\right)^{-\beta}\Big\|_{t=1}$ $= {}_tO^{\frac{1}{2}-\frac{\beta}{2}}_{1-\beta}\,F_2(\alpha,\beta/2,\beta/2;\gamma,\gamma';x/t,y/(w-t))\|_{t=1}$

Table A.3. Appell's hypergeometric functions of two variables expressed with operators ${}_zO^{\alpha}_{\beta}$ and ${}_{z,w}O^{\alpha,\alpha'}_{\beta,\beta'}$

Name	Representation with the Well-Posed Fractional Calculus Operators ${}_{g(z)}O^{\alpha}_{\beta}$ and ${}_{g(z),h(w)}O^{\alpha,\alpha'}_{\beta,\beta'}$
First ϕ_1 confluent function of Humbert	$\phi_1(\alpha,\beta;\gamma;xt,yt) = {}_tO^{\alpha}_{\gamma}\,(1-xt)^{-\beta}e^{yt}$ $= {}_tO^{\beta}_{\gamma}\,(1-xt)^{-\alpha}\,{}_1F_1\left(\alpha;\gamma-\beta;\frac{y(1-t)}{1-xt}\right)\Big\|_{t=1}$
Second ϕ_2 confluent function of Humbert	$\phi_2(\alpha,\beta;\gamma;x,y) = {}_yO^{\beta}_{\gamma-\alpha}\,e^{yt}\,{}_1F_1(\alpha;\alpha+\beta;t(x-y))$ $\phi_2(\alpha,\beta;\gamma;xt,yt) = {}_tO^{\alpha+\beta}_{\gamma}\,e^{yt}\,{}_1F_1(\alpha;\gamma;t(x-y))$
Third ϕ_3 confluent function of Humbert	$\phi_3(\alpha,\beta;\gamma;xt,yt) = {}_tO^{\frac{\gamma}{2}}_{\gamma}\,{}_1F_1(\alpha;\gamma/2;xt)\,{}_0F_1(-;\gamma/2;y(w-t))\|_{w=t}$ $\phi_3(\alpha,\beta;\gamma;x,y) = {}_tO^{\alpha}_{\gamma}\,e^{xt}\,{}_0F_1(-;\gamma-\alpha;y(1-t))\|_{t=1}$
First Ψ_1 confluent function of Humbert	$\Psi_1(\alpha,\beta;\gamma,\gamma';x,y) = {}_{x,y}O^{\beta,\alpha}_{\gamma,\gamma'}\,(1-x)^{-\alpha}\,e^{y/(1-x)}$ $= {}_{x,t}O^{\alpha,1+\alpha-\gamma'}_{\gamma',\gamma}\,e^{yt}\left(1+\frac{xt}{1-t}\right)^{-\beta}\Big\|_{t=1}$
Second Ψ_2 confluent function of Humbert	$\Psi_2(\alpha,\beta;\gamma;x,y) = {}_{x,y}O^{\alpha,\alpha}_{\beta,\gamma'}\,e^{x+y}\,{}_0F_1(-;\alpha;xy)$ $= {}_{x,t}O^{1+\alpha-\gamma,\alpha}_{\beta,\gamma}\,e^{\frac{t(y-x-yt)}{1-t}})\Big\|_{t=1}$
First Ξ_1 confluent function of Humbert	$\Xi_1(\alpha,\alpha',\beta;\gamma;xt,yt) = {}_{x,y,t}O^{\beta,\alpha',\frac{\gamma}{2}}_{\frac{\gamma}{2},\frac{\gamma}{2},\gamma'}\,(1-xt)^{-\alpha}\,e^{y(w-t)}\|_{w=t}$ $\Xi_1(\alpha,\alpha',\beta;\gamma;x,y) = {}_{x,t}O^{\beta,\alpha'}_{\gamma-\alpha,\gamma}\,(1-x+xt)^{-\alpha}\,e^{yt}\|_{t=1}$
Second Ξ_2 confluent function of Humbert	$\Xi_2(\alpha,\beta;\gamma;xt,yt) = {}_{x,t}O^{\beta,\frac{\gamma}{2}}_{\frac{\gamma}{2},\gamma}\,(1-xt)^{-\alpha}\,{}_0F_1(-;\tfrac{\gamma}{2};y(w-t))\|_{w=t}$ $\Xi_2(\alpha,\beta;\gamma;x,y) = {}_{x,t}O^{\alpha,\beta}_{\gamma-\beta,\gamma}\,e^{x-xt+yt}\|_{t=1}$

Table A.4. Confluent Hypergeometric Functions of two Variables expressed with operators ${}_zO^{\alpha}_{\beta}$ and ${}_{z,w}O^{\alpha,\alpha'}_{\beta,\beta'}$

No.	Function	Summation formula $f(w-z)=\sum_i a_i g_i(w) h_i(z)$
1	$\sin(z)$	$\sin(z)\cos(w)-\cos(w)\sin(z)$
2	$\cos(z)$	$\cos(z)\cos(w)+\sin(w)\sin(z)$
3	$\ln(z)$	$\ln(w)+\ln(1-z/w)$
4	Laguerre polynomial $L_n^{(1+a-b)}(z)$	$\sum_{s=0}^n L_s^{(a)}(w) L_{n-s}^{(b)}(-z)$
5	Hermite polynomial $H_n(z)$	$\sum_{s=0}^n \binom{n}{s} H_s(\sqrt{2}w) H_{n-s}(-\sqrt{2}z)$
6	Bessel function $J_n(z)$	$\sum_{-\infty}^{+\infty} J_s(w) J_{n-s}(-z)$
7	Newman polynomial with $-1/z$	$O_0(w)J_0(z)+2\sum_{s=1}^{\infty} O_s(w)J_s(z)$

Table A.5. Addition formulas

No.	Summation formulas
1	$\frac{\alpha z}{\beta}\,{}_2F_3\left[\begin{matrix}\alpha/2+1/2,\alpha/2+1\\ \beta/2+1/2,\beta/2+1,3/2\end{matrix}\middle\vert -\frac{z^2}{4}\right]$ $= \sin(z)\,{}_2F_3\left[\begin{matrix}\beta/2-\alpha/2,\beta/2-\alpha/2+1/2\\ \beta/2,\beta/2+1/2,1/2\end{matrix}\middle\vert -\frac{z^2}{4}\right] - \frac{(\beta-\alpha)z}{\beta}\cos(z)\,{}_2F_3\left[\begin{matrix}\beta/2-\alpha/2+1/2,\beta/2-\alpha/2+1\\ \beta/2+1/2,\beta/2+1,3/2\end{matrix}\middle\vert -\frac{z^2}{4}\right]$
2	${}_2F_3\left[\begin{matrix}\alpha/2,\alpha/2+1/2\\ \beta/2,\beta/2+1/2,1/2\end{matrix}\middle\vert -\frac{z^2}{4}\right]$ $= \cos(z)\,{}_2F_3\left[\begin{matrix}\beta/2-\alpha/2,\beta/2-\alpha/2+1/2\\ \beta/2,\beta/2+1/2,1/2\end{matrix}\middle\vert -\frac{z^2}{4}\right] - \frac{(\beta-\alpha)z}{\beta}\sin(z)\,{}_2F_3\left[\begin{matrix}\beta/2-\alpha/2+1/2,\beta/2-\alpha/2+1\\ \beta/2+1/2,\beta/2+1,3/2\end{matrix}\middle\vert -\frac{z^2}{4}\right]$
3	$\psi(\alpha)-\psi(\beta) = \frac{(\alpha-\beta)}{\beta}\,{}_3F_2\left[\begin{matrix}1,1,1+\beta-\alpha\\ 2,1+\beta\end{matrix}\middle\vert 1\right]$
4	$\frac{(2+a+b)_n}{n!}\,{}_2F_2\left[\begin{matrix}-n,\alpha\\ 2+a+b,\beta\end{matrix}\middle\vert z\right] = \sum_{s=0}^n L_s^{(a)}(z)\frac{(1+b)_{n-s}}{(n-s)!}\,{}_2F_2\left[\begin{matrix}-n+s,\beta-\alpha\\ 1+b,\beta\end{matrix}\middle\vert -z\right]$
5	$(2z)^n \frac{(\alpha)_n}{(\beta)_n}\,{}_4F_2\left[\begin{matrix}-n/2,-n/2+1/2,1/2-\beta/2-n/2,1-\beta/2-n/2\alpha\\ 1/2-\alpha/2-n/2,1-\alpha/2-n/2\end{matrix}\middle\vert -\frac{1}{z^2}\right]$ $= \sum_{s=0}^n \binom{n}{s}\frac{(\beta-\alpha)_s}{(\beta)_s}(-2\sqrt{2}z)^s H_{n-s}(\sqrt{2}z)\,{}_4F_2\left[\begin{matrix}-s/2,-s/2+1/2,1/2-\beta/2-s/2,1-\beta/2-s/2\\ 1/2-\beta/2+\alpha/2-s/2,1-\beta/2+\alpha/2-s/2\end{matrix}\middle\vert -\frac{1}{2z^2}\right]$
6	$\frac{(\alpha)_n}{(\beta)_n}\frac{(z/2)^n}{n!}\,{}_2F_3\left[\begin{matrix}\alpha/2+n/2,\alpha/2+n/2+1/2\\ \beta/2+n/2,\beta/2+n/2+1/2,1+n\end{matrix}\middle\vert -\frac{z^2}{4}\right]$ $= \sum_{-\infty}^{+\infty}\frac{(\beta-\alpha)_s}{(\beta)_s}\frac{(-z/2)^s}{s!}J_{n-s}(z)\,{}_2F_3\left[\begin{matrix}\beta-\alpha/2+s/2,\beta-\alpha/2+s/2+1/2\\ \beta/2+s/2,\beta/2+s/2+1/2,1+s\end{matrix}\middle\vert -\frac{z^2}{4}\right]$
7	$\frac{\beta-1}{\alpha-1}$ $= O_0(z)\,{}_2F_3\left[\begin{matrix}\beta/2-\alpha/2,\beta/2-\alpha/2+1/2\\ \beta/2,\beta/2+1/2,1\end{matrix}\middle\vert -\frac{z^2}{4}\right]\frac{1}{z} + 2\sum_{s=1}^{\infty}\frac{(\beta-\alpha)_s}{(\beta)_s}\frac{(z/2)^s}{s!}O_s(z)\,{}_2F_3\left[\begin{matrix}\beta/2-\alpha/2+s/2,\beta/2-\alpha/2+s/2+1/2\\ \beta/2+s/2,\beta/2+s/2+1/2,1+s\end{matrix}\middle\vert -\frac{z^2}{4}\right]$

Table A.6. Series associated to addition formulas of Table 5

$f_n(z)$	h_n	k_n	*Remarks*
$P_n^{(\alpha,\beta)}(z)$	$\frac{2^{1+\alpha+\beta}}{1+\alpha+\beta+2n}\frac{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!\Gamma(1+\alpha+\beta+n)}$	$\frac{1}{2^n}\binom{\alpha+\beta+2n}{n}$	$\alpha>-1,\beta>-1$
$L_n^{(\alpha)}(z)$	$\frac{\Gamma(1+\alpha+n)}{n!}$	$\frac{(-1)^n}{n!}$	$\alpha>-1$
$H_n(z)$	$\sqrt{\pi}2^n n!$	2^n	
$C_n^{(\alpha)}(z)$	$\frac{\pi 2^{1-2\alpha}\Gamma(\alpha+2n)}{n!(\alpha+n)(\Gamma(\alpha))^2}$	$\frac{2^n\Gamma(\alpha+n)}{n!\Gamma(\alpha)}$	$\alpha>-\frac{1}{2}$
$P_n(z)$	$\frac{2}{2n+1}$	$\frac{1}{2^n}\binom{2n}{n}$	
$T_n(z)$	$\frac{\pi}{2}(1+\delta_{n,0})$	2^{n-1}	$\delta_{i,j}=\begin{cases}0, & \text{if } i\neq j\\ 1, & \text{if } i=j.\end{cases}$
$U_n(z)$	$\frac{\pi}{2}$	2^n	

Table A.7. Christoffel-Darboux parameters