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Orthogonal Polynomials Concerning to the Abel and Lindelöf Weights and Their Modifications on the Real Line

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Abstract

Orthogonal polynomials related to Abel and Lindelöf weight functions on $\mathbb{R}$, as well as ones related to some products of these weight functions, are considered. Using the moments of the weight functions, the coefficients in the three-term recurrence relations are determined in the explicit form. Also, some connections with Meixner-Pollaczek polynomials with real parameters are presented.

Keywords: Orthogonal polynomials, Three-term recurrence relation, Weight functions of Abel and Lindelöf, Logistic weights, Moments, Hankel determinants, Meixner-Pollaczek polynomials, Gaussian quadrature

2010 MSC: 33C45, 33C47, 33D45, 33F10

1. Introduction

In this paper we denote the space of all algebraic polynomials defined on $\mathbb{R}$ by $\mathcal{P}$, and by $\mathcal{P}_N \subset \mathcal{P}$ the space of polynomials of degree at most N ($N \in \mathbb{N}$). Also, a nonnegative function $x \mapsto w(x)$ on $\mathbb{R}$ for which all moments $\mu_k = \int_{\mathbb{R}} x^k w(x)\,\mathrm{d}x$, $k \geq 0$, exist, are finite and $\mu_0 > 0$, we called the *weight function*. Then, for each $N \in \mathbb{N}$, there exists the N-point Gauss-Christoffel quadrature rule (*cf.* [22]-[24])

$$\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{N} A_\nu^{(N)} f(x_\nu^{(N)}) + R_N(f), \tag{1.1}$$

which is exact for all polynomials of degree $\leq 2N-1$ ($f \in \mathcal{P}_{2N-1}$).

We start this paper with two weight functions on $\mathbb{R}$:

- The Abel weight
$$w_A(x) = \frac{x}{\mathrm{e}^{\pi x} - \mathrm{e}^{-\pi x}} = \frac{x}{2\sinh \pi x}; \tag{1.2}$$
- The Lindelöf weight
$$w_L(x) = \frac{1}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}} = \frac{1}{2\cosh \pi x}. \tag{1.3}$$

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In 1823 Niels Henrik Abel [1] proved an interesting summation formula for the finite *"alternating sum"*

$$\sum_{k=m}^{n}(-1)^k f(k) = \frac{1}{2}[(-1)^m f(m) + (-1)^n f(n+1)] - \int_{\mathbb{R}}[(-1)^m \psi_m(y) + (-1)^n \psi_{n+1}(y)] w_A(y)\,\mathrm{d}y, \tag{1.4}$$

where the function $\psi_m(y)$ in (1.4) is given by

$$\psi_m(y) = \frac{f(m+\mathrm{i}y) - f(m-\mathrm{i}y)}{2\mathrm{i}y}.$$

When $n \to +\infty$ (1.4) reduces to the *Abel summation formula* for the alternating series

$$\sum_{k=m}^{+\infty}(-1)^{k-m} f(k) = \frac{1}{2} f(m) - \int_{\mathbb{R}} \frac{f(m+\mathrm{i}y) - f(m-\mathrm{i}y)}{2\mathrm{i}y} w_A(y)\,\mathrm{d}y. \tag{1.5}$$

As an alternative formula to (1.5) there is the *Lindelöf summation formula* [17]

$$\sum_{k=m}^{+\infty}(-1)^{k-m} f(k) = \int_{\mathbb{R}} f(m - 1/2 + \mathrm{i}y) w_L(y)\,\mathrm{d}y, \tag{1.6}$$

where the *Lindelöf* weight function $w_L(x)$ is given by (1.3).

In order to construct quadrature formulas of Gaussian type with respect to the weight functions $w_A(x)$ and $w_L(x)$, for integrals which appear in (1.4) and (1.6), respectively, we need the corresponding (monic) orthogonal polynomials π_k, i.e., their three-term recurrence relations

$$\pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, \ldots, \tag{1.7}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$, where recursion coefficients $\{\alpha_k\}$ and $\{\beta_k\}$ depend only on the weight function $w(x)$ (in our case, $w(x) = w_A(x)$ or $w(x) = w_L(x)$). The coefficient β_0 may be arbitrary, but is conveniently defined by $\beta_0 = \mu_0 = \int_{\mathbb{R}} w(x)\,\mathrm{d}x$.

For even weights on $\mathbb{R}$, such as our weight functions (1.2) and (1.3), the coefficients α_k are zero, so that (1.7) becomes

$$\pi_{k+1}(x) = x\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0, 1, \ldots\,. \tag{1.8}$$

Remark 1.1. The quadrature *nodes* $x_\nu^{(N)}$, $\nu = 1, \ldots, N$, in (1.1) are eigenvalues of the *Jacobi* matrix

$$J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{N-1}} \\ \mathbf{O} & & & \sqrt{\beta_{N-1}} & \alpha_{N-1} \end{bmatrix},$$

and the first components of the corresponding normalized eigenvectors $\mathbf{v}_\nu = [v_{\nu,1} \ \ldots \ v_{\nu,N}]^{\mathrm{T}}$ (with $\mathbf{v}_\nu^{\mathrm{T}}\mathbf{v}_\nu = 1$) give the *weight coefficients* (*Christoffel numbers*) $A_\nu^{(N)} = \beta_0 v_{\nu,1}^2$, $\nu = 1, \ldots, N$. Such a construction of the Gauss-Christoffel quadrature rule (1.1) is done by the Golub-Welsch algorithm [13].

Unfortunately, the recursion coefficients are known explicitly only for some narrow classes of orthogonal polynomials, as e.g. for the classical orthogonal polynomials (Jacobi, the generalized Laguerre, and Hermite polynomials). However, for a large class of the so-called *strongly non-classical polynomials* these coefficients can be constructed numerically. Basic procedures for generating these coefficients are the *method of (modified) moments*, the *discretized Stieltjes–Gautschi procedure*, and the *Lanczos algorithm* and they play a central role in the so-called *constructive theory of orthogonal polynomials*, which was developed by Walter Gautschi in the eighties on the last century. In [10] he starts with an arbitrary positive measure $\mathrm{d}\mu(t)$, which is given explicitly, or implicitly via moment information, and considers the basic computational problem: *For a given measure* $\mathrm{d}\mu$ *and for given* $n \in \mathbb{N}$, *generate the first* n *coefficients* $\alpha_k(\mathrm{d}\mu)$ *and* $\beta_k(\mathrm{d}\mu)$ *for* $k = 0, 1, \ldots, n-1$. The problem is very sensitive with respect to small perturbations in the data. The basic references are [10, 11, 19] and [25].

By the progress in *symbolic computation* and *variable-precision arithmetic* it is possible to generate the recurrence coefficients α_k and β_k directly by using the original Chebyshev method of moments in sufficiently high precision. The corresponding software for such a purpose, as well as many other calculations with orthogonal polynomials and different quadrature rules, is now available: Gautschi's package `SOPQ` in MATLAB, and our MATHEMATICA package `OrthogonalPolynomials` (see [5] and [28]). These packages are downloadable from Web Sites:

`http://www.cs.purdue.edu/archives/2002/wxg/codes/`

and

`http://www.mi.sanu.ac.rs/~gvm/`,

respectively. Thus, all that is required is a procedure for the symbolic calculation of moments or their calculation in variable-precision arithmetic.

For a given sequence of moments (`mom`), our MATHEMATICA Package `OrthogonalPolynomials` enables us to get recurrence coefficients {`al,be`} in a symbolic form

```
{al,be}=aChebyshevAlgorithm[mom, Algorithm -> Symbolic];
```

The moments for the Abel weight function (1.2) can be expressed in terms of Bernoulli numbers as (*cf.* [24])

$$\mu_k^A = \int_{\mathbb{R}} \frac{x^{k+1}}{2\sinh(\pi x)}\,\mathrm{d}x = \begin{cases} 0, & k \text{ odd}, \\ (2^{k+2}-1)\dfrac{(-1)^{k/2}B_{k+2}}{k+2}, & k \text{ even}. \end{cases} \tag{1.9}$$

Using the package `OrthogonalPolynomials` we get the coefficients in the three-term recurrence relation (1.8) for the Abel polynomials $\pi_k^A(x)$ in explicit form (see [19, p. 159])

$$\beta_0 = \mu_0 = \frac{1}{4}, \quad \beta_k = \frac{k(k+1)}{4}, \quad k = 1, 2, \ldots\,. \tag{1.10}$$

For the Lindelöf weight (1.3) the moments can be expressed in terms of the generalized Riemann zeta function $z \mapsto \zeta(z, a)$, defined by

$$\zeta(z, a) = \sum_{\nu=0}^{+\infty} (\nu + a)^{-z},$$

as (*cf.* [24])

$$\mu_k^L = \int_{\mathbb{R}} \frac{x^k\,\mathrm{d}x}{2\cosh(\pi x)} = \begin{cases} \dfrac{1}{2}, & k = 0, \\ 0, & k \text{ odd}, \\ \dfrac{2k!}{(4\pi)^{k+1}}\left[\zeta\left(k+1, \tfrac{1}{4}\right) - \zeta\left(k+1, \tfrac{3}{4}\right)\right], & k \text{ even } (\geq 2). \end{cases}$$

Then we can obtain the recurrence coefficients for the Lindelöf polynomials $\pi_k^L(x)$ (see also [19, p. 159])

$$\beta_0 = \mu_0 = \frac{1}{2}, \quad \beta_k = \frac{k^2}{4}, \quad k = 1, 2, \ldots\,.$$

Some additional information on the Abel and Lindelöf orthogonal polynomials $\pi_k^A(x)$ and $\pi_k^L(x)$ can be found in [7]-[9], [27, 29].

Remark 1.2. The term *Abel polynomial* (not orthogonal!) also met as a polynomial $A_k(x; a) = x(x - ak)^{k-1}$ of degree k, given by by the generating function

$$\sum_{k=0}^{+\infty} \frac{A_k(x; a)}{k!} t^k = \mathrm{e}^{xW(at)/a},$$

where $x \mapsto W(x)$ is the *Lambert W-function* (i.e., the the inverse function of $f(W) = W\mathrm{e}^W$). For details on this subject, as well as on the associated Sheffer sequence, see [34, p. 29 & p. 73].

In the next section we consider orthogonal polynomials with respect to weights obtained as a product of the previous weight functions (1.2) and (1.3). Some symmetric Meixner-Pollaczek polynomials with a real parameter will be analyzed in Section 3.

2. Product weight functions (1.2) and (1.3)

In this section we consider three products of the weight functions (1.2) and (1.3):

- The Abel^2 weight

$$w_{A^2}(x) = w_A(x)^2 = \left(\frac{x}{2\sinh \pi x}\right)^2; \tag{2.1}$$

- The Abel-Lindelöf weight

$$w_{AL}(x) = w_A(x)w_L(x) = \frac{x}{4\sinh \pi x \cosh \pi x} = \frac{x}{2\sinh(2\pi x)},$$

$$w_{AL}(x) = \frac{1}{2}w_A(2x); \tag{2.2}$$

- The Lindelöf^2 weight

$$w_{L^2}(x) = w_L(x)^2 = \frac{1}{4\cosh^2 \pi x}. \tag{2.3}$$

As we can see, the weight function in (2.2) is again the Abel weight, but the moments given in (1.9) should be divided by 2^{k+2}. According to (1.10), the corresponding coefficients in the three-term recurrence relation (1.8), in this case are given by

$$\beta_0 = \mu_0 = \frac{1}{16}, \quad \beta_k = \frac{k(k+1)}{16}, \quad k = 1, 2, \ldots .$$

2.1. *The Abel^2 weight*

Here we consider the moments of the weight function $w_{A^2}(x)$ defined by (2.1),

$$\mu_k \equiv \mu_k^{A^2} = \int_{\mathbb{R}} \frac{x^{k+2}}{4\sinh^2 \pi x}\,\mathrm{d}x.$$

It is easy to find $\mu_0 = 1/(12\pi)$, as well as that $\mu_k = 0$ for odd k.

In order to determine μ_k for even $k \geq 2$, we use the equality (*cf.* [33, Eq. 2.4.9.2, p. 361])

$$\int_0^\infty x^{\alpha-1}(\coth ax - 1)\,\mathrm{d}x = \frac{2^{1-\alpha}}{a^\alpha}\Gamma(\alpha)\zeta(\alpha), \quad a > 0,\ \mathrm{Re}\,\alpha > 1,$$

where the Riemann zeta function $s \mapsto \zeta(s)$ is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \mathrm{Re}\, s > 1.$$

An integration by parts of the previous integral gives

$$\int_0^\infty x^{\alpha-1}(\coth ax - 1)\,\mathrm{d}x = \frac{a}{\alpha}\int_0^\infty \frac{x^\alpha}{\sinh^2 ax}\,\mathrm{d}x, \quad a > 0,\ \alpha > 2.$$

Putting $a = \pi$ and $\alpha = k + 2$, we conclude that for each even $k \geq 2$

$$\mu_k = \frac{k+2}{\pi} \cdot \frac{\Gamma(k+2)\zeta(k+2)}{(2\pi)^{k+2}}.$$

Since for even k, the value of zeta function can be expressed in terms of Bernoulli numbers,

$$\zeta(k+2) = (-1)^{k/2}\frac{B_{k+2}(2\pi)^{k+2}}{2(k+2)!},$$

we get finally

$$\mu_k = \int_{\mathbb{R}} \frac{x^{k+2}}{4\sinh^2(\pi x)}\,\mathrm{d}x = \begin{cases} 0, & k \text{ odd}, \\ (-1)^{k/2}\dfrac{B_{k+2}}{2\pi}, & k \text{ even}. \end{cases} \tag{2.4}$$

Theorem 2.1. *The polynomials* $\pi_k(x) \equiv \pi_k^{A^2}(x)$, $k = 0, 1, \ldots$, *orthogonal with respect to the weight function* $w_{A^2}(x)$ *given by* (2.1) *satisfy the following three-term recurrence relation*

$$\pi_{k+1}^{A^2}(x) = x\pi_k^{A^2}(x) - \frac{k(k+1)^2(k+2)}{4(2k+1)(2k+3)}\pi_{k-1}^{A^2}(x), \quad k = 0, 1, 2, \ldots, \tag{2.5}$$

where $\pi_0^{A^2}(x) = 1$ *and* $\pi_{-1}^{A^2}(x) = 0$.

Proof. Using the moments given by (2.4), we consider the corresponding Hankel determinats

$$\Delta_0 = 1, \quad \Delta_k = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{k-1} \\ \mu_1 & \mu_2 & & \mu_k \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-2} \end{vmatrix}, \quad k = 1, 2, \ldots, \tag{2.6}$$

as well as two determinants (with non-zero elements) as in [6]

$$E_m = \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2m-2} \\ \mu_2 & \mu_4 & & \mu_{2m} \\ \vdots & & & \\ \mu_{2m-2} & \mu_{2m} & & \mu_{4m-4} \end{vmatrix}, \quad F_m = \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2m} \\ \mu_4 & \mu_6 & & \mu_{2m+2} \\ \vdots & & & \\ \mu_{2m} & \mu_{2m+2} & & \mu_{4m-2} \end{vmatrix}.$$

Our purpose is to evaluate the moment determinants Δ_k, which can be expressed in terms of the determinants E_m and F_m.

Similarly as in [12, 21] and [6], using Laplace expansion for determinants (2.6), we can get (see [6, Lemma 2.2])

$$\Delta_{2m} = E_m F_m \quad \text{and} \quad \Delta_{2m+1} = E_{m+1}F_m. \tag{2.7}$$

Depending of parity of m, we can calculate the determinants E_m and F_m, as well as their quotients, but these processes are technical and can be given by an expansion of determinants in the last row, using a very long computation, which is partly done using the symbolic capabilities of Mathematica. We omit the procedure due to space limitations and we mention only quotients of the determinants E_m and F_m:

$$\begin{cases} \dfrac{E_m}{F_m} = \dfrac{5\times 4^{m-1}\left(\frac{7}{4}\right)_{m-1}\left(\frac{9}{4}\right)_{m-1}}{\left(\frac{3}{2}\right)_{m-1}((2)_{m-1})^2\left(\frac{5}{2}\right)_{m-1}} = \dfrac{1}{(2m)!}\dbinom{4m+1}{2m}, \\ \dfrac{E_m}{F_{m-1}} = \dfrac{(1)_{m-1}(\left(\frac{3}{2}\right)_{m-1})^2(2)_{m-1}}{3\times 4^m\pi\left(\frac{5}{4}\right)_{m-1}\left(\frac{7}{4}\right)_{m-1}} = \dfrac{(2m-1)!}{4\pi}\dbinom{4m-1}{2m}^{-1}. \end{cases} \tag{2.8}$$

The recurrence coefficients β_k in (2.5) for the weight function (2.1) can be expressed in terms of the Hankel determinants (2.6) (*cf.* [19, p. 97]) as

$$\beta_k = \frac{\Delta_{k-1}\Delta_{k+1}}{\Delta_k^2}, \quad k \geq 1. \tag{2.9}$$

According to (2.7) and (2.9) for $k = 2m$ and $k = 2m + 1$, we have

$$\beta_{2m} = \frac{E_m F_{m-1}}{E_m F_m} \cdot \frac{E_{m+1} F_m}{E_m F_m} = \frac{E_{m+1}}{F_m} \left(\frac{E_m}{F_{m-1}} \right)^{-1}$$

and

$$\beta_{2m+1} = \frac{E_m F_m}{E_{m+1} F_m} \cdot \frac{E_{m+1} F_{m+1}}{E_{m+1} F_m} = \frac{E_m}{F_m} \left(\frac{E_{m+1}}{F_{m+1}} \right)^{-1},$$

respectively. Finally, using (2.8), from these equalities we get

$$\beta_0 = \mu_0 = \frac{1}{12\pi}, \quad \beta_k = \frac{1}{4} \cdot \frac{k(k+1)^2(k+2)}{(2k+1)(2k+3)}, \quad k = 1, 2, \ldots,$$

which proves the recurrence relation (2.5). □

Remark 2.2. Explicit expressions for orthogonal polynomials $\pi_k^{A^2}(x)$ are

$$\begin{aligned}
&\pi_0^{A^2}(x) = 1, \quad \pi_1^{A^2}(x) = x, \quad \pi_2^{A^2}(x) = x^2 - \frac{1}{5}, \quad \pi_3^{A^2}(x) = x^3 - \frac{5x}{7}, \\
&\pi_4^{A^2}(x) = x^4 - \frac{5x^2}{3} + \frac{4}{21}, \quad \pi_5^{A^2}(x) = x^5 - \frac{35x^3}{11} + \frac{14x}{11}, \\
&\pi_6^{A^2}(x) = x^6 - \frac{70x^4}{13} + \frac{707x^2}{143} - \frac{60}{143}, \\
&\pi_7^{A^2}(x) = x^7 - \frac{42x^5}{5} + \frac{189x^3}{13} - \frac{3044x}{715}, \\
&\pi_8^{A^2}(x) = x^8 - \frac{210x^6}{17} + \frac{609x^4}{17} - \frac{5260x^2}{221} + \frac{4032}{2431}, \\
&\pi_9^{A^2}(x) = x^9 - \frac{330x^7}{19} + \frac{25179x^5}{323} - \frac{31240x^3}{323} + \frac{96624x}{4199}, \\
&\pi_{10}^{A^2}(x) = x^{10} - \frac{165x^8}{7} + \frac{2937x^6}{19} - \frac{103015x^4}{323} + \frac{385836x^2}{2261} - \frac{43200}{4199},
\end{aligned}$$

etc.

2.2. *The Lindelöf² weight*

Here we consider the weight function $w_{L^2}(x)$ defined by (2.3), i.e.,

$$w_{L^2}(x) = \frac{1}{4\cosh^2 \pi x} = \frac{1}{(e^{\pi x} + e^{-\pi x})^2} = \frac{e^{-2\pi x}}{(1 + e^{-2\pi x})^2}.$$

As we can see, this function is the so-called *logistic weight* (*cf.* [24, p. 49]). Exactly,

$$w_{L^2}(x) = w^{\log}(2x),$$

for which the moments are

$$\mu_k = \int_{\mathbb{R}} x^k w^{\log}(2x)\,dx = \begin{cases} 0, & k \text{ odd}, \\ (-1)^{k/2-1} \dfrac{(2^{k-1} - 1)B_k}{2^k \pi}, & k \text{ even}. \end{cases}$$

The corresponding coefficients in the three-term recurrence relation (1.8), in this case are given by

$$\beta_0 = \mu_0 = \frac{1}{2\pi}, \quad \beta_k = \frac{k^4}{4(4k^2 - 1)}, \quad k = 1, 2, \ldots.$$

Remark 2.3. One-side logistic weight function, i.e., the hyperbolic function $x \mapsto 1/\cosh^2 x$ on $\mathbb{R}_+$ was used in a method for summation of slowly convergent series [20].

3. Some symmetric Meixner-Pollaczek polynomials with real parameter

In a recent joint paper with Gupta [14] we have provided a solution to the open problem on the exponential type operators, connected with $1+x^2$, using the Meixner-Pollaczek polynomials $p_k^{(\lambda)}(x)$ defined by the following three-term recurrence relation (*cf.* [4, 16])

$$(k+1)p_{k+1}^{(\lambda)}(x) = xp_k^{(\lambda)}(x) - (k-1+2\lambda)p_{k-1}^{(\lambda)}(x), \quad k = 0,1,\ldots, \tag{3.1}$$

with $p_0^{(\lambda)}(x) = 1$, $p_{-1}^{(\lambda)}(x) = 0$, and the parameter $\lambda > 0$. Polynomials $p_k^{(\lambda)}(x)$ are orthogonal on $\mathbb{R}$ with respect to the weight function

$$W_\lambda(x) = \frac{1}{2\pi}\left|\Gamma\left(\lambda + \mathrm{i}\frac{x}{2}\right)\right|^2. \tag{3.2}$$

Also, it is known the generating function for these polynomials is given by

$$G_\lambda(x,t) = \frac{\mathrm{e}^{x\arctan t}}{(1+t^2)^\lambda} = \sum_{k=0}^{\infty} p_k^{(\lambda)}(x)t^k.$$

The Meixner-Pollaczek polynomials were first invented by Meixner [18] and independently later by Pollaczek [32]. Many details can be found in [2]-[4], [15, 16, 30].

Taking $\lambda/2$ instead of λ and $2x$ instead of x in (3.2), we consider the following modified weight function

$$w_\lambda(x) = W_{\lambda/2}(2x) = \frac{1}{2\pi}\left|\Gamma\left(\frac{\lambda}{2} + \mathrm{i}\,x\right)\right|^2.$$

Then, the corresponding three-term recurrence relation (3.1) for the monic polynomials $P_k^{(\lambda/2)}(x) = a_k p_k^{(\lambda/2)}(2x)$, where $a_k = 2^{-k}k!$ $(k \geq 0)$, becomes

$$P_{k+1}^{(\lambda/2)}(x) = xP_k^{(\lambda/2)}(x) - \beta_k P_{k-1}^{(\lambda/2)}(x), \quad k = 0,1,\ldots,$$

where $P_0^{(\lambda/2)}(x) = 1$ and $P_{-1}^{(\lambda/2)}(x) = 0$, and

$$\beta_0 = \int_{\mathbb{R}} w_\lambda(x)\,\mathrm{d}x, \quad \beta_k = \frac{1}{4}k(k-1+\lambda), \quad k \geq 1.$$

Explicit expressions for the monic orthogonal polynomials $P_k^{(\lambda/2)}(x)$ can be done in terms of the Gauss hypergeometric function,

$$P_k^{(\lambda/2)}(x) = \frac{(\lambda)_k \mathrm{i}^k}{2^k}\, {}_2F_1\left(\begin{matrix} -k,\ \lambda/2+\mathrm{i}x \\ \lambda \end{matrix}\ \middle|\ 2\right), \quad k = 0,1,2,\ldots,$$

with the generating function

$$G_{\lambda/2}(2x,t) = \frac{\mathrm{e}^{2x\arctan t}}{(1+t^2)^{\lambda/2}} = \sum_{k=0}^{\infty} P_k^{(\lambda/2)}(x)\frac{(2t)^k}{k!}.$$

For example, these Meixner-Pollaczek polynomials $P_k^{(\lambda/2)}(x)$ for $k \leq 6$ are:

$$\begin{aligned}
&P_0^{(\lambda/2)}(x) = 1, \quad P_1^{(\lambda/2)}(x) = x, \quad P_2^{(\lambda/2)}(x) = x^2 - \frac{1}{4}\lambda,\\
&P_3^{(\lambda/2)}(x) = x^3 - \frac{1}{4}(3\lambda+2)x, \quad P_4^{(\lambda/2)}(x) = x^4 - \frac{1}{2}(3\lambda+4)x^2 + \frac{3}{16}\lambda(\lambda+2),\\
&P_5^{(\lambda/2)}(x) = x^5 - \frac{5}{2}(\lambda+2)x^3 + \frac{1}{16}\left(15\lambda^2 + 50\lambda + 24\right)x,\\
&P_6^{(\lambda/2)}(x) = x^6 - \frac{5}{4}(3\lambda+8)x^4 + \frac{1}{16}\left(45\lambda^2 + 210\lambda + 184\right)x^2 - \frac{15}{64}\lambda(\lambda+2)(\lambda+4).
\end{aligned}$$

Because of (*cf.* [31, Eq. 5.4.4] and [26, p. 113])

$$\left|\Gamma\left(\frac{1}{2}+\mathrm{i}\,y\right)\right|^2=\frac{\pi}{\cosh\pi y}\quad\text{and}\quad|\Gamma(1+\mathrm{i}\,y)|^2=\frac{\pi y}{\sinh\pi y}$$

the modified weight function $w_\lambda(x)$ for $\lambda = 1$ and $\lambda = 2$ reduces to the Lindelöf and the Abel weight function,

$$w_1(x)=\frac{1}{2\cosh\pi x}=w_L(x)\quad\text{and}\quad w_2(x)=\frac{x}{2\sinh\pi x}=w_A(x),$$

respectively, so that

$$P_k^1(x)=\pi_k^A(x)\quad\text{and}\quad P_k^{1/2}(x)=\pi_k^L(x).$$

Remark 3.1. The cases with $\lambda \le 0$ were also investigated with respect to certain non-standard inner product (*cf.* [4]).

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