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Generalized Metric Spaces

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Abstract

In the paper we present some generalization of the Paluszyński, Stempak method of producing an "induced" metric by a b-metric, by using Cauchy multiplicative functional equation.

Keywords: Metric space, metrizability, b-metric space, generalized b-metric space, completion of a generalized b-metric space, fixed point

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Contents

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1. Preface

Fixed point theory has become on very important branch of mathematics, because it is useful also in applications both in ordinary and partial differential equations, for example.

Many aspects of fixed point theory can be found in the book of Dugundi and Granas [29]. They are also journals, devoted to fixed point theory.

In this survey article we try to present so-called b-metric spaces, generalized b-metric spaces as well as some recent results in the fixed point theory and their applications.

We use the notations the same as in the relevant mathematical papers. For example, the symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Z}_+, \mathbb{Q}, \mathbb{R}$, $\mathbb{R}_+, \mathbb{C}$, denote the sets of natural numbers, natural numbers with zero, integers, positive integers, rationals, real numbers, positive real numbers and complex numbers, respectively.

The list of references at the end of the survey article, is by no means complete; we have presented only those, which we referred to in the text.

2. Quasi-metric spaces

The ideas of metric and metric spaces are very important both in the theory as well as in applications in many areas of mathematics and other sciences.

We present in this survey article some generalizations of these ideas.

A b-metric on a set X (nonempty) is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$,

for all $x, y, z \in X$, and for some fixed real number $s \geqslant 1$. The pair (X, d) is called a b-metric space.
Clearly, for $s = 1$ we obtain a metric on X. If d can take infinite values, we call d the generalized b-metric and (X, d) the generalized b-metric space (see also e.g. [12]).

The idea of b-metric spaces were considered by several authors. We present some of these papers:

1. Coiman and de Guzman [13],
2. Macias and Segovia [40], [41],
3. Bakhtin [9],
4. Czerwik [16], [17],
5. Khamsi and Hussain [36],
6. Xia [51].

Similarly, if X is a linear space, we can define a b-norm as follows:

(iv) $\|x\| = 0 \Leftrightarrow x = 0$,

(v) $\|\lambda x\| = |\lambda|\|x\|$,

(vi) $\|x + y\| \leqslant s(\|x\| + \|y\|)$,

for all $x, y \in X$, $\lambda \in \mathbb{C}$ and some $s \geqslant 1$. Also if $\|\cdot\|$ can take infinite values, we call $\|\cdot\|$ the generalized b-norm and $(X, \|\cdot\|)$ the generalized b-normed space.

A typical example of b-normed spaces (generalized b-normed spaces), one can obtain from metric spaces.

Example 2.1. Let (X, d) be a metric space and let $\alpha > 0$. Then d^α is a b-metric such that

$$d^\alpha(x, y) \leqslant 2^\alpha[d^\alpha(x, z) + d^\alpha(z, y)],$$

for all $x, y, z \in X$.
In fact, for $x, y, z \in X$, one has

$$\begin{aligned} d^\alpha(x, y) &\leqslant \left[d(x, z) + d(z, y)\right]^\alpha \leqslant \left(2 \max\left[d(x, z), d(z, y)\right]\right)^\alpha \\ &\leqslant 2^\alpha[d^\alpha(x, z) + d^\alpha(z, y)]. \end{aligned}$$

The conditions (i) and (ii) are obvious.

3. Minkowski's inequalities for integrals and series

We start with the following Lemma.

Lemma 3.1. *Let $a \geqslant 0$, $b \geqslant 0$, $0 < p < 1$. Then*

$$(a + b)^p \leqslant a^p + b^p. \tag{3.1}$$

Proof. Consider two cases:

1. $a = 0$ or $b = 0$,

then (3.1) is obvious; and

2. $a > 0$ and $b > 0$.

Take (see also [27])

$$H(x) := \frac{1 + x^p}{(1 + x)^p}, \quad 0 \leqslant x \leqslant 1.$$

Since $H'(x) \geqslant 0$, so H is increasing. Let

$$x = \frac{a}{b} \quad \text{if} \quad 0 < \frac{a}{b} \leqslant 1.$$

One has

$$\frac{1 + x^p}{(1 + x)^p} \geqslant H(0) = 1,$$

and therefore,

$$\frac{1 + (\frac{a}{b})^p}{(1 + \frac{a}{b})^p} \geqslant 1,$$

where we get (3.1). If $\frac{b}{a} \leqslant 1$, the proof is the same (see also [27]). □

Lemma 3.2. *If $a \geqslant 0$, $b \geqslant 0$ and $p \geqslant 1$, then*

$$(a + b)^p \leqslant 2^{p-1}(a^p + b^p). \tag{3.2}$$

Proof. Take $f(x) = x^p$, $x \in \mathbb{R}_+$. Since f is convex, one has

$$f\Big(\frac{a + b}{2}\Big) = \Big(\frac{a + b}{2}\Big)^p \leqslant \frac{1}{2} f(a) + \frac{1}{2} f(b) = \frac{1}{2}(a^p + b^p),$$

i.e. (3.2). □

Lemma 3.3. *Let $f : E \to \mathbb{R}$, $g : E \to \mathbb{R}$, $E = [0, 1]$, f, g are measarable functions satisfying the conditions*

$$\int_E |f(x)|^p < \infty \quad \text{and} \quad \int_E |g(x)|^p < \infty,$$

for $0 < p < 1$, then

$$\|f + g\| \leqslant 2^{\frac{1-p}{p}} \Big(\|f\| + \|g\|\Big), \tag{3.3}$$

where

$$\|f\| := \Big(\int_E |f(x)|^p \Big)^{\frac{1}{p}}.$$

Proof. By (3.1) we have

$$\int_E |f(x) + g(x)|^p \leqslant \int_E \Big(|f(x)|^p + |g(x)|^p\Big) \leqslant \int_E |f(x)|^p + \int_E |g(x)|^p.$$

Next, by (3.2) one has

$$\begin{aligned} \Big[\int_E \big|f(x) + g(x)\big|^p \Big]^{\frac{1}{p}} &\leqslant \Big[\int_E |f(x)|^p + \int_E |g(x)|^p \Big]^{\frac{1}{p}} \\ &\leqslant 2^{\frac{1}{p}-1} \Big[\Big(\int_E |f(x)|^p \Big)^{\frac{1}{p}} + \Big(\int_E |g(x)|^p \Big)^{\frac{1}{p}} \Big] \end{aligned}$$

i.e. (3.3) for $0 < p < 1$. □

Remark 3.4. The inequality 3.3 we call the Minkowski inequality for integrals, for $0 < p < 1$.

Example 3.5. Let $f(x) = 1$, $g(x) = x$, $x \in E$, $p = \frac{1}{2}$. Then the triangle inequality does not hold for $s = 1$. In fact, left-hand side $L = \frac{4}{9}(9 - 4\sqrt{2})$, but right-hand side $R = \frac{13}{9}$, and $L > R$.

Theorem 3.6. *Let* $u = (x_1, \ldots, x_n)$, $v = (y_1, \ldots, y_n)$, $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, n$, $0 < p < 1$, $n \in \mathbb{N}$ *and*

$$\|u\| := \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{\frac{1}{p}}.$$

Then

$$\|u + v\| \leqslant 2^{\frac{1}{p}-1}(\|u\| + \|v\|). \tag{3.4}$$

Proof. We use the mathematical induction principle. For $n = 2$, by Lemma 3.1 and Lemma 3.2, we easily get the proof. Assume that for $n = k \geqslant 2$ the inequality (3.4) is true. For $n = k + 1$ one has (by Lemma 3.1 and Lemma 3.2)

$$\begin{aligned} \|u + v\| &= \Big(|x_1 + y_1|^p + \ldots + |x_{k+1} + y_{k+1}|^p\Big)^{\frac{1}{p}} \\ &\leqslant \Big(|x_1|^p + \ldots + |x_{k+1}|^p + |y_1|^p + \ldots + |y_{k+1}|^p\Big)^{\frac{1}{p}} \\ &\leqslant 2^{\frac{1}{p}-1}\Big[(|x_1|^p + \ldots + |x_{k+1}|^p)^{\frac{1}{p}} + (|y_1|^p + \ldots + |y_{k+1}|^p)^{\frac{1}{p}}\Big] \\ &\leqslant 2^{\frac{1}{p}-1}(\|u + v\|), \end{aligned}$$

i.e. (3.4). □

Remark 3.7. Let $l^p, 0 < p < 1$ be the space of all sequences $x = (x_k)$ of real (or complex) numbers such that

$$\sum_{k=1}^{\infty} |x_k|^p < \infty,$$

with the norm

$$\|x\| := \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}, \quad 0 < p < 1. \tag{3.5}$$

Then, for $x = (x_k)$, $y = (y_k)$, $x, y \in l^p$, $0 < p < 1$,

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p}-1} \left[\left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \right]. \tag{3.6}$$

In fact, one gets (3.6) from (3.4) for $n \to \infty$.

Remark 3.8. The inequality (3.6) is called the Minkowski inequality for series and $0 < p < 1$.

Example 3.9. Let

$$u = (1, 1), \quad v = (1, 2), \quad p = \frac{1}{2}.$$

Then $L = 5 + 2\sqrt{6}$, $R = 7 + 2\sqrt{2}$, and $L > R$, which means that for $0 < p < 1$ the triangle inequality (3.4) does not hold with $s = 1$.

Now we consider the problem of separability of l^p, $0 < p < 1$.

Definition 3.10. A b-metric space (X, d) is separable, iff there exists a sequence $\xi = (\xi_1, \xi_2, \ldots)$, $\xi_k \in X$ for $k \in \mathbb{N}$ such that for each $x \in X$ and each $\epsilon > 0$ there exists an element $\xi_k \in \xi$ such that

$$d(x, \xi_k) < \epsilon,$$

(see also [38]).

Lemma 3.11. *If b-metric space* l^p, $0 < p < 1$, *is separable.*

Proof. Let $x = (x_1, x_2, \ldots) \in l^p$, $0 < p < 1$, i.e.

$$\sum_{k=1}^{\infty} |x_k|^p < \infty, \quad 0 < p < 1.$$

For each $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} |x_k|^p < \frac{\epsilon}{2}^p, \quad 0 < p < 1.$$

In view of the fact that the set $\mathbb{Q}$ is dence in $\mathbb{R}$, there exist sequences of rational numbers

$$\{x_k^m\}, \quad k = 1, \ldots, n, \quad x_k^m \to x_k, \quad \text{as} \quad m \to \infty, \quad k = 1, \ldots, n.$$

Take $\epsilon_1 = \frac{\epsilon^p}{2n} > 0$. Then there exists $m \in \mathbb{N}$ such that

$$|x_k^m - x_k|^p < \frac{\epsilon^p}{2n}, \quad k = 1, \ldots, n.$$

Let

$$w^m = (x_1^m, \ldots, x_n^m, 0, \ldots),$$

then

$$d^p(x, w^m) = \sum_{k=1}^{n} |x_k^m - x_k|^p + \sum_{k=n+1}^{\infty} |x_k|^p < \frac{\epsilon^p}{2} + \frac{\epsilon^p}{2} = \epsilon^p,$$

and consequently

$$d(x, w^m) < \epsilon.$$

Since $E = (w_1, \ldots, w_n, 0, \ldots)$, $w_k \in \mathbb{Q}$, $k = 1, \ldots, n$, $n \in \mathbb{N}$, is also countable, this means that l^p, $0 < p < 1$ is a separable space. If $x_k, k = 1, \ldots$, is a complex number, the proof is very similar. □

Remark 3.12. The b-metric space l^p, $0 < p < 1$, with the b-metric

$$d(x, y) := \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{\frac{1}{p}}, \quad 0 < p < 1,$$

is also complete.

Remark 3.13. By Paluszyński and Stempak [48] inequalities

$$d_r \leqslant d^r \leqslant 4d_r$$

one has:

a) (X, d) complete $\Leftrightarrow$ (X, d_r) complete,

b) (X, d) separable $\Leftrightarrow$ (X, d_r) separable.

Note that d means a b-metric and d_r the metric "induced" by the b-metric d.

Question:

Let (X, d) be a given b-metric space. What is the smolest constant $s > 1$ for which the triangle inequality

$$d(x, y) \leqslant s[d(x, z) + d(z, y)],$$

holds true for all $x, y, z \in X$?

Note that for b-metric space defined in the Example 3.5, such $s \leqslant 2^{\alpha}$, $\alpha > 0$ but for the space l^p, $0 < p < 1$, defined in the Remark 3.7, $s \leqslant 2^{\frac{1}{p}-1}$ (see also the inequality (3.6)).

4. B-metric and induced metric

In this part we present a method of building a metric by a given b-metric. A main result will be based on the paper by Paluszyński - Stempak [48].

Let (X, d) be a b-metric space, i.e. d satisfies the conditions (i), (ii) and (iii) for all $x, y, z \in X$ and for some real number $s \geqslant 1$. Then one has also for $x, y, z \in X$

$$d(x,y) \leqslant s\Big[d(x,z) + d(z,y)\Big] \leqslant 2s \max\left[d(x,z), d(z,y)\right],$$

i.e.

$$d(x,y) \leqslant 2s \max\left[d(x,z), d(z,y)\right] \tag{4.1}$$

for all $x, y, z \in X$.

Two b-metrics d_1 and d_2 are equivalent iff

$$\alpha d_1(x,y) \leqslant d_2(x,z) \leqslant p d_1(x,y) \tag{4.2}$$

for all $x, y \in X$ where α, β are two positive real numbers.

Let $0 < p \leqslant 1$ be given. Define (see [48]) for a b-metric d:

$$d_p(x,y) := \inf\left\{\sum_{i=1}^{n} d^p(x_{i-1}, x_i)\Big| x_0 = x, x_1, \ldots, x_n = y, n \geqslant 1\right\}. \tag{4.3}$$

It is obious that d_p is symmetric and satisfies the triangle inequality

$$d_p(x,y) \leqslant d_p(x,z) + d_p(z,y), \quad x, y, z \in X,$$

as well as

$$d_p(x,y) \leqslant d^p(x,y), \quad x, y \in X. \tag{4.4}$$

Under some additional assumptions d_p becomes a metric and a metric equivalent to d^p. In fact, we have the following interesting result.

Proposition 4.1 (*cf.* [48])**.** *Let (X, d) be a b-metric space and let $0 < p \leqslant 1$ be given by the condition*

$$(2s)^p = 2. \tag{4.5}$$

Then d_p defined by (4.3) *is a metric on X, equivalent to d^p.*

Proof. As it was pointed out, it is sufficient to consider the case where d satisfies (4.1) with $K = (2s)^p \geqslant 2$. We already now that d_p defined by (4.3) is symmetric, fulfils the triangle inequality and also the inequality

$$d_p(x,y) \leqslant d_p(x,z), \quad x, y \in X.$$

We verify that also

$$d^p(x,y) \leqslant 4 d_p(x,y), \quad x, y \in X. \tag{4.6}$$

To do that, we apply the induction principle. In fact we verify that for any given sequence of $n+1$ points belonging to X:

$$x = x_0, x_1, \ldots, x_n = y, \quad n \geqslant 2, n \in \mathbb{N}$$

one has

$$d^p(x,y) \leqslant 2\Big(d^p(x,x_1) + 2\sum_{i=1}^{n-2} d^p(x_i, x_{i+1}) + d(x_{n-1}, y)\Big). \tag{4.7}$$

First of all, let three points x, x_1, y from X be given, then in view of (4.5) one gets

$$\begin{aligned} d^p(x,y) &\leqslant \Big\{2s \max\left[d(x,x_1), d(x_1,y)\right]\Big\}^p \\ &\leqslant (2s)^p\big(\max[d(x,x_1), d(x_1,y)]\big)^p \\ &\leqslant (2s)^p \max[d^p(x,x_1), d^p(x_1,y)] \\ &\leqslant 2\max[d^p(x,x_1), d^p(x_1,y)]. \end{aligned}$$

This means that (4.7) holds true for $n = 2$. Now let us assume that (4.7) is true for n. Consider a sequence of $n+2$ points: $x = x_0, x_1, \ldots, x_n, x_{n+1} = y$, from X. Define m as the largest number among $\{0, 1, \ldots, n\}$ with the following property

$$d^p(x,y) \leqslant 2d^p(x_m, y), \tag{4.8}$$

(such point x_m exists).
Since we have by (4.1)

$$d^p(x,y) \leqslant 2\max\{d^p(x,x_{m+1}), d^p(x_{m+1},y)\},$$

therefore, clearly by (4.8)

$$d^p(x,y) \leqslant 2d^p(x, x_{m+1}). \tag{4.9}$$

We can also verify this easily for $m = n$ or $m \leqslant n-1$ and the definition of m. Taking into account (4.8) and (4.9) one gets (for $x, y \in X$)

$$\begin{aligned} d^p(x,y) &\leqslant 2\min\left[d^p(x_m,y), d^p(x,x_{m+1})\right] \\ &\leqslant d^p(x,x_{m+1}) + d^p(x_m,y). \end{aligned}$$

In the cases: $m = 0$ or $m = n$, we get our inequality (4.7) for $n+1$, i.e. the points $x = x_0, \ldots, x_n, x_{n+1} = y$ from the last inequality. Therefore if $0 < m < n$, consider the points

$$x = x_0, \ldots, x_m, x_{m+1} \quad \text{and} \quad x_m, x_{m+1}, \ldots, x_n, x_{n+1} = y,$$

both of length less than or egual to $n+1$. From the last inequality, onegets by the inequality (4.7),

$$\begin{aligned} d^p(x,y) &\leqslant d^p(x,x_{m+1}) + d^p(x_m,y) \\ &\leqslant 2\Big[d^p(x_0,x_1) + 2\sum_{i=1}^{m-1} d^p(x_i,x_{i+1}) + d^p(x_m,x_{m+1})\Big] \\ &\quad + 2\Big[d^p(x_m,x_{m+1}) + 2\sum_{i=m+1}^{n-1} d^p(x_i,x_{i+1}) + d^p(x_n,x_{n+1})\Big] \\ &\leqslant 2\Big[d^p(x_0,x_1) + 2\sum_{i=1}^{n-1} d^p(x_i,x_{i+1}) + d^p(x_n,x_{n+1})\Big]. \end{aligned}$$

This is the inequality (4.7) for $n+1$, which completes the induction proof.
Since the inequality (4.7) holds for all "chains" $x = x_0, x_1, \ldots, x_{n+1} = y$, $n \in \mathbb{N}$, connecting x and y, it follows

$$d^p \leqslant d^p(x,y) \leqslant 4d_p(x,y), \tag{4.10}$$

for all $x, y \in X$. Consequently d_p is a metric in X and d_p is equivalent to d^p. □

5. Cauchy's functional equation, b-metric and induced metric

A function $f : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, is called additive iff it satisfies Cauchy's functional equation

$$f(x + y) = f(x) + f(y), \tag{5.1}$$

for all $x, y \in \mathbb{R}^n$, whereas a function $f : \mathbb{R} \to \mathbb{R}$ satisfying Cauchy's functional equation

$$f(xy) = f(x)f(y), \tag{5.2}$$

for all $x, y \in \mathbb{R}$, is referred to as multiplicative. We shall use the following result.

Proposition 5.1 (*cf.* [37])**.** *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a solution of Cauchy's functional equation* (5.2)*. If, moreover, φ is increasing and*

$$\varphi(2s) = 2, \quad s \geqslant 1, \tag{5.3}$$

then there exists an additive function $g : \mathbb{R} \to \mathbb{R}$, such that φ is given by the formula

$$\varphi(x) = \begin{cases} \exp g(lgx), & x > 0, \\ 0, & x = 0. \end{cases}$$

Remark 5.2. Note that $\varphi(x) = x^p$, $x \geqslant 0$, $0 < p \leqslant 1$, satisfies all the assumptions of Proposition 5.1.

Let (X, d) be a b-metric and $\varphi : [0, \infty) \to \mathbb{R}$ be as in Proposition 5.1. Define

$$d_\varphi(x, y) := \inf \left\{ \sum_{i=1}^{n} \varphi[d(x_{i-1}, x_i)], \quad x = x_0, \ldots, x_n = y, \quad n \geqslant 1 \right\}. \tag{5.4}$$

We have the following

Theorem 5.3. *Let (X, d) be a b-metric space. If $\varphi : [0, \infty) \to \mathbb{R}$ is an increasing solution of Cauchy's functional equation* (5.2)*, satisfying the condition* (5.3)*, then*

a) $d_\varphi(x, y) \leqslant \varphi[d(x, y)]$,

b) $d_\varphi(x, y) = d_\varphi(y, x)$,

c) $d_\varphi(x, y) \leqslant d_\varphi(x, z) + d_\varphi(z, y)$,

for all $x, y, z \in X$.

Proof. The proof follows directly from the definition (4.3) of d_φ.
Next, one has the following Paluszyński - Stempak [48] inequality. □

Lemma 5.4. *For any given sequence of $n + 1$ points*

$$x = x_0, \ldots, x_n = y, \quad x_k \in X, k = 0, 1, \ldots, n, n \in \mathbb{N}, n \geqslant 2,$$

where (X, d) is a b-metric space and φ satisfies the assumptions of Theorem 5.3*, the following inequality holds true:*

$$\varphi[d(x, y)] \leqslant 2\Big(\varphi[d(x_0, x_1)] + 2 \sum_{i=1}^{n-2} \varphi[d(x_i, x_{i+1}] + \varphi[d(x_{n-1}, x_n)]\Big). \tag{5.5}$$

Proof. The proof is the same as in [48], so we state only some basic parts.
By virtue of d and φ, we have for $n = 2$ and $x, y, z \in X$:

$$\begin{aligned} \varphi[d(x, y)] &\leqslant \varphi\Big[s(d(x, z) + d(z, y))\Big] \\ &\leqslant \varphi(2s)\varphi\Big[\max\,(d(x, z), d(z, y))\Big] \\ &\leqslant \varphi(2s) \max \Big[\varphi[d(x, z)], \varphi[d(z, y)]\Big] \\ &\leqslant 2\Big[\varphi[d(x, z)] + \varphi[d(z, y)]\Big], \end{aligned}$$

i.e. (5.5) for $n = 2$.
Assume now that the induction hypothesis holds, i.e. (5.5) holds true. Consider a sequence of $n + 2$ points from $X : x = x_0, x_1, \ldots, x_n, x_{n+1} = y$. Let m be the largest number from $\{0, 1, 2, \ldots, n\}$ such that

$$\varphi[d(x, y)] \leqslant 2\varphi[d(x_m, y)]. \tag{5.6}$$

Note that such number m exists. Since we have

$$\varphi[d(x, y)] \leqslant 2 \max\Big(\varphi[d(x, x_{m+1})], \varphi[d(x_{m+1}, y)]\Big),$$

so by (5.6), one has

$$\varphi[d(x, y)] \leqslant 2\varphi[d(x, x_{m+1})]. \tag{5.7}$$

Therefore, by (5.6) and (5.7), we got

$$\varphi[d(x, y)] \leqslant 2 \min\Big(\varphi[d(x_m, y)], \varphi[d(x, x_{m+1})]\Big), \tag{5.8}$$

i.e.

$$\varphi[d(x, y)] \leqslant \varphi[d(x, x_{m+1})] + \varphi[d(x_m, y)]. \tag{5.9}$$

For $m = 0$, we have, by (5.8)

$$\varphi[d(x, y)] \leqslant 2\varphi[d(x, x_1)],$$

but for $m = n$, one has also by (5.8)

$$\varphi[d(x, y)] \leqslant 2\varphi[d(x_n, y)],$$

and in both cases the inequality (5.5) is obvious for n replaced by $n + 1$. Therefore, for $1 < m \leqslant n - 1$ and points

$$x = x_0, \ldots, x_m, x_{m+1} = y \quad \text{and} \quad x_m, \ldots, x_n, x_{n+1} = y$$

we have, using the induction assumption and (5.8)

$$\begin{aligned}
\varphi[d(x, y)] &\leqslant \varphi\Big[d(x, x_{m+1})\Big] + \varphi\Big[d(x_m, y)\Big] \\
&\leqslant 2\Bigg(\varphi\Big[d(x_0, x_1)\Big] + 2\sum_{i=1}^{m-1} \varphi\Big[d(x_i, x_{i+1})\Big] + \varphi\Big[d(x_m, x_{m+1})\Big]\Bigg) \\
&\quad + 2\Bigg(\varphi\Big[d(x_m, x_{m+1})\Big] + 2\sum_{i=m+1}^{n-1} \varphi\Big[d(x_i, x_{i+1})\Big] + \varphi\Big[d(x_n, x_{n+1})\Big]\Bigg) \\
&\leqslant 2\Bigg(\varphi\Big[d(x_m, x_{m+1})\Big] + 2\sum_{i=m+1}^{n-1} \varphi\Big[d(x_i, x_{i+1})\Big] + \varphi\Big[d(x_n, x_{n+1})\Big]\Bigg)
\end{aligned}$$

i.e. the inequality (5.5) for $n + 1$, which ends the proof. □

Lemma 5.5. *Let (X, d) be a b-metric. Then for all $x, y \in X$, we have*

$$d_\varphi(x, y) \leqslant \varphi[d(x, y)] \leqslant 4d_\varphi(x, y). \tag{5.10}$$

The proof follows directly from the Paluszyñski-Stempak inequality (5.5).

In fact, since the inequality (5.5) holds true for all "chains" $x = x_0, \ldots, x_{n+1} = y$, $n \in \mathbb{N}$ connecting x and y, we got the inequality

$$\varphi[d(x, y)] \leqslant 4d_\varphi(x, y),$$

for all $x, y \in X$.
Now we can state

Theorem 5.6. *Let the assumptions of Theorem* 5.1 *be satisfied. Then*

(iv) *d is a metric,*

(v) *d is equivalent to $\varphi[d]$.*

Remark 5.7. Are there other methods leading from a b-metric d to (equivalent) metric g "induced" by a b-metric d?

6. On b-metric spaces and Brower and Schauder fixed point principles

Let's note (see [3, 12])

Definition 6.1. A mapping $d : X \times X \to \mathbb{R}_+$, we call a strong b-metric iff it satisfies (i) and (ii) from the definition presented in section 2, and

(vii) $d(x, y) \leqslant d(x, z) + sd(z, y)$,

for all $x, y, z \in X$ and some fixed $s \geqslant 1$.
It is clear, by the symetry of d, that also

(viii) $d(x, y) \leqslant sd(x, z) + d(z, y)$,

for all $x, y, z \in X$ and $s \geqslant 1$.

Remark 6.2. A strong b-metric fulfils the condition

$$d(x_0, x_n) \leqslant s[d(x_0, x_1) + \ldots + d(x_{n-1}, x_n)], \tag{6.1}$$

for all $x_k \in X$, $k = 0, 1, \ldots, n$, $n \in N_0$.
Clearly, one has

$$\begin{aligned} d(x_0, x_n) &\leqslant sd(x_0, x_1) + d(x_1, x_n) \\ &\leqslant sd(x_0, x_1) + sd(x_1, x_2) + d(x_2, x_n) \\ &\leqslant s\Big[d(x_0, x_1) + \ldots + d(x_{n-2}, x_n)\Big] + d(x_{n-1}, x_n) \\ &\leqslant s\Big[d(x_0, x_1) + \ldots + d(x_{n-1}, x_n)\Big]. \end{aligned}$$

We say that d satisfies the s-relaxed triangle inequality iff the condition is fulfilled, and d satisfies the s-relaxed polygonal inequality iff the condition (6.1) holds true. Therefore, strong b-metric satisfies also the s-relaxed poligonal inequality (see also [12]).

Remark 6.3. We can cosider a strong b-norm and a strong b-normed space.

Remark 6.4. Let $\|\cdot\|$ be a strong b-norm in a linear space X. The function

$$d(x, y) := \|x - y\|, \quad x, y \in X \tag{6.2}$$

is a strong b-metric in X.
In fact, for all $x, y, z \in X$, one has

$$\begin{aligned} d(x, y) = \|x - y\| &= \|(x - z) + (z - y)\| \\ &\leqslant \|x - z\| + s\|z - y\| = d(x, z) + sd(z, y), \end{aligned}$$

i.e. we get (6.2).

Remark 6.5. Let $\|\cdot\|$ be a strong b-norm in a linear space X. Then

$$\|x_1 + x_2\| \leqslant s\|x_1\| + \|x_2\|, \tag{6.3}$$

$$\|x_1 + \ldots + x_n\| \leqslant \|x_1\| + s(\|x_2\| + \ldots + \|x_n\|), \tag{6.4}$$

$$\|x_1 + \ldots + x_n\| \leqslant s(\|x_1\| + \ldots + \|x_{n-1}\|) + \|x_n\|, \tag{6.5}$$

$$\|x_1 + \ldots + x_n\| \leqslant s(\|x_1\| + \ldots + \|x_n\|), \tag{6.6}$$

for all $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}$, and fixed $s \geqslant 1$.
Since the inequalities are very simply, we verify only e.g. (6.4). One has

$$\begin{aligned} \|x_1 + \ldots + x_n\| &\leqslant \|x_1 + \ldots + x_{n-1}\| + s\|x_n\| \\ &\leqslant \|x_1 + \ldots + x_{n-2}\| + s(\|x_{n-1}\| + \|x_n\|) \\ &\ldots \\ &\leqslant \|x_1\| + s(\|x_2\| + \ldots + \|x_n\|). \end{aligned}$$

Remark 6.6. A strong b-metric is a continuous function. In fact, if $d(x_n, x) \to 0$ as $n \to \infty$ and $d(y_n, y) \to 0$ as $n \to \infty$ for all sequences $\{x_n\}, \{y_n\}, x_n, y_n \in X$, $n \in \mathbb{N}$ tending to $x, y \in X$, respectively as $n \to \infty$, then

$$\begin{aligned} d(x_n, y_n) &\leqslant sd(x_n, x) + d(x, y_n) \\ &\leqslant sd(x_n, x) + d(x, y) + sd(y, y_n) \end{aligned}$$

and hence

$$d(x_n, y_n) - d(x, y) \leqslant s[d(x_n, x) + d(y, y_n)].$$

Also

$$d(x, y) - d(x_n, y_n) \leqslant s[d(x_n, x) + d(y_n, y)],$$

so we have

$$|d(x, y) - d(x_n, y_n)| \leqslant s[d(x_n, x) + d(y_n, y)], \tag{6.7}$$

and the proof is completed.

Lemma 6.7. *Assume that* $(X, s, \|\cdot\|)$, $s \geqslant 1$ *is a strong b-normed space. Then* $\|\cdot\|$ *is a continuous function.*

Proof. Take $x_n \to x$, $n \to \infty$, i.e. $\|x_n - x\| \to 0$ as $n \to \infty$, $x_n, x \to X$, $n \in \mathbb{N}$ We have

$$\|x_n\| = \|x + (x_n - x)\| \leqslant \|x\| + s\|x_n - x\|,$$

and hence

$$\|x_n\| - \|x\| \leqslant s\|x_n - x\|. \tag{6.8}$$

Similarly

$$\|x\| = \|x_n + (x - x_n)\| \leqslant \|x_n\| + s\|x_n - x\|,$$

so

$$\|x_n\| - \|x\| \geqslant -s\|x_n - x\|. \tag{6.9}$$

In view of (6.8) and (6.9) one has

$$\Big|\|x_n\| - \|x\|\Big| \leqslant s\|x_n - x\|. \tag{6.10}$$

By (6.10) for $n \to \infty$, we have

$$\|x_n\| \to \|x\| \quad \text{as} \quad n \to \infty,$$

i.e. $\|\cdot\|$ is a continuous function on X. □

Definition 6.8. We say that a mapping $d : X \times X \to \mathbb{R}_+$ satisfies the s-relaxed strong polygonal inequality, iff for all $x_0, x_1, \ldots, x_n \in X$, $n \in \mathbb{N}_0$ and some fixed $s \geqslant 1$, one has

$$d(x_0, x_n) \leqslant d(x_0, x_1) + s[d(x_1, x_2) + \ldots + d(x_{n-1}, x_n)]. \tag{6.11}$$

Lemma 6.9. *If* $d : X \times X \to \mathbb{R}_+$ *satisfies* (6.11) *where d is a strong b-metric, then*

$$d(x_0, x_n) \leqslant s\Big[d(x_0, x_1) + \ldots + d(x_{n-2}, x_{n-1})\Big] + d(x_{n-1}, x_n), \tag{6.12}$$

for all $x_0, x_1, \ldots, x_n \in X$ *and* $n \in \mathbb{N}_0$.

Proof. We have by (6.11) and (viii)

$$\begin{aligned} d(x_0, x_n) &\leqslant sd(x_0, x_1) + d(x_1, x_n) \\ &\leqslant sd(x_0, x_1) + sd(x_1, x_2) + d(x_2, x_n) \\ &\ldots \\ &\leqslant s[d(x_0, x_1) + \ldots + d(x_{n-2}, x_{n-1})] + d(x_{n-1}, x_n), \end{aligned}$$

i.e. we get (6.12). □

Remark 6.10. Clearly, the s-relaxed strong polygonal inequality (6.11) implies the s-relaxed strong triangle inequality (vii).

Remark 6.11. Let $x = x_0, \dots, x_n = y$, $n \in \mathbb{N}_0$. The inequality (vii) is equivalent to (6.11).
Indeed, it is enough to verity any that (vii) implies (6.11).
In fact,

$$\begin{aligned} d(x,y) &\leqslant d(x,x_{n-1}) + sd(x_{n-1},x_n) \\ &\leqslant d(x,x_{n-2}) + sd(x_{n-2},x_{n-1}) + sd(x_{n-1},x_n) \\ &\dots \\ &\leqslant d(x,x_1) + s\Big[d(x_1,x_2)] + d(x_2,x_3) + \dots + d(x_{n-1},x_n)\Big] \end{aligned}$$

i.e. (6.11).

7. Compactness in b-metric spaces

Definition 7.1. Let $(X; d, s)$ be a b-metric space with $s \geqslant 1$. A set $M \subset X$ is compact, iff any $\{x_n\}$ in M contains a subsequence $\{x_{n_k}\}$ which converger (with respect to d) to some $x \in X$. If $x \in M$, then M is said to be strongly compact.

Theorem 7.2. *Let $(X; d, s)$ be a b-metric space and let $M \subset X$. Let M be strongly compact and $f : M \to \mathbb{R}$ be continuous. Then*

(a) *f is bounded on M,*

(b) *there exist $x_1, x_2 \in M$ such that*
$f(x_1) = \inf\{f(x) : x \in M\}$,
$f(x_2) = \sup\{f(x) : x \in M\}$.

Proof. The proof runs very similarly to that one presented in the book [38]. Now we show that f is bounded below. For the contrary, assume that for every $n \in \mathbb{N}$, there exists $x_n \in M$ such that

$$f(x_n) < -n. \tag{7.1}$$

However, by the compactness of M and continuity of f, there exists a subsequence $\{x_{n_k}\} \subset M$ such that $x_{n_k} \to x_0 \in M$ and $f(x_{n_k}) \to f(x_0) \in \mathbb{R}$. □

According to (7.1), we get contradiction. To prove (b), assume

$$\alpha = \inf\Big\{f(x) : x \in M\Big\}.$$

So for every $\epsilon_n = \frac{1}{n}$ there exists $x_n \in M$ such that

$$\alpha \leqslant f(x_n) < \alpha + \frac{1}{n}.$$

Consequently, there exists $\{x_{n_k}\}$, $x_{n_k} \to x_0 \in M$ and

$$\alpha \leqslant f(x_{n_k}) < \alpha + \frac{1}{n_k},$$

and $f(x_{n_k}) \to f(x_0)$ as $k \to \infty$.
Thus

$$f(x_{n_k}) \to \alpha \quad \text{and} \quad f(x_{n_k}) \to f(x_0),$$

which means that $\alpha = f(x_0)$, $x_0 \in M$. The proof is complete.
To verify the rest statements, one can do the same.

Remark 7.3. If the assumptions of Theorem 7.2 are not satisfied, the result may not be true (see [38]).

Definition 7.4. A set $E \subset X$, a subset of a b-metric space $(X; d)$ is said to be an ϵ-net for a set $M \subset X$, iff for every $x \in M$ there exists $u \in E$ such that $d(x, u) < \epsilon$.

Theorem 7.5. *Let $(X; d, s)$ be a b-metric complete space with $s \geqslant 1$. Let for each $\epsilon > 0$ there exists finite ϵ-net with points belonging to $M \subset X$. Then M is a compact set.*
The proof can be done by using very similar arguments as in [38]*, so the details are left to the Reader.*

Theorem 7.6. *Let $(X; d, s)$ be a b-metric space. If $M \subset X$ is compact, then for every $\epsilon > 0$ there exists finite ϵ-net $\{c_1, \ldots, c_n\} \subset M$ for the set M.*
The proof can be done similarly to the proof presented in [38] *for a metric space.*

Remark 7.7. Till now, the existence of completion of b-metric spaces is still an important and open problem.

Theorem 7.8. *Let X be a b-metric space. If $M \subset X$ is compact, then M is bounded.*

Proof. Let

$$T = \{x_1, \ldots, x_n\}$$

be 1-net for M (see Theorem 7.6) and let $a \in X$. Then for $x \in M$, $x_i \in T$, $i = 1, \ldots, n$, one has

$$\begin{aligned} d(x, a) &\leqslant s[d(x, x_i) + d(x_i, a)] \\ &\leqslant s[1 + \max_i d(x_i, a)] \leqslant K < \infty. \end{aligned}$$

□

Theorem 7.9. *Every compact b-metric space $(X; d)$, is separable.*

Proof. Let $\{\epsilon_n\}$ be a sequence of positive, tending to zero, decreasing sequence and lot

$$T_n = \{x_1^n\}, \quad i = 1, \ldots, i_n,$$

be an ϵ_n-not for X. Put

$$E := U_{i=1}^{\infty} T_n.$$

Then T is a countable set. Moreover, for each $x \in X$, $\epsilon_n < \epsilon$ there exists $x_i^n \in T_n$ for some $i \in \mathbb{N}$, such that

$$d(x, x_i^n) < \epsilon_n < \epsilon,$$

i.e. the set E is dense in X. Therefore we get the desired conclusion and the proof is finished. □

8. Finite dimensional b-normed spaces

Let $(X; \|\cdot\|)$ with $s \geqslant 1$ be an n-dimensional b-normed linear space over K (the set K is the set of all real or all complex numbers; K be also any other ring or field, but even the idea of linearly dependent set of elements requires that K is the field). Let $\{e_1, \ldots, e_n\}$ be a base of X. It is well known that any $x \in X$ has a unique representation

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n, \quad \alpha_i \in K = \mathbb{R}, \quad i = 1, \ldots, n. \tag{8.1}$$

Let's define

$$\|x\|_0 := \sum_{i=1}^{n} |\alpha_i|, \tag{8.2}$$

where x is given by formula (8.1).
One has the following

Theorem 8.1 (*cf.* [38])**.** *Assume that* $(X, \|\cdot\|)$ *is an n-dimensional b-normed linear space. Then there exists* $\beta > 0$ *such that for all* $x \in X$

$$\|x\| \leqslant \beta \|x\|_0. \tag{8.3}$$

Proof. One has

$$\begin{aligned}\|x\| &= \|\sum_{i=1}^{n} \alpha_i e_i\| \\ &\leqslant s|\alpha_1|\|e_i\| + \ldots + s^n|\alpha_n|\|e_n\| \\ &\leqslant s^n\Big(|\alpha_1|\|e_i\| + \ldots + |\alpha_n|\|e_n\|\Big) \\ &\leqslant s^n \max_i \Big(\|e_i\| \sum_{i=1}^{n} |\alpha_i|\Big) \\ &\leqslant (s^n M) \sum_{i=1}^{n} |\alpha_i| = \beta\|x\|_0,\end{aligned}$$

where $\beta := s^n M$, $M = \max_i \|e_i\|$, i.e. (8.3). The next result is the following □

Theorem 8.2 (*cf.* [38])**.** *Let* $(X; \|\cdot\|)$ *be an n-dimensional strong b-normed linear space. Then there exist* $\alpha > 0$ *and* $\beta > 0$ *such that for all* $x \in X$,

$$\alpha\|x\|_0 \leqslant \|x\| \leqslant \beta\|x\|_0. \tag{8.4}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a base of X and for $x \in X$,

$$X = \alpha_1 x_1 + \ldots + \alpha_n x_n, \quad \alpha_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Let $U = \{x \in X : \|x\|_0 = 1\}$. We say that U is bounded: for if $x_1, x_2 \in U$, $x_k = \alpha_1^k e_1 + \ldots + \alpha_n^k e_n$, $\alpha_i^k \in \mathbb{R}$, $i = 1, \ldots, n$, $k = 1, 2$, then

$$\begin{aligned}\|x_1 - x_2\| &= \|\sum_{i=1}^{n}(x_i^1 - x_i^2)e_i\| \\ &\leqslant s^n \sum_{i=1}^{n} |\alpha_i^1 - \alpha_i^2|\|e_i\| \\ &\leqslant s^n \max_i \Big(\|e_i\|\Big[\sum_{i=1}^{n} |\alpha_i^1| + \sum_{i=1}^{n} |\alpha_i^2|\Big]\Big) \\ &\leqslant 2Ms^n,\end{aligned}$$

where $M = \max_i \|e_i\|$, $1 = \sum_{i=1}^{n} |\alpha_i^k|$, $k = 1, 2$.

By Lemma 6.7, the function $f(x) := \|x\|$, $f : U \to \mathbb{R}_+$, is a continuous one. However, $U = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |\alpha_i| = 1\}$ as a bounded and closed subset of n-dimensional space $\mathbb{R}^n$, is the strongly compact set. So by Theorem 7.2, f has infimum α in U, different from zero, because $\inf f$ on U is equal to $f(x_0)$, $x_0 \in U$, so $x_0 \neq 0$.
Therefore,

$$\alpha = \inf_{x \in U} f(x) = \inf_{x \in U} \|x\| = f(\overline{x}), \quad \overline{x} \in U, \overline{x} \neq 0.$$

In view of this

$$\|\frac{x}{\|x\|_0}\| \geqslant \alpha > 0 \quad \text{forall} \quad x \in X, \tag{8.5}$$

so

$$\alpha\|x\|_0 \leqslant \|x\|, \quad x \in X \tag{8.6}$$

The inequality (8.5) is true, because

$$\|\frac{x}{\|x\|_0}\|_0 = 1 \quad \text{forall} \quad x \in X.$$

Finally, by (8.6) and (8.3) we get (8.4) and the proof is complete. □

Remark 8.3. Note that if b-norms $\|\cdot\|_1$ and $\|\cdot\|_2$ satisfy (8.4), so we have also (8.4) with b-norms $\|\cdot\|_1$, and $\|\cdot\|_2$. Indeed, if

$$\alpha\|x\|_0 \leqslant \|x\|_1 \leqslant \beta\|x\|_0, \quad \alpha, \beta > 0, \quad x \in E,$$

and

$$\alpha_1\|x\|_0 \leqslant \|x\|_2 \leqslant \beta_1\|x\|_0, \quad \alpha_1, \beta_1 > 0, \quad x \in X,$$

then

$$\frac{\alpha_1}{\beta}\|x\|_1 \leqslant \|x\|_2 \leqslant \frac{\beta_1}{\alpha}\|x\|_1, \quad x \in X. \tag{8.7}$$

This is a desired conclusion.

Lemma 8.4. *Let $(X; \|\cdot\|)$ be n-dimensional strong b-normed linear space and let $U \subset X$ be a bounded set. Then U is compact (in X).*

Proof. Put

$$x = \sum_{i=1}^{n} \alpha_i e_i, \quad \alpha_1 \in \mathbb{R}, \quad i = 1, \ldots, n, \quad n \in \mathbb{N}, \quad x \in X,$$

and

$$\overline{x} = (\alpha_1, \ldots, \alpha_n), \quad \overline{x} \in \mathbb{R}^n. \tag{8.8}$$

Take

$$f(x) := \overline{x}, \quad x \in U.$$

Then $f(U) = \overline{U}$. By the inequalities

$$\alpha\|\overline{x}\|_1 \leqslant \|x\| \leqslant \beta\|\overline{x}\|_1, \quad \alpha > 0, \beta > 0,$$

where $\|x\|$ is the norm of x in X, and $\|\overline{x}\|_1$ is the norm of corresponding $\overline{x}$ in $\mathbb{R}^n$, one gets

a) U bounded in X iff $\overline{U}$ bounded in $\mathbb{R}^n$,

b) a sequence $\{x_n\}$ of elements in X, is convergent in $(X; \|\cdot\|)$ iff the corresponding sequence $\{\overline{x}_n\}$ is convergent in $\mathbb{R}^n$.

Therefore, the compactness of U bounded in X, follows from the compactness of $\overline{U}$ bounded in $\mathbb{R}^n$. The proof is complete. □

Definition 8.5. The space $\overline{U}$ consistent with all $\overline{x}$ defined by (8.8) we call the "induced space" $\mathbb{R}^n$ (depending on the base $\{e_1, \ldots, e_n\}$.

Theorem 8.6. *If the induced space $(\mathbb{R}^n; \|\cdot\|_0)$ for the strong b-normed n-dimensional linear space $(X; \|\cdot\|_0)$ is complete, then also $(X; \|\cdot\|_0)$ is a complete space.*

Proof. Let (x_m), $x_m = \alpha_1^m e_1 + \ldots + \alpha_n^m e_n$, $\alpha_i^m \in \mathbb{R}$, $i = 1, \ldots, n$, $n, m \in \mathbb{N}$, be a Cauchy sequence of elements from X. Then $\{\overline{x}_m\} = \{\alpha_1^m + \ldots + \alpha_n^m\}$ is a Cauchy sequence in $(\mathbb{R}^n; \|\cdot\|_0)$: for if $\epsilon > 0$ and

$$\|x_m - x_k\|\epsilon, \quad \text{for} \quad m, k > n_0,$$

then by (8.6) one has for $m, k > n_0$

$$\|x_m - x_k\|_0 = \sum_{i=1}^{n} |\alpha_i^m - \alpha_i^k| \leqslant \frac{1}{\alpha}\|x_m - x_k\| < \frac{\epsilon}{\alpha}.$$

Since $(\mathbb{R}^n; \|\cdot\|_0)$ is complete, so for $x_m \to x$ as $m \to \infty$, with respect to $\|\cdot\|_0$, by (8.4) we get

$$\|x_m - x\| \leqslant \beta\|x_m - x\|_0 \to 0 \quad \text{as} \quad m \to \infty,$$

and $x = \alpha_1 e_1 + \ldots + \alpha_n e_n \in (X; \|\cdot\|)$, which means that $(X; \|\cdot\|)$ is a complete space, and the proof of the theorem is complete. □

9. Brower fixed point principle in b-normed spaces

It is known that

Theorem 9.1 (Brower). *Let U be a nonempty bounded convex closed subset of $\mathbb{R}^n$, and let $T : U \to U$ be a continuous map. Then T has a fixed point $u \in U$. Now we prove.*

Theorem 9.2 (Brower). *Let $(X; \|\cdot\|)$ be an n-dimensional strong b-normed linear space, and lot $A \subset X_n$ be a nonempty bounded convex closed set. If, moreover, $\varphi : A \to A$ is continuous (with respect to b-norm $\|\cdot\|$), then there exists $y \in A$ such that $\varphi(y) = y$.*

Proof. Assume that $x \in A$, then $x = \alpha_1 e_1 + \ldots + \alpha_n e_n$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $\{e_1, \ldots, e_n\}$ is a base of X_n. Denote $\overline{x} = (\alpha_1, \ldots, \alpha_n) \in \overline{A} \subset \mathbb{R}^n$, and

$$\phi : A \to \overline{A}, \quad \phi(x) = \overline{x}, \quad x \in A. \tag{9.1}$$

We show that ϕ, is a homeomorphism of A onto $\overline{A} = \phi(A)$.
In fact, ϕ is one to one. Moreover, ϕ and ϕ^{-1} are continuous. Really, we verify that for $x, x_0 \in A$,

$$\|x - x_0\| \to 0 \quad \text{implies} \quad \|\phi(x) - \phi(x_0)\|_0 \to 0.$$

In fact, by (8.4) Theorem 8.2,

$$\|\phi(x) - \phi(x_0)\|_0 = \|\overline{x} - \overline{x_0}\|_0 \leqslant \frac{1}{\alpha}\|x - x_0\| \to 0,$$

i.e. $\phi(x) - \phi(x_0)$ as $x \to x_0$. It is clear that $\overline{x} - \overline{x_0} = \overline{x - x_0}$.
Similarly,

$$\phi^{-1}(\overline{x}) = x, \quad \phi^{-1} : \overline{A} \to A,$$

and if $\overline{x} \to \overline{x}_0$ then $\phi^{-1}(\overline{x}) \to \phi^{-1}(\overline{x}_0)$. In fact,

$$\|\phi^{-1}(\overline{x}) - \phi^{-1}(\overline{x}_0)\| = \|x - x_0\| \leqslant \beta\|\overline{x - x_0}\|_0 = \beta\|\overline{x} - \overline{x}_0\|_0 \to 0,$$

i.e. ϕ^{-1} is continuous in $\overline{A}$. Now we verify that $\overline{A} = \phi(A)$ is convex. For, let $\overline{x}, \overline{y} \in \phi(A)$, so

$$\overline{x} = (\alpha_1, \ldots, \alpha_n), \quad \overline{y} = (\beta_1, \ldots, \beta_n), \quad \alpha_i, \beta_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Since A is convex, for all $0 \leqslant \lambda \leqslant 1$, one has

$$\begin{aligned}\lambda\overline{x} + (1-\lambda)\overline{y} &= \Big(\lambda\alpha_1 + (1-\lambda)\beta_1, \ldots, \lambda\alpha_n + (1-\lambda)\beta_n\Big) \\ &= \phi[\lambda x + (1-\lambda)y] \in \phi(A).\end{aligned}$$

Therefore
$$\lambda\overline{x} + (1-\lambda)\overline{y} \in \phi(A), \quad 0 \leqslant \lambda \leqslant 1,$$
i.e. $\phi(A)$ is convex.
There is no problem to show that $\phi(A)$ is bounded: for if $\overline{x}, \overline{y} \in \phi(A)$, so by the boundedness of A and Theorem 8.2,
$$d(\overline{x}, \overline{y}) = \|\overline{x} - \overline{y}\|_0 \leqslant \frac{1}{\alpha}\|x - y\| \leqslant \frac{1}{\alpha} \cdot M = K,$$
where M is the constant such that
$$\|x - y\| \leqslant M, \quad x, y \in A.$$
Thus $\phi(A)$ is bounded.
At the end we prove that $\phi(A)$ is closed. Let
$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n, \quad \overline{x} = (\alpha_1, \ldots, \alpha_n), \quad \overline{x} \in \phi(A),$$
$$x_0 = \gamma_1 e_1 + \ldots + \gamma_n e_n, \quad \overline{x}_0 = (\gamma_1, \ldots, \gamma_n), \quad \alpha_i, \gamma_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$
We verify that
$$\Big(\|\overline{x} - \overline{x}_0\|_0 \to 0\Big) \Rightarrow \overline{x}_0 \in \phi(A).$$
In fact, by the closedness of A, one gets
$$\Big(\|x - x_0\| \leqslant \beta\|\overline{x} - \overline{x}_0\|_0 \Rightarrow 0\Big) \quad \text{implies} \quad x \to x_0,$$
i.e. $x_0 \in A$, so $\overline{x}_0 \in \phi(A) = \overline{A}$.
To finish the proof, define
$$T := \phi\varphi\phi^{-1}, \quad T : \overline{A} \to \overline{A}.$$
Let's verify, that T is continuous. In fact, for $\|\overline{x} - \overline{x}_0\|_0 \to 0$, we have, by continuity of φ and (8.4)
$$\|T(\overline{x}) - T(\overline{x}_0)\|_0 = \|\phi\varphi(x) - \phi\varphi(x_0)\|_0 = \|\overline{\varphi(x)} - \overline{\varphi(x_0)}\|_0 \leqslant \frac{1}{\alpha}\|\varphi(x) - \varphi(x_0)\|,$$
i.e. T is continuous.
In view of Theorem 9.1, there exists $x \in \overline{A}$ with
$$T(x) := \phi\varphi\phi^{-1}(x) = x,$$
so $\varphi[\phi^{-1}(x)] = \varphi^{-1}(x)$. Take $y = \phi^{-1}(x)$, then $\varphi(y) = y$, $y \in A$. This concludes the proof. □

10. Schauder fixed point theorem in b-normed spaces

This part we start with

Definition 10.1 (*cf.* [42, P. 54]). Let $N := \{c_1, \ldots, c_n\}$ be a finite subset of a strong b-normed linear space E, and for any fixed $\epsilon > 0$, let
$$(N, \epsilon) := U\Big\{B(c_i, \epsilon) : i = 1, \ldots, n\Big\},$$
where
$$B(c_i, \epsilon) = \{x \in E : \|x - c_i\| < \epsilon\}, \quad i = 1, \ldots, n.$$
For each $i = 1, \ldots, n$, define $\mu_i : (N, \epsilon) \to \mathbb{R}$ as
$$\mu_i(x) := \max\Big[0, \epsilon - \|x - c_i\|\Big].$$
Then the Schauder - Dugundi - Granas projection (see [42])
$$p_\epsilon : (N, \epsilon) \to conv(N)$$

is given by

$$p_\epsilon(x) := \Big[\sum_{i=1}^{n}\mu_i(x)\Big]^{-1}\sum_{i=1}^{n}\mu_i(x)c_i. \tag{10.1}$$

It is clear, that p_ϵ is well-defined, since each $x \in (N,\epsilon)$ belongs to some $B(c_i,\epsilon)$, so

$$\sum_{i=1}^{n}\mu_i(x) \neq 0.$$

Moreover, $p_\epsilon[(N,\epsilon)] \subset conv(N)$ as a convex combination of points $c_1,\ldots,c_n$.

Remark 10.2. The values of p_ϵ are in a finite dimensional b-normed linear space contained in E.

Definition 10.3 (*cf.* [42])**.** Let X and Y be topological spaces. A continuous map $F : X \to Y$ is called compact iff $F(X)$ is contained in a compact subset of Y.

We have the following

Lemma 10.4 (*cf.* [42])**.** *Let $E = (E; \|\cdot\|)$ be a strong b-normed linear space, and let $c_1,\ldots,c_n \in U \subset E$, U - convex subset. Then*

(ix) $\|x - p_\epsilon(x)\| < \epsilon s,\ x \in (N,\epsilon)$,

(x) $p_\epsilon : (N,\epsilon) \to conv(N) \subset U$ *is a continuous compact map.*

Proof. For $x \in (N,\epsilon)$, by Definition 10.1 and Remark 7.3, one has

$$\begin{aligned}\|x - p_\epsilon(x)\| &= \Big\|\Big(\sum_{i=1}^{n}\mu_i(x)\Big)^{-1}\sum_{i=1}^{n}\mu_i(x) - \Big(\sum_{i=1}^{n}\mu_i(x)\Big)^{-1}\sum_{i=1}^{n}\mu_i(x)c_i\Big\| \\ &\leqslant s\Big[\sum_{i=1}^{n}\mu_i(x)\Big]^{-1}\sum_{i=1}^{n}\mu_i(x)\|x-c_i\| \\ &\leqslant s\epsilon\Big[\sum_{i=1}^{n}\mu_i(x)\Big]^{-1}\Big(\sum_{i=1}^{n}\mu_i(x)\Big) = s\epsilon.\end{aligned}$$

i.e. we have (ix).
For (x): the continuity of p_ϵ is a consequence of the fact, that p_ϵ is a finite sum of continuous functions (for more details see Lemma 6.7); but compactness we can derive from Lemma 8.4. □

Lemma 10.5. *Suppose that X is a topological space and E a strong b-normed linear space. Let U be a convex subset of E and let $F : X \to U$ be a compact map. For every $\epsilon > 0$ there exists a finite set*

$$N = \{c_1,\ldots,c_n\} \subset F(x) \subset U,$$

and a finite - dimensional map $F_\epsilon : X \to U$ such that

(xi) $\|F_\epsilon(x) - F(x)\| < s\epsilon,\ x \in X$,

(xii) $F_\epsilon(X) \subset conv(N) \subset U$.

Proof. Since $F(X)$ is compact in E, by Theorem 7.6 there exists a finite ϵ-not $\{c_1,\ldots,c_n\} \subset F(X)$. Also $F(x) \subset (N,\epsilon)$: for if $y \in F(X)$, then $d(y,c_i) < \epsilon$ for some $i \in \{1,\ldots,n\}$, so hence $y \in B(c_i,\epsilon)$, i.e. $y \in (N,\epsilon)$.
Therefore $F(X) \subset (N,\epsilon)$.
Now, let $F_\epsilon(x) := p_\epsilon[F(x)]$, for $x \in X$. If $y = F(x)$, $x \in X$, then

$$\|F_\epsilon(x) - F(x)\| = \|p_\epsilon(y) - y\| < s\epsilon,$$

because for $y = F(x) \in (N, \epsilon)$, $x \in X$, by Lemma 10.4,

$$\|y - p_\epsilon(y)\| < s\epsilon,$$

and consequently,

$$\|F_\epsilon(x) - F(x)\| < s\epsilon, \quad x \in X.$$

To prove (xii), let $y \in F_\epsilon(X)$, so $y = p_\epsilon(z)$, $z = F(x) \in (N, \epsilon)$, for some $x \in X$. Let

$$y = p_\epsilon(z) = \sum_{i=1}^{n} \lambda_i c_i, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \in \mathbb{R}, \quad i = 1, \dots, n.$$

Hence $y \in conv(N) \subset U$. Therefore, since U is convex,

$$F_\epsilon(X) \subset conv(N) \subset U,$$

i.e. we get (xii), as desired. □

Definition 10.6 (*cf.* [42])**.** Let $(E; d, s)$ be a b-metric space, and let $U \subset E$. If for a given $\epsilon > 0$, there exists a point $x \in U$ such that $d(x, F(x)) < \epsilon$ for a map $F : U \to E$, then we say that x is an ϵ-fixed point for F.

We shall use the following

Theorem 10.7. *Let $(X; d, s)$ be a strong b-metric space, and $A \subset X$ be a closed set. Let $F : A \to X$ be a compact map. Then F has a fixed point $a \subset A$ iff for each $\epsilon > 0$ it has an ϵ-fixed point.*

Proof. The necessary condition is trivial. Therefore we verify the sufficient condition only. Assume $\epsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$ and let for each $n \in \mathbb{N}$ there exists an $a_n \in A$, $n \in \mathbb{N}$, an ϵ_n-fixed point for F, i.e.

$$d(a_n, F(a_n)) < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{10.2}$$

Since $F(x) \subset U \subset X$, and U is compact (in X), then there exists subsequence $\{a_{n_k}\}$, such that

$$F(a_{n_k}) \to a \quad \text{as} \quad k \to \infty, \quad a \in X.$$

In view of (10.2), for $k \geqslant m_0$ and $\epsilon > 0$

$$\begin{aligned} d(a_{n_k}, a) &\leqslant s\Big[d(a_{n_k}), F(a_{n_k}) + d(F(a_{n_k}, a))\Big] \\ &< s\Big[\frac{1}{n_k} + \epsilon\Big] < 2s\epsilon, \end{aligned}$$

i.e. $a_{n_k} \to a$ as $k \to \infty$, and $a \in A$. Therefore, $F(a_{n_k}) \to a$ and $F(a_{n_k}) \to F(a)$, because F is continuous. Consequently $a = F(a)$, which finishes the proof. □

Now we are in a good position to state the following main result of this part.

Theorem 10.8. *(Schauder fixed point principle) Let $(X; \|\cdot\|)$ be a strong b-normed linear space, and let $U \subset X$ be a nonempty convex closed subset. Moreover, lot $F : U \to U$ be a compact map. Then there exists a $u \in U$ such that $u = F(u)$.*

Proof. On account of Theorem 10.7 it is enough to show that for each $\epsilon > 0$, F has an ϵ-fixed point in U. In view of Lemma 10.5, for every $\epsilon > 0$ there exists $F : U \to U$ with

(a) $\|F_\epsilon(x) - F(x)\| < \epsilon$, $x \in U$,

(b) $F_\epsilon(U) \subset conv(N) \subset U$,

(N - is as in the Lemma 10.5).
However, $F_\epsilon : conv(N) \to conv(N)$. Indeed,

$$conv(N) \subset U \quad \text{and} \quad F_\epsilon[conv(N)] \subset F_\epsilon(U) \subset conv(N).$$

Finally, by Theorem 9.2 (Brower) and Lemma 10.5 there exists $u \in U$ with $F(u) = u$, which was to be shown. □

11. Banach fixed point theorem in generalized b-metric spaces

Following the ideas of the author of this book, containing in the paper [31], we present the fixed point theorems for (ϵ, λ)-uniformly locally contractive mappings and ϵ-chainable generalized b-metric spaces, extensions of a famous S. Banach contraction principle for complete metric spaces, used both in the theory and applications of mathematics, in many areas of sciences.

A function $T : X \to X$, where $(X; d)$ is a b-metric space, is said to be an (ϵ, λ)-uniformly locally contractive mapping, with $\epsilon > 0$ and $0 \leqslant \lambda < 1$, iff (see [14, 31]),

$$d[T(x), T(y)] \leqslant \lambda d(x, y) \text{ for all } x, y \in X, d(x, y) < \epsilon. \tag{11.1}$$

A generalized b-metric space (X, d) is called an ϵ-chainable, for $\epsilon > 0$, iff for all $x, y \in X$ such that $d(x, y) < \infty$, there exists an ϵ-chain from x to y (that is a finite set of points $x = x_0, x_1, \ldots, x_n = y$, $x_k \in X$, $k = 0, 1, \ldots, n$, with $d(x_{k-1}, x_k) < \epsilon$ for $k = 1, \ldots, n$.

The following very interesting result will be used latter on (see also [11, 12, 48]).

Proposition 11.1 (*cf.* [12, 18, 26])**.** *Assume that d is a b-metric on a nonempty set X. If the number* $p \in (0, 1]$ *is given by the equation*

$$(2s)^p = 2,$$

then the mapping $g : X \times X \to [0, \infty)$*, defined by*

$$g(x, y) := \inf \left\{ \sum_{i=1}^{n} d^p(x_{i-1}, x_i); x = x_0, \ldots, x_n = y \right\}, \tag{11.2}$$

is a metric on X, satisfying the inequalities

$$g(x, y) \leqslant d^p(x, y) \leqslant 4g(x, y), \tag{11.3}$$

for all $x, y \in X$.

Now we can state the main results of this part.

Theorem 11.2 (Banach)**.** *Let* $(X; d)$ *be a complete b-metric space. If* $T : X \to X$ *is a contraction mapping*

$$d[T(x), T(y)] \leqslant \lambda d(x, y), \quad x, y \in X \tag{11.4}$$

with

$$0 \leqslant \lambda < 1, \tag{11.5}$$

then

a) *T has in X exactly one fixed point u*;

b) $T^n(x) \to u$ *as* $n \to \infty$*,* $x \in X$*,*

Proof. We shall present the proof little bit different that the presented in [12]. In the case $\lambda = 0$ the proof is trivial, so let's consider $0 < \lambda < 1$.
Take $x \in X$ and put $x_n = T^n(x)$, $n \in \mathbb{N}$. Then

$$\begin{aligned} &d[T(x), T^2(x)] \leqslant \alpha d[x, T(x)], \\ &d[T^n(x), T^{n+1}2(x)] \leqslant \alpha^n d[x, T(x)], \quad x \in X, n \in N. \end{aligned} \tag{11.6}$$

In view of [12], Theorem 2.2 (see also [11, 26, 48]) for $0 < p \leqslant 1$ such that $(2s)^p = 2$, there exists the metric $g : X \times X \to [0, \infty)$, $g \sim d^p$, given by

$$g(x, y) := \inf \left\{ \sum_{i=1}^{n} d^p(x_{i-1}, x_i); \quad x = x_0, ..., x_n = y \right\}. \tag{11.7}$$

From the definition, one can see that g is symmetric, satisfies the triangle inequality and $g \leqslant d^p$.
Of course,

$$g(x, x_1) \leqslant d^p(x, x_1), g(x_1, x_2) \leqslant \left[\lambda d(x, T(x)\right]^p,$$

and consequently,

$$g[T^n(x), T^{n+1}(x)] \leqslant \xi^n a; n \in \mathbb{N}, \xi = \lambda^p < 1, a = d^p(x, T(x)). \tag{11.8}$$

Since $g \sim d^p$, for $m, n \in \mathbb{N}$, by (12.5),

$$\begin{aligned} g(x_n, x_{n+m}) &\leqslant g[T^n(x), T^{n+m}(x)] \\ &\leqslant g(x_n, x_{n+1}) + \ldots + g(x_{n+m-1}, x_{n+m}) \\ &\leqslant (1-\xi)^{-1}\xi^n a \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This means that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X; g)$ and by the relation $g \sim d^p$, in (X, d) as well. Thus by the completeness of $(X; g)$ and the Banach contraction principle, $x_n \to u \in (X; g)$. Moreover, by (12.4), it follows that $T^n(x) \to T(u)$, so $T(u) = u$. Since the uniqueness part is trivial, this complete the proof. □

Remark 11.3. By the inequality

$$g(x_n, x_{n+m}) \leqslant (1-\xi)^{-1}\xi^n a,$$

we get the estimation

$$g[T^n(x), u] \leqslant (1-\xi)^{-1}\xi^n d^p(x, T(x)), \tag{11.9}$$

for all $x \in X$, $n \in \mathbb{N}$, $0 < p \leqslant 1$, which is usefull for applications.

Theorem 11.4 (*cf.* [31])**.** *Let $(X; d)$ be a generalized complete b-metric space and let $x_0 \in X$. Let $T : X \to X$ be an (λ, ϵ) - uniformly locally contractive mapping. The following alternative holds: either*

(a) *for each iterative sequence $\{x_n\}_{n=1}^{\infty}$ at $x_0 \in X$, $d(x_{n-1}, x_n) \geqslant \epsilon$, $\epsilon > 0$, for each $n \in \mathbb{N}$,*
or

(b) *the iterative sequence $\{x_n\}_{n=1}^{\infty}$ at $x_0 \in X$, converges to a fixed point of T.*

Proof. Assume that (a) does not hold. Therefore, there exists $m \in \mathbb{N}$ such that

$$d(x_m, x_{m+1}) < \epsilon,$$

for a given fixed $\epsilon > 0$. Thus we have

$$d[T(x_m), T(x_{m+1})] \leqslant \lambda d(x_m, x_{m+1}) < \lambda\epsilon < \epsilon. \tag{11.10}$$

Thus, by (11.10), one has

$$d[T(x_{m+1}), T(x_{m+2})] \leqslant \lambda d(x_{m+1}, x_{m+2}) < \lambda^2 d(x_m, x_{m+1}) < \epsilon,$$

and, by the induction principle, for $n \in \mathbb{N}$,

$$d[T^n(x_m)T^n(x_{m+1})] \leqslant \lambda^n d(x_m, x_{m+1}) < \lambda^m \epsilon. \tag{11.11}$$

It is known, by Paluszyński and Stempak [48], that there exists a generalized complete metric $g : X \times X \to [0, \infty]$, given by the formula

$$g(x, y) := \begin{cases} \inf\Big\{ \sum\limits_{i=1}^{n} d^p(x_{i-1}, X_i) : x = x_0, \ldots, x_n = y, d(x, y) < \infty \Big\}, \\ +\infty, \quad \text{if} \quad d(x, y) = \infty, \end{cases}$$

with $0 < p \leqslant 1$ and such that $(2s)^p = 2$, $s \geqslant 1$. One can verify that $g \sim d^p$ (please see also [48]). It is clear that g is a symmetric function and satisfies the triangle inequality and $g \leqslant d^p$ as well. We have also

$$\begin{aligned} g(x_{m+1}, x_{m+2}) &\leqslant d^p(x_{m+1}, x_{m+2}) \leqslant (\lambda d(x_m, x_{m+1}))^p \\ &\leqslant \lambda^p d^p(x_m, x_{m+1}) = \xi z, \end{aligned}$$

where $\xi = \lambda^p < 1$, $z = d^p(x_m, X_{m+1})$.
Very similarly,

$$g(T^2(x_m), T^2(x_{m+1})) \leqslant \xi^2 z,$$

and, by the induction principle,

$$g(T^n(x_m), T^n(x_{m+1})) \leqslant \xi^n z, \quad n \in \mathbb{N}. \tag{11.12}$$

Moreover, one has, for $n, k \in \mathbb{N}$,

$$\begin{aligned} g(x_n, x_{n+k}) &= g[T^n(x_m), T^{n+k-1}(x_{m+1})] \\ &\leqslant g(x_n, x_{n+1}) + \ldots + g(x_{n+k-1}, x_{n+k}) \\ &\leqslant \xi^n z + \ldots + \xi^{n+k-1} z \\ &\leqslant (1-\xi)^{-1} \xi^n z. \end{aligned}$$

This means that the sequence $\{x_n\}_{n=1}^{\infty}$, is a Cauchy sequence in a generalized metric space $(X; g)$. By the relation $g \sim d^p$, it is also true in $(X; d^p)$. Thus $x_n \to u \in (X, g)$ and clearly, $d^p(x_n, u) < \epsilon$, for n sufficiently large.
We have also

$$\begin{aligned} g(x_{nu}, T(u)) &= g[T(x_n), T(u)] \\ &\leqslant d^p[T(x_n), T(u)] \leqslant \lambda^p d^p(x_n, u), \end{aligned}$$

i.e. $x_n \to T(u)$, therefore $T(u) = u$. This completes the proof of the Theorem. □

The next result reads as follows.

Theorem 11.5 (*cf.* [31, 48]). *Let $(X; d)$ be a generalized complete b-metric space and let $x_0 \in X$. Let $T : X \to X$ be a contraction mapping*

$$d[T(x), T(y)] \leqslant \lambda d(x, y), \quad x, y \in X, \quad d(x, y) < \infty \tag{11.13}$$

and

$$0 \leqslant \lambda < 1. \tag{11.14}$$

Then the following alternative holds: either

(c) *for the iterative sequence $\{x_n\}_{n=1}^{\infty}$, of T at $x_0 \in X$, $d(x_{n-1}, x_n) = \infty$ for each $n \in \mathbb{N}$,*
or

(d) *the iterative sequence $\{x_n\}_{n=1}^{\infty}$, of T at $x_0 \in X$, converges to a fixed point of T.*

Proof. Assume that (c) does not hold. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ of T at $x_0 \in X$ has the property that $d(x_{m-1}, x_m) < \infty$ for some $m \in \mathbb{N}$. Consider $\epsilon > 0$ such that $d(x_{m-1}, x_m) < \epsilon$. Thus T is an (λ, ϵ) - uniformly locally contractive mapping. Thus T satisfies (b) of Theorem 11.2 i.e. the (d) of our theorem, and the proof is completed. □

The further result is the following.

Theorem 11.6 (*cf.* [31]). *Let $(X; d)$ be a complete ϵ-chainable generalized b-metric space and let $x_0 \in X$. If $T : X \to X$ is an (λ, ϵ)-uniformly locally contractive mapping, then the following alternative holds: either*

(e) *for the iterative sequence $\{x_n\}_{n=1}^{\infty}$, of T at $x_0 \in X$, $d(x_{n-1}, x_n) = \infty$ for each $n \in \mathbb{N}$;*
or

(f) *the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *of* T *at* $x_0 \in X$ *converges to a fixed point of* T.

Proof. Let (e) does not hold. Then define $d_\epsilon : X \to [0, \infty]$ by (see also [48])

$$d_\epsilon(x,y) := \begin{cases} \inf\Big\{ \sum\limits_{i=1}^{n} d^p(x_{i-1}, x_i) : x = x_0, \dots, x_n = y, \\ \quad \text{is an } \epsilon \text{ - chain from x to y}, \\ \quad \text{for } d(x,y) < \infty, \\ \infty, \quad \text{if } d(x,y) = \infty, \end{cases}$$

where $(2s)^p = 2$, $0 < p \leqslant 1$, $s \geqslant 1$.
By the definition we see that d_ϵ is symmetric and satisfies the triangle inequality and is the function with values from $[0, \infty]$, $d_\epsilon \leqslant d^p$ as well. Similarly as in [48], one can verify that d_ϵ is a generalized complete metric and $d_\epsilon \sim d^p \sim d$.
Let $d(x,y) < \infty$. For $x = x_0, \dots, x_n = y$, $d(x_{i-1}, x_i) < \epsilon$, for $i = 1, \dots, n$, we get

$$d_\epsilon[T(x), T(y)] \leqslant \sum_{i=1}^{n} d_\epsilon[T(x_{i-1}), T(x_i)]$$
$$\leqslant \lambda^p \sum_{i=1}^{n} d^p(x_{i-1}, x_i) \leqslant \lambda^p d_\epsilon(x,y),$$

since $x_0, \dots, x_n$ is any ϵ-chain from x to y.
Therefore

$$d_\epsilon[T(x), T(y)] \leqslant \xi d_\epsilon(x,y), \quad x, y \in X, \quad d(x,y) < \infty, \quad \xi = \lambda^p < 1, \tag{11.15}$$

where $(2s)^p = 2$, $0 < p \leqslant 1$.
Since $(X; d_\epsilon)$ is also complete our statement follows directly from Theorem 11.5. The proof is finished. □

12. Completion of b-metric spaces

There are several type of spaces, considered by mathematicians. Let's note the following one (see [26])

(xiii) Let $d : X \times X \to \mathbb{R}_+$ satisfy the conditions (i) and (ii). For every $\epsilon > 0$ there exists $\phi(\epsilon) > 0$ such that if for $a, b, c \in X$, $d(a,b) < \phi(\epsilon)$ and $d(c,b) < \phi(\epsilon)$, then $d(a,c) < \epsilon$.

Remark 12.1. If $\phi(\epsilon) = \frac{\epsilon}{2}$ we got important case, since $\phi(\epsilon)$ may be replaced by any $\psi(\epsilon) \leqslant \phi(\epsilon) \leqslant \frac{\epsilon}{2}$. This special case we denote by (xiv) (see also [26]).

Remark 12.2. If $(X; d)$ is a b-metric space with $s \geqslant 1$: so

$$d(x,y) \leqslant s[d(x,z) + d(z,y)], \quad x, y, z \in X; \tag{12.1}$$

then it implies (xiii) for $\phi(\epsilon) \leqslant \frac{\epsilon}{2s}$.
Really, we have for $\phi(\epsilon) = \frac{\epsilon}{2s}$, $\epsilon > 0$, if $d(x,z) < \frac{\epsilon}{2s}$ and $d(z,y) < \frac{\epsilon}{2s}$, $x, y, z \in X$, then by (12.1) we got

$$d(x,y) < s\Big[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\Big] = \epsilon,$$

i.e.

$$d(x,y) < \epsilon, \quad x, y \in X.$$

Lemma 12.3. *If* d *is a b-metric with* $s \geqslant 1$, *then* d^p, $p > 0$ *is also a b-metric with* $\overline{s} = (2s)^p$, $p > 0$.

Proof. Let $x, y, z \in X$ and $p > 0$. Then

$$d(x,y) \leqslant s[d(x,z) + d(z,y)] \leqslant 2s \max[d(x,z), d(z,y)].$$

Thus

$$d^p(x,y) \leqslant (2s)^p\Big\{\max\big[d(x,z),d(z,y)\big]\Big\}^p$$
$$\leqslant (2s)^p\Big[d^p(x,z)+d^p(z,y)\Big]$$

i.e. d^p is a b-metric with $\overline{s}=(2s)^p$, $p>0$. □

Remark 12.4. Let d be a b-metric in X with $s \geqslant 1$ and d^p be a b-metric in X with $\overline{s}=(2s)^p$, $s \geqslant 1$, $p>0$. Then $d^p = \xi$ satisfies for $x,y,z \in X$, $\epsilon > 0$;

$$\xi(x,z) < \frac{\epsilon}{2\overline{s}} \quad \text{and} \quad \xi(z,y) < \frac{\epsilon}{2\overline{s}}, \quad \text{then} \quad \xi(x,y) < \epsilon. \tag{12.2}$$

To verify this, see Lemma 12.3 and Remark 12.2.

Let ξ satisfy (i), (ii) and (xiii). As in Frink [26], one can define d satisfying (xiv) and such that $d \sim \xi$. It is also possible to prove (see [26], p.134),

$$d(a,b) \leqslant 2d(a,x_1)+4d(x_1,x_2)+\ldots+4d(x_{n-2},x_{n-1})+2d(x_{n-1},b) \tag{12.3}$$

for all chains $a=x_0,x_1,\ldots,x_{n-1},x_n=b$ in X.
Define

$$D(a,b) := \inf\left\{\sum_{i=1}^{n} d(x_{i-1},x_i) : a=x_0,\ldots,x_n=b\right\}. \tag{12.4}$$

Therefore, as in [26], one can prove, by (12.3),

$$4^{-1}d(a,b) \leqslant D(a,b) \leqslant d(a,b), \tag{12.5}$$

for all $a,b \in X$.

We get

Lemma 12.5. *We have*

α) $D \sim d \sim \xi$,

β) D is a metric in X,

where D is defined by (12.4).

Proof. Please see (12.5), (12.4) and (12.2). Also consult [26]. □

Remark 12.6. The formula (12.4) (very interesting one) gives the possibility to produce several metrics from b-metric d^p, $p>0$. Note that in [48] there is only one metric build from a given b-metric d for p such that $(2s)^p=2$. We call this metric "induced" metric.
Let's note also the following.

Lemma 12.7. *Let d satisfy* (i), (ii) *and* (xiii) *and let $s>1$ be a given fixed number. Then, for $x,y,z \in X$*

$$d(x,y) \leqslant s[d(x,z)+d(z,y)].$$

Proof. For $x=y$ the proof is trivial. So assume that $x \neq y$, $x,y \in X$. Take

$$\epsilon_1 := sd(x,z), \quad \epsilon_2 = sd(z,y), \quad \epsilon := \max[\epsilon_1,\epsilon_2] > 0.$$

By (xiii) one gets for $x,y \in X$,

$$d(x,y) < \epsilon \leqslant \epsilon_1+\epsilon_2 \leqslant s[d(x,z)+d(z,y)],$$

i.e. our thesis. □

Remark 12.8. The distance function d is in fact, a b-metric with $s > 1$. But for $s = 1$, Lemma 12.7 in not true (there are distance functions which are not metrics but b-metrics with $s > 1$).
Now we can state the basic result of this part.

Proposition 12.9. *Every b-metric space $(X; d)$ can be extended to a complete b-metric space.*

Proof. Let (X, D) be a metric space, with the metric D induced by d (see e.g. [26, 48]), equivalent to d^p, $p > 0$. Let $I : (X; d) \to (X; D)$ be the identity mapping. As we know, (X, D) has a completion Y (see e.g. [38]), so therefore this complete metric space Y we consider as the completion of the b-metric space $(X; d)$. Let us note that such space Y is exactly one (in accuracy to the isometry), that is any other such space is isometric to Y.

Lot us note also that if $\{x_n\}$ is a Cauchy sequence in $(X; d)$, then $\{y_n\} = \{Ix_n\}$, $I : (X; d) \to (X; D)$, I - identity map, is also a Cauchy sequence in (X, D) in view of the equivalence relation

$$d \sim d^p, \quad p > 0.$$

□

13. System of mappings

This part is based on the paper [24].

In the sequal, we shall be using the following result (see also [7, 50]).

Theorem 13.1. *Let $(X; d)$ be a complete b-metric space and $T : X \to X$ satisfy*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)], \quad x, y \in X \tag{13.1}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that

$$\lim_{n\to\infty} \varphi^n(t) = 0, \tag{13.2}$$

for each $t > 0$. Then T has exactly one fixed point $u \in X$, and

$$\lim_{n\to\infty} d[T^n(x), u] = 0 \tag{13.3}$$

for each $x \in X$.
One can apply the ideas contained in [28]*, to get the following result for a system of mappings in generalized b-metric spaces.*

Theorem 13.2. *Let $(X_i; d_i)$, $i = 1, \ldots, n$ be complete generalized b-metric spaces. Assume that there exist nonnegative real numbers $d_{i,k}$, $i, k = 1, \ldots, n$ such that the mappings $T_i : X_1 \times \ldots \times X_n \to X_i$, $i = 1, \ldots, n$ satisfy*

$$d_i[T_i(x_1, \ldots, x_n), T_i(z_1, \ldots, z_n)] \leqslant \sum_{k=1}^{n} d_{i,k} d_k(x_k, z_k), \tag{13.4}$$

for $x_k, z_k \in X_k$, $d_k(x_k, z_k) < \infty$, $k = 1, \ldots, n$.
Moreover, there exists a system of positive numbers $r_i, i = 1, \ldots, n$ satisfying the inequalities

$$\sum_{i=1}^{n} r_i a_{i,k} < r_k, \quad k = 1, \ldots, n. \tag{13.5}$$

For any fixed $x^0 \in X = X_1 \times \ldots \times X_n$, consider the sequence of successive approximations

$$x_i^{m+1} = T_i(x_1^m, \ldots, x_n^m), \quad m = 0, 1, \ldots, \quad i = 1, \ldots, n \tag{13.6}$$

Then, either

(A) *for any non-negative integer u, there exists an $i \in \{1, \ldots, n\}$, such that*

$$d_i[x_i^u, T_i(x_1^u, \ldots, x_n^u)] = \infty, \tag{13.7}$$

or

(B) *there exists a non-negative integer u such that for every $i = 1, \ldots, n$,*

$$d_i[x_i^u, x_i^{u+1}] < \infty. \tag{13.8}$$

In (B) the sequence $x^m = (x_1^m, \ldots, x_n^m)$ given by (13.6), converges to a fixed point $u = (u_1, \ldots, u_n) \in X$ of $T = (T_1, \ldots, T_n)$, i.e.

$$T_i(u_1, \ldots, u_n) = u_i, \quad i = 1, \ldots, n.$$

In the space $B = B_1 \times \ldots \times B_n$, where

$$B_i = \Big\{x_i \in X_i; \quad d_i(x_i^u, x_i) < \infty\Big\}, \quad i = 1, \ldots, n, \tag{13.9}$$

the point u is the unique fixed point of T.

Proof. Assume that the case (A) does not hold. So in (B), in view of (13.8), one has for $i = 1, \ldots, n$:

$$d_i[T_i(x^u), T_i(x^{u+1})] \leqslant \sum_{k=1}^{n} a_{i,k} d_k(x_k^u, x_k^{u+1}) < \infty.$$

Thas, by mathematical induction principle,

$$d_i(x_i^{u+l}, x_i^{u+l+1}) < \infty, \quad l \in \mathbb{N}_0, \quad i = 1, ..., n.$$

Consider the number

$$\alpha := \max_k \Big[\frac{1}{r_k} \sum_{i=1}^{n} r_i d_{i,k}\Big].$$

It is obvious that

$$0 \leqslant \alpha < 1 \tag{13.10}$$

and

$$\sum_{i=1}^{n} r_i a_{i,k} \leqslant \alpha_k^r, \quad k = 1, \ldots, n. \tag{13.11}$$

We verify that $T : B \to B$. For let $x \in B$, then

$$\begin{aligned} d_k[x_k^u, T_k(x)] &\leqslant s_k\Big[d_k(x_k^u, T_k(x^u) + d_k(T_k(x^u), T_k(x)\Big] \\ &\leqslant s_k\Big[d_k(x_k^u, x_k^{u+1}) + \sum_{l=1}^{n} a_{k,l} d_l(x_l^u, x_l)\Big] < \infty. \end{aligned}$$

Let's now define

$$D(x, y) := \sum_{i=1}^{n} r_i d_i(x_i, y_i), \quad x, y \in B. \tag{13.12}$$

One can show that D is a b-metric with $s = \max_i(s_i) > 0$. Also we can prove that $(X; D)$ is a complete b-metric space (for details you can see [28]).

Furthemore we verify that T is a contraction mapping in B. In fact, take $x, z \in B$, so

$$\begin{aligned} D[T(x), T(z)] &= \sum_{i=1}^{n} r_i d_i[T_i(x), T_i(z)] \\ &\leqslant \sum_{i=1}^{n} r_i\Big[\sum_{i=1}^{n} a_{i,k} d_k(x_k, z_k)\Big] \\ &\leqslant \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} r_i a_{i,k}\Big) d_k(x_k, z_k) \\ &\leqslant \sum_{k=1}^{n} \alpha r_k d_k(x_k, z_k) \leqslant \alpha D(D(x, z)). \end{aligned}$$

This means exactly

$$D[T(x), T(z)] \leqslant \alpha D(x, z), \quad x, z \in B. \tag{13.13}$$

Thus, by (13.10), T is a strict contraction in B.
Eventually, in view of Theorem 13.1 for $\varphi(t) = \alpha t$, $t \geqslant 0$, T has in B exactly one fixed point u, which is the limit of the successive approximation for any initial element from B, which means that T has exactly one fixed point in X. The proof is complete. □

Next is the following.

Corollary 13.3. *Let the assumptions of Theorem* 13.2 *be satisfied. Let, moreover,*

$$s\alpha < 1. \tag{13.14}$$

Then

$$D(x, u) \leqslant \frac{s}{1 - s\alpha} D(x, T(x)), \quad x \in B \tag{13.15}$$

Proof. For the case (B) for $x \in B$ and $u = T(u)$, one has

$$\begin{aligned} D(x, u) &\leqslant s\Big[D(x, T(x)) + D(T(x), T(u))\Big] \\ &\leqslant s\Big[D(x, T(x)) + \alpha D(x, u)\Big], \end{aligned}$$

whence

$$D(x, u) \leqslant \frac{s}{1 - s\alpha} D(x, T(x)), \quad x \in B.$$

i.e. the condition (13.15). □

We have also

Corollary 13.4. *Let the assumptions of Corollary* 13.3 *are satisfied. If, moreover, $d_i, i = 1, \ldots, n$ are continous at least with respect to one variable, then*

$$D(T^m(x), u) \leqslant \frac{s\alpha^m}{1 - s\alpha} D(x, T(x)), \quad x \in B. \tag{13.16}$$

Proof. By Theorem 13.2, for $\varphi(t) = \alpha t$, $t \geqslant 0$, $x \in B$ and continaily of D, one gots

$$\begin{aligned} D[T^m(x), u] &\leqslant \sum_{k=0}^{\infty} s^{k+1} \varphi^{m+k}\Big[D(x, T(x)\Big] \\ &= \sum_{k=0}^{\infty} s^{k+1} \alpha^{m+k} D(x, T(x)) = \frac{s\alpha^m}{1 - s\alpha} D(x, T(x)), \end{aligned}$$

i.e. the inequality (13.16), which ends the proof. □

Remark 13.5. A b-metric D may not be continuous (*cf.* e.g. [39]).

Remark 13.6. From Theorem 13.2 we get theorem of Diaz - Margolis [28], Luxemberg [39], Banach [10], Matkowski [43], Czerwik [16, 17].

14. Local results

In this part we are concerned on some local results for a system of mappings in generalized b-metric spaces. We start with the following.

Theorem 14.1. *Let $(X; d)$, $i = 1, \dots, n$ be complete generalized b-metric spaces. Assume that there exist non-negative real numbers $d_{i,k}$, $i, k = 1, \dots, n$ and $c > 0$ such that the mappings $T_i : X_1 \times \dots \times X_n \to X_i$, $i = 1, \dots, n$ fulfil the inequalities*

$$d_i[T_i(x_1, \dots, x_n), T_i(z_1, \dots, z_n)] \leqslant \sum_{k=1}^{n} \alpha_{i,k} d_k(x_k, z_k) \tag{14.1}$$

for $d_k(x_k, z_k) < c$, $x_k, z_k \in X_k$, $k = 1, \dots, n$. Moreover, let the characteristic roots $\xi_i, i = 1, \dots, n$ of the matrix $[a_i, k]$ satisfy

$$\alpha = \max\left[|\xi_i| : i = 1, ..., n\right] < 1. \tag{14.2}$$

Let $x^0 \in X = X_1 \times \dots \times X_n$, be arbitrarily fixed. Consider the sequence of successive approximations (13.6)*. Then the following alternative holds: either*

(C) *for any $u \in \mathbb{N}_0$, there exists $i \in \{1, \dots, n\}$ such that*

$$d_i[x_i^u, T_i(x_i^u, \dots, x_n^u)] \geqslant C, \tag{14.3}$$

or

(D) *there exists a non-negative integer u such that for every $i = 1, \dots, n$,*

$$d_i(x_i^u, x_i^{u+1}) < C. \tag{14.4}$$

In the case (D), if moreover, the numbers on the left hand side of (14.4) *are sufficiently small and* (13.14) *holds true, then $T = (T_1, \dots, T_n)$ has a fixed point $u \in X$.*

Proof. The proof with details can be found in [24]. Also some number of remarks concerned the assumptions, can be found in [24]. □

15. Fixed point theorems in generalized b-metric spaces

First of all we present the result from which we got the famous result of Stefan Banach and Theorem 13.1.

Theorem 15.1 (*cf.* [31])**.** *Let $(X; d)$ be a complete generalized b-metric space and let $T : X \to X$ be such that*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)], \tag{15.1}$$

for all $x, y \in X$, $d(x, y) < \infty$, where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing and

$$\lim_{n\to\infty} \varphi^n(z) = 0, \text{ for } z > 0. \tag{15.2}$$

Let $x \in X$ be arbitrarily fixed. Then the following alternative holds: either

(E) *for every nonnegative integer $n \in \mathbb{N}_0$*

$$d[T^n(x), T^{n+1}(x)] = \infty,$$

or

(F) *there exists a $k \in \mathbb{N}_0$ such that*

$$d[T^k(x), T^{k+1}(x)] < \infty.$$

In (F)

1. *the sequence $\{T^m(x)\}$ is a Cauchy sequence in X;*
2. *there exists a point $u \in X$ such that*

$$\lim_{n\to\infty} d[T^n(x), u] = 0 \text{ and } T(u) = u;$$

3. *u is the unique fixed point of T in*

$$B := \{t \in X : d[T^k(x), t] < \infty\};$$

4. *for every $t \in B$,*

$$\lim_{n\to\infty} d[T^n(t), u] = 0.$$

If, moreover, d is continuous (with respect to one variable) and

$$\sum_{i=1}^{\infty} s^i \varphi^i(t) < \infty \text{ for } t > 0,$$

then for $t \in B$

$$d[T^m(t), u] \leqslant \sum_{i=0}^{\infty} s^{i+1} \varphi^{m+i}[d(t, T(t))], \quad m \in \mathbb{N}_0. \tag{15.3}$$

Please note that $\varphi^n(t)$ means the n-th iteration of at φ at t.

Proof. Let's consider the following steps.

1^o) Let's $x \in X$ and $\epsilon > 0$ be fixed. Consider $n \in \mathbb{N}$ with

$$\varphi^n(t) < \frac{\epsilon}{2s},$$

and take

$$F = T^n, \quad \alpha = \varphi^n \text{ and } x_m = F^m(x) \text{ for } m \in \mathbb{N}.$$

Then for all $x, y \in X$ satisfying $d(x, y) < \infty$, we got

$$d[F(x), F(y)] = d[T^n(x), T^n(y)] \leqslant \varphi^n[d(x, y)] \leqslant \alpha[d(x, y)]. \tag{15.4}$$

2^o) we can prove that $(B; d)$ is a complete b-metric space. Also we have $T^k(x), T^{k+1}(x)$ are elements of B

3^o) Now we verify that $T : B \to B$. For it, let $t \in B$, i.e. $d[T^k(x), t] < \infty$. Then one has

$$\begin{aligned} d[T^k(x), T(t)] &\leqslant s\Big[d(T^k(x), T^{k+1}(x)) + d(T^{k+1}(x), T(t))\Big] \\ &\leqslant s\Big[\epsilon_1 + \varphi[d(T^k(x), t]\Big] \\ &\leqslant s\Big[\epsilon_1 + \epsilon_2\Big] < \infty, \end{aligned}$$

where of course ϵ_1, ϵ_2 are some positive real numbers. This means that $T : B \to B$, and hence also $F : B \to B$ as well.

4^o) If $t \in B$, then $\{F^m(t)\} \subset B$ for all $m \in \mathbb{N}_0$. We verify also that $\{F^m(t)\}$ is a Cauchy sequence. For it, let $y_m = F^m(t)$, $m \in \mathbb{N}_0$, $t \in B$ Then by (15.4) and 3^o)

$$d[F(x), F^2(t)] \leqslant \alpha[d(t, F(t)]$$

(because t, $F(t)$ are in B, so $d[t, F(x)] < \infty$). Consequently

$$d[F^m(t), F^{m+1}(t)] \leqslant \alpha^m[d(t, F(t)],$$

i.e.

$$d(y_m, y_{m+1}) \leqslant \alpha^m[d(t), F(t)], \quad m \in \mathbb{N}_0,$$

whence

$$d(y_m, y_{m+1}) \to 0 \text{ as } m \to \infty.$$

Consider m such that

$$d(y_m, y_{m+1}) < \frac{\epsilon}{2s}.$$

Therefore, for every $z \in K(y_m, \epsilon) := \{y \in X; d(y_m, y) \leqslant \epsilon\}$ one gets

$$d[F(z), F(y_m)] \leqslant \alpha[d(z, y_m)] \leqslant \alpha(\epsilon) = \varphi^n(\epsilon) < \frac{\epsilon}{2s},$$

and

$$d[F(y_m), y_m] = d(y_{m+1}, y_m) < \frac{\epsilon}{2s},$$

so

$$d[F(z), y_m] \leqslant s[d(F(z), T(y_m)) + d(F(y_m, y_m)] \leqslant s[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}] = \epsilon.$$

This means that F maps $K(y_m, \epsilon)$ into itself. So

$$d(y_r, y_l) \leqslant 2s\epsilon \text{ for all } r, l \geqslant m,$$

and consequently, the sequence $\{y_r\} = \{F^r(t)\}$, $t \in B$, is a Cauchy sequence.

5^o) Since B is complete, so there exists $u \in BCX$ such that $y_r \to u$ as $r \to \infty$. Moreover, by the continuity of F (see the condition (15.4)),

$$F(u) = \lim_{r\to\infty} F(y_r) = \lim_{r\to\infty} y_{r+1} = u,$$

i.e. u is the fixed point of F, and $u \in X$.

By the condition $\alpha(t) = \varphi^n(t) < t$ for all $t > 0$, it is clear that F has only one fixed point in B. Moreover, by the continuity of T on B (see (15.1)), one gets

$$T(u) = \lim_{n\to\infty} T(F^r(t)) = \lim F^r(T(t)) = u,$$

whence we see that u is a fixed point of T as well. Clearly, in view of (15.1), such point is only one in B.

Finally, since for every $t \in B$ and every $r = 0, 1, \ldots, n-1$,

$$T^m(t) = T^{nl+r}(t) = F^l[T^r(t)] \to u \text{ as } l \to \infty,$$

so

$$d[T^m(t), u] \to 0 \text{ as } m \to \infty$$

for every $t \in B$. This shows 4).

6^o) Let $t \in B$ and $m, n \in \mathbb{N}_0$. Then

$$\begin{aligned} d[T^m(t), T^{m+n}(t)] &\leqslant s\Big[d(T^m(t), T^{m+1}(t)) + d(T^{m+1}(t), T^{m+n}(t))\Big] \\ &\leqslant sd\Big[T^m(t), T^{m+1}(t)\Big] + \cdots + s^n d\Big[T^{m+n-1}(t), T^{m+n}(t)\Big] \\ &\leqslant s\varphi^m[d(t, T(t))] + \cdots + s^n\varphi^{m+n-1}[d(t, T(t))] \\ &\leqslant \sum_{r=0}^{\infty} s^{r+1}\varphi^{m+r}[d(t, T(t))]. \end{aligned}$$

Therefore,

$$d[T^m(t), T^{m+n}(t)] \leqslant \sum_{r=0}^{\infty} s^{r+1}\varphi^{m+r}[d(t, T(t)].$$

Eventually, for $n \to \infty$, if d is continuous function (with respect to one variable), then for all $t \in B$ and $m \in \mathbb{N}_0$, we got the estimation

$$d[T^m(t), u] \leqslant \sum_{r=0}^{\infty} s^{r+1}\varphi^{m+r}[d(t, T(t)],$$

which ends the proof of the theorem.

□

Remark 15.2. If X is a b-metric space, then $B = X$, and we got from Theorem 15.1 previous result, Theorem 13.1 (see also [7, 16, 17]).

Remark 15.3. Consider the series of iterates

$$\sum_{n=1}^{\infty} \varphi^n(t).$$

We know that the sufficient condition for convergent of such series of iterates is the following one

$$\limsup_{n\to\infty} \frac{\varphi^{n+1}(t)}{\varphi^n(t)} = q(t) < 1, \quad t > 0. \tag{15.5}$$

The another one is the following

$$\liminf_{n\to\infty} \Big[-\frac{\ln \varphi^n(t)}{\ln n}\Big] = \alpha(t) > 1, \quad t > 0 \tag{15.6}$$

For more details, the reader may consult [23].

Let's note also the following result (for quasi-linear contraction in b-metric spaces). For more details and the proof, see [7].

Theorem 15.4. *Let $(X; d)$ be a complete generalized b-metric space. Let $T : X \to X$ be continuous and satisfies the condition*

$$d[T(x), T^2(x)] \leqslant \alpha d(x, T(x)), \tag{15.7}$$

for all $x \in X$ such that $d(x, T(x)) < \infty$, and

$$\alpha s = q < 1. \tag{15.8}$$

Let $x \in X$ be arbitrarily fixed. Then the following alternative holds: either

(G) *for every nonnegative integer $n \in \mathbb{N}_0$,*

$$d[T^n(x), T^{n+1}(x)] = \infty,$$

or

(H) *there exists a* $k \in \mathbb{N}_0$ *such that*

$$d[T^k(x), T^{k+1}(x)] < \infty.$$

In (H)

1. *the sequence* $\{T^m(x)\}$ *is a Cauchy sequence in* X*;*
2. *there exists a point* $u \in X$ *such that*

$$\lim_{m\to\infty} d[T^m(x), u] = 0 \text{ and } T(u) = u.$$

The proof is omitted here (for details see [23]*).*

16. Nadler's fixed point theorem for set-valued mappings in b-metric spaces

In this section we present theorems on set-valued mappings in b-metric spaces, which generalize the famous Nadler's fixed point result presented in the papers [45] and [46]. See also [21].

First of all we present some usefull ideas.

Definition 16.1. A set $YC(X, d)$, where d is a generalized b-metric in X, is said to be closed, iff for every sequence of points $x_n \in Y$ for $n \in \mathbb{N}$, x_n tends to $x \in X$ (with respect to d) implies $x \in Y$.
Let's note: by $CL(X)$ we denote the family of all nonempty and closed subsets of b-metric space X (see also [14]).

Definition 16.2. Let

$$H(A, B) := \begin{cases} \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A), \text{if the max exists,} \\ \infty, \text{otherwise,} \end{cases} \tag{16.1}$$

for all $A, B \in CLX$, where $D(x, A)$ denotes the distance between x and A.
In such case, the function H is called the generalized Hausdorff distance, induced by d (see also [21]).

Let's note the following lemma.

Lemma 16.3. *For* $a \in X$ *and* $A \in CL(X, d)$*, where* (X, d) *is a generalized b-metric space, one has*

$$D(a, A) = 0 \Rightarrow a \in A.$$

Proof. Let $D(a, A) = 0$, that is $\inf_{\xi\in A} d(a, \xi) = 0$, then for $\epsilon_n > 0$, there exists $\xi_n \in A$ with $d(a, \xi_n) < \epsilon_n$. Assume that $\epsilon_n \to 0$ as $n \to \infty$, then $d(a, \xi_n) \to 0$. Clearly $\{\xi_n\}$ is a Cauchy sequence of real positive numbers. Indeed,

$$d(\xi_n, \xi_{n+m}) \leqslant s[d(\xi_n, a) + d(a, \xi_{n+m})] \to s[010],$$

for $n, m \in \mathbb{N}$, $s \geqslant 1$. Therefore $\xi_n \to \xi \in X$, and since $A \in CL(X, d)$, $\xi \in A$.
But

$$d(a, \xi) \leqslant s[d(a, \xi_n) + d(\xi_n, \xi)] \to s[0 + 0],$$

so

$$[d(a, \xi) \to 0] \Rightarrow a = \xi \in A,$$

which means that $a \in A$.
Of course, $a \in A \Rightarrow D(a, A) = 0$.
Also one has (see also [14]) □

Lemma 16.4. *If* (X, d) *is a generalized b-metric space, then* H *defined by* (16.1) *is a generalized b-metric* (*gbm*), *i.e.*

$$H(A, B) = 0 \Leftrightarrow A = B, \tag{16.2}$$

$$H(A, B) = H(B, A), \tag{16.3}$$

$$H(A, B) \leqslant s[H(A, U) + H(U, B)], \tag{16.4}$$

for all $A, B, U \in CL(X, d)$ *(except that* H *may have infinite values).*

Proof. Assume that for $A, B \in CL(X, d)$

a) $H(A, B) < \infty, H(A, B) = \sup_{a\in A} D(a, B)$.

One has

$$\begin{aligned} H(A, B) = \sup_{a\in A} D(a, B) &\leqslant \sup_{a\in A}\{\inf_{b\in B} d(a, b)\} \\ &\leqslant \sup_{a\in A}\inf_{b\in B}\{s[d(a, \xi) + d(\xi, b)\}, (\xi \in U, \in CL(X, d)) \\ &\leqslant s\sup_{a\in A}\{d(a, \xi) + d(\xi, b)\} \\ &\leqslant s\sup_{a\in A}\{d(a, \xi) + D(U, b)\} \leqslant s\sup_{a\in A}\{D(a, U) + D(U, b)\} \\ &\leqslant s[\sup_{a\in A} D(a, U) + DU, b)] \leqslant s[H(A, U) + H(U, B)]. \end{aligned}$$

Thus we have

$$H(A, B) \leqslant s[H(A, U) + H(U, B)], \quad A, B, U \in CL(X, d).$$

For the case

b) $H(A, B) = \sup_{b\in B} D(b, A)$,

the proof is exactly the same.

For (16.2): by Lemma 16.3 and the Definition 16.2,

$$H(A, B) = 0 \Rightarrow \sup_{a\in A} D(a, B) = 0 \Rightarrow D(a, B) = 0 \Rightarrow a \in B \Rightarrow ACB.$$

Similarly, BCA, i.e. $A = B$.

Conversely, if $A = B \Rightarrow H(A, B) = 0$ strictly by the definition. The condition (16.3) is obvious. □

Let's note also the following simple remark (one can see also [46]). Generalized b-metric can be reduce to b-metric by taking the minimum of the generalized b-metric and the real positive number 1. Such now b-metric preserves the topology but changes the Lipschitz structure of the generalized b-metric. Since we work with contraction mappings with respect to b-metric, we are not in a position to do such operation during the proofs.

The next result is the following

Lemma 16.5. *Let* (X, d) *be a generalized b-metric in* X. *Define*

$$\xi(x, y) := \min[d(x, y), 1] = \begin{cases} d(x, y), & d(x, y) < 1, \\ 1, & d(x, y) \geqslant 1. \end{cases} \tag{16.5}$$

The function (16.5) *is a b-metric in* X.

Proof. Consider the case

1) $d(x, y) \geqslant 1$.

Therefore one has $\xi(x, y) = 1$. Also

$$d(x, y) \leqslant s[d(x, z) + d(z, y)], \quad s \geqslant 1, \quad x, y, z \in X, \quad d(x, y) < \infty.$$

Let $d(x, z) < 1, d(z, y) < 1$, then $\xi(x, z) = d(x, z)$, $\xi(z, y) = d(z, y)$ and

$$\xi(x, y) = 1 \leqslant s[d(x, z) + d(z, y)] \leqslant s[\xi(x, z) + \xi(z, y)].$$

If $d(x, z) \geqslant 1$, then $\xi(x, z) = 1$, and consequently

$$\xi(x, y) \leqslant 1 \leqslant s \leqslant s[1 + \xi(z, y)] \leqslant s[\xi(x, z) + \xi(z, y)].$$

Let

2) $d(x, y) < 1$.

So $\xi(x, y) = d(x, y)$ and

$$\xi(x, y) = d(x, y) \leqslant s[d(x, z) + d(z, y)], \quad x, y, z \in X, \quad s \geqslant 1.$$

Let $d(x, z) < 1$, $d(z, y) < 1$, then $\xi(x, z) = d(x, z)$, $\xi(z, y) = d(z, y)$ and $\xi(x, y) = d(x, y) \leqslant s[\xi(x, z) + \xi(z, y)]$.
If $d(x, z) \geqslant 1$, then $\xi(x, z) = 1$

$$\xi(x, y) = d(x, y) < 1 \leqslant s \leqslant s\xi(x, z) \leqslant s[\xi(x, z) + \xi(z, y)].$$

In the rest cases, the proof can be done quite similarly, so the inequality

$$\xi(x, y) \leqslant s[\xi(x, z) + \xi(z, y)], \quad x, y, z \in X, \quad s \geqslant 1$$

holds true in any case. □

Lemma 16.6. *For every $u \in A$, $A \in CL(X, d)$, where d is α generalized metric, and every $\epsilon > 0$, there exists a $b \in B$, $B \in CL(X, d)$ such that*

$$d(u, b) \leqslant H(A, B) + \epsilon. \tag{16.6}$$

Proof. Let

α) $H(A, B) < \infty$, $H(A, B) = \sup_{a \in A} D(a, B)$.

Then

$$\sup_{a \in A} D(a, B) \leqslant H(A, B) \leqslant H(A, B) + \epsilon.$$

Therefore there exists $b \in B$ such that

$$d(a, b) \leqslant H(A, B) + \epsilon.$$

If not, so for every $b \in B$, $d(a, b) > H(A, B) + \epsilon$, who $\epsilon > 0$ is fixed. Hence

$$\inf_{b \in B} d(a, b) = D(a, B) \geqslant H(A, B) + \epsilon,$$

so

$$\sup_{a \in A} D(a, B) \geqslant H(A, B) + \epsilon,$$

and therefore

$$H(A, B) = \sup_{a \in A} D(a, B) \geqslant H(A, B) + \epsilon,$$

what is impossible. □

Definition 16.7 (*cf.* [14]). A function $T : X \to CL(X)$ is said to be a multi-valued contraction mapping (mvcm) iff there exists a real number $0 \leqslant \lambda < 1$ such that

$$H[T(x), T(y)] \leqslant \lambda d(x, y), \quad x, y \in X, \quad d(x, y) < \infty.$$

Similarly, a mapping $T : X \to CL(X)$ is said to be an (λ, ϵ)-uniformly locally contractive multi-valued mapping, where $\epsilon > 0$ and $0 \leqslant \lambda < 1$, iff

$$H[T(x), T(y)] \leqslant \lambda d(x, y), \quad x, y \in X, \quad d(x, y) < \epsilon.$$

Definition 16.8 (*cf.* [14]). Let (X, d) be a generalized b-metric space $x \in X$, and let $T : X \to CL(X)$ be a set-valued mapping. It sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in X$, $n \in \mathbb{N}$, is said to be an iterative sequence of T at x_0, iff $x_{n+1} \in T(x_n)$ for $n \in \mathbb{N}_0$. A point $x \in X$ is called a fixed point of T iff $x \in T(x)$.

Definition 16.9 (*cf.* [14, 15]). A generalized b-metric space (X, d) is said to be ϵ-chainable, $\epsilon > 0$, iff for every $x, y \in X$, $d(x, y) < \infty$, there exists an ϵ - chain from x to y, i.e. a finite set of points $x_0 = x, x_1, \ldots, x_n = y$, $x_k \in X$, $k \in \mathbb{N}_0$, such that $d(x_{k-1}, x_k) < \epsilon$ for $k = 1, \ldots, n$. Let's note the following interesting results presented by Covitz and Nadler, Ir. in [14].

Proposition 16.10 (*cf.* [14]). *Let (X, d) be a generalized complete metric space, and let $x_0 \in X$. If $T : X \to CL(X)$ is an (λ, ϵ) - uniformly locally contractive multi-valued mapping, then the following alternative holds: either*

(a) *for each iterative sequence $\{x_n\}_{n=1}^{\infty}$, of T at $x_0, d(x_{n-1}, x_n) \geqslant \epsilon$ for each $n \in \mathbb{N}$,*
or

(d) *there exists an iterative sequence $\{x_n\}_{n=1}^{\infty}$, of T at x_0, such that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T.*

Proposition 16.11 (*cf.* [45]). *Let (X, d) be a complete metric space and lot $x_0 \in X$. If $T : X \to CL(X)$ is a mvcm, then there exists an iterative sequence $\{x_n\}_{n=1}^{\infty}$ of T at x_0 such that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T.*

Now we are in the position to present the following main result of this section.

Theorem 16.12. *Let (X, d) be a generalized complete b-metric space and let $x_0 \in X$. If $T : X \to CL(X, d)$ is a (λ, ϵ) - uniformly locally contractive multi-valued mapping, i.e.*

$$H[T(x), T(y)] \leqslant \lambda d(x, y), \quad x, y \in X, \quad d(x, y) < \epsilon, \tag{16.7}$$

and

$$0 \leqslant \lambda < 1, \tag{16.8}$$

then the following alternative holds: either

(c) *for each iterative sequence $\{x_n\}_{n=1}^{\infty}$, of T at $x_0, d(x_{n-1}, x_n) \geqslant \epsilon$ for $n \in \mathbb{N}$,*

or

(d) *there exists an iterative sequence $\{x_n\}_{n=1}^{\infty}$ of T at x_0, such that $x_n \to u \in X$ and $u \in T(u)$, i.e. u is a fixed point of T.*

Proof. Assume that (c) does not hold. So there exists $x_m \in T(x_{m-1})$, $x_m, x_{m-1} \in X$ and $d(x_m, x_{m-1}) < \epsilon$ for some $m \in \mathbb{N}$. Therefore,

$$H\big[T(x_{m-1}), T(x_m)\big] \leqslant \lambda d(x_{m-1}, x_m) < \lambda\epsilon < \epsilon.$$

Since $x_m \in T(x_{m-1})$, there exists an $x_{m+1} \in T(x_m)$ such that (by Lemma 16.6)

$$d(x_m, x_{m+1}) \leqslant H\big[T(x_{m-1}), T(x_m)\big] + \lambda\overline{\epsilon},$$

with $\overline{\epsilon} > 0$. Let $\overline{\epsilon} < \min\left[\epsilon - \lambda\epsilon, 1\right]$. Therefore

$$d(x_m, x_{m+1}) < \lambda\epsilon + \lambda\overline{\epsilon} < \lambda\epsilon + \overline{\epsilon} \leqslant \lambda\epsilon + (\epsilon - \lambda\epsilon) \leqslant \epsilon.$$

so we have

$$d(x_m, x_{m+1}) < \epsilon$$

□

Now we show that there exists an iterative sequence $\{x_n\}_{n=1}^{\infty}$, for T such that

α) $x_{m+k+1} \in T(x_{m+k})$,

β) $d(x_{m+k}, x_{m+k+1}) < \epsilon$,

γ) $d(x_{m+k}, x_{m+k+1}) < \epsilon\lambda^k(1+k)$,

for each $k \in \mathbb{N}_0$.
If $k = 0$, the conditions α, β are satisfied (γ is trivial). So assume now that α, β, γ are true for $k \in \mathbb{N}_0$. By mathematical induction, for $k+1$ one has (see the Lemma 16.6):

$$\begin{aligned} d[x_{m+k+1}, x_{m+k+2}] &\leqslant H[T(x_{m+k}), T(x_{m+k+1})] + \lambda^{k+1}\overline{\epsilon} \\ &\leqslant \lambda[\lambda^k\epsilon(1+k)] + \lambda^{k+1}\overline{\epsilon} \\ &\leqslant \lambda^{k+1}(1+k)\epsilon + \lambda^{k+1}\overline{\epsilon} \\ &\leqslant \lambda^{k+1}[\epsilon(1+k) + \overline{\epsilon}] \\ &< \lambda^{k+1}[\epsilon(1+k) + \epsilon] \leqslant \epsilon\lambda^{k+1}(z+k), \end{aligned}$$

i.e.

$$d(x_{m+k+1}, x_{m+k+2}) < \epsilon\lambda^{k+1}(z+k).$$

Obviously,

$$\begin{aligned} d(x_{m+k+1}, x_{m+k+2}) &\leqslant H[T(x_{m+k}), T(x_{m+k+1})] + \lambda^{k+1}\overline{\epsilon} \\ &\leqslant \lambda d(x_{m+k}, x_{m+k+1}) + \overline{\epsilon} \\ &\leqslant \lambda\epsilon + \overline{\epsilon} < \lambda\epsilon + (\epsilon - \lambda\epsilon) = \epsilon. \end{aligned}$$

Therefore, α, β, γ are true for $k+1$. From induction principle, α, β, γ are true for all $k \in \mathbb{N}_0$.

In the next step we verify that such constracted sequence is a Cauchy sequence. To do that we shall use the contraction of Paluszyński and Stempak [48], of a metric from a b-metric.

For $x, y \in X$, $d(x, y) < \infty$, let

$$d_\epsilon := \begin{cases} \inf \sum_{i=1}^{n} d^p(x_{i-1}, x_i) : x = x_0, x_1, \ldots, x_n = y, \\ \infty, \quad \text{if } d(x, y) = \infty, \end{cases} \tag{16.9}$$

where $0 < p \leqslant 1$ is such that

$$(2s)^p = 2.$$

Clearly, d_ϵ is symmetric, satisfies the triangle inequality and $d_\epsilon \leqslant d^p$. Paluszyński and Stempak also proved that d_ϵ is a metric (in little bite different assumptions) and $d_\epsilon \sim d^p$.

Also for $m, r \in \mathbb{N}$, by the definition (16.9) one has

$$\begin{aligned}
d_\epsilon(x_m, x_{m+r}) &\leqslant d_\epsilon(x_m, x_{m+1}) + \ldots + d_\epsilon(x_{m+r-1}, x_{m+r}) \\
&\leqslant d^p(x_m, x_{m+1}) + \ldots + d^p(x_{m+r-1}, x_{m+r}) \\
&\leqslant \sum_{i=m}^{\infty} d^p(x_i, x_{i+1}) \\
&\leqslant \sum_{i=m}^{\infty} [\epsilon\lambda^i(1+i)]^p \leqslant \epsilon^p \sum_{i=m}^{\infty} \xi^i(1+i), \quad \xi = \lambda^p < 1 \\
&\leqslant \epsilon^p \sum_{i=m}^{\infty} \xi^i(1+i)^p \leqslant \epsilon^p \sum_{i=m}^{\infty} \xi^i(1+i), \quad \xi = \lambda^p < 1 \\
&\leqslant \epsilon^p \sum_{i=m}^{\infty} (\xi^{i+1})' \leqslant \epsilon^p (\sum_{i=m}^{\infty} \xi^{i+1})' \\
&\leqslant \epsilon^p \Big[\frac{\xi^{i+1}}{1-\xi}\Big]' \leqslant \epsilon^p \frac{(m+1)\xi^m(1-\xi) + \xi^{m+1}}{(1-\xi)^2} \\
&\leqslant \epsilon^p (1-\xi)^{-2}[(m+1)\xi^m(1+\xi) + \xi^m] \\
&\leqslant \epsilon^p \xi^m[(m+1)(1+\xi) + 1] \leqslant \epsilon^p \lambda^{pm}[2(m+1)+1](1-\xi)^{-2} \\
&\leqslant \epsilon^p (1-\xi)^{-2}\lambda^{pm}(2m+3) \to 0, \quad m \to \infty \text{ since } \lambda < 1
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in (X, d_ϵ), so $\{x_n\}$ is a Cauchy sequence in (X, d), since $d_\epsilon \sim d^p$ (for more details see Jung [33] and [19, 23]). Consequently, $x_n \to u$ as $n \to \infty$ in (X, d), but (X, d) is complete, so $u \in X$.
At last, we prove that $u \in T(u)$. Indeed, for n sufficiently large, we have

$$\begin{aligned}
D(u, T(u)) = \inf_{\xi \in T(u)} d(u, \xi) &\leqslant d(u, \xi) \\
&\leqslant s[d(u, x_{n+1}) + d(x_{n+1}, \xi)] \\
&\leqslant s[d(u, x_{n+1}) + D(x_{n+1}, T(u))] \\
&\leqslant s[d(u, x_{n+1}) + H(T(x_n), T(u))] \\
&\leqslant s[d(u, x_{n+1}) + \lambda d(x_n, u)] \to s[0+0] = 0, \quad n \to \infty.
\end{aligned}$$

Hence $D(u, T(u)) = 0$, but since $T(u)$ is closed, so $u \in T(u)$ and the proof is complete.
We have also the following

Theorem 16.13. *Let (X, d) be a complete ϵ-chainable generalized metric space and let $x_0 \in X$. If $T : X \to CL(X, d)$ is a (λ, ϵ)-uniformly locally contractive multi-valued mapping, then the following alternative holds: either*

(e) *for each iterative sequence $\{x_n\}_{n=1}^{\infty}$ of T at x_0, $d(x_{n-1}, x_n) = \infty$ for $n \in \mathbb{N}$*

or

(f) *there exists an iterative sequence $\{x_n\}_{n=1}^{\infty}$ of T at x_0 such that $x_n \to x$ as $n \to \infty$, $x \in X$ and $x \in T(x)$.*

Proof. Suppose (e) does not hold. By Paluszyński and Stempak [48], define $d_\epsilon(x, y)$ for $x, y \in X$ by

$$d_\epsilon(x, y) := \begin{cases} \inf \sum_{i=1}^{n} & d^p(x_{i-1}, x_i), \quad x_0 = x, x_1, \ldots, x_n = y, \\ & d(x_{i-1}, x_i) < \infty \text{ for } i = 1, \ldots, n, \\ \infty, & \text{if } d(x, y) = \infty, \end{cases}$$

where

$$(2s)^p = 2.$$

Then, it is clear, that (X, d_ϵ) is a generalized complete metric space, $d_\epsilon \sim d^p$, $0 < p \leqslant 1$ (one can repeat the proof presented by Paluszyński and Stempak for ϵ-chainable b-metric space). Let by H_ϵ, denote the generalized Hausdorff metric on $CL(X, d_\epsilon)$ induced by d_ϵ.
One can verify that $CL(X, d) = CL(X, d_\epsilon)$. Really, if $U \in CL(X, d)$, then by the Definition 16.1, one has $[x_n \in U$ and $x_n \overset{d}{\to} x \in X] \Rightarrow x \in U$, so as well $[x_n \in U \quad x_n \overset{d_\epsilon}{\to} x] \Rightarrow x \in U$, and consequently $CL(X, d) \subset CL(X, d_\epsilon)$. Conversely we have the same, since $d_\epsilon \sim d^p$.
Furthermore we prove that

$$H_\epsilon(A, B) \leqslant H^p(A, B), \quad A, B \in CL(X, d_\epsilon).$$

Let $H_\epsilon(A, B) = \sup\limits_{a \in A} D_\epsilon(a, B)$, then we have for $H_\epsilon(A, B) < \infty$:

$$\begin{aligned} H_\epsilon(A, B) = \sup_{a \in A} D_\epsilon(a, B) &\leqslant \sup_{a \in A} \Big\{ \inf_{b \in B} d_\epsilon(a, b) \Big\} \\ &\leqslant \sup_{a \in A} \inf_{b \in B} d^p(a, b) \leqslant \sup_{a \in A} \Big\{ \inf_{b \in B} d(a, b) \Big\}^p \\ &\leqslant \sup_{a \in A} D^p(a, B) \leqslant \Big[\sup_{a \in A} D(a, B) \Big]^p \leqslant H^p(A, B), \end{aligned}$$

i.e.

$$H_\epsilon(A, B) \leqslant H^p(A, B), \quad A, B \in CL(X, d) = CL(X, d_\epsilon).$$

Assume now that $x, z \in X$ and $d(x, z) < \infty$. For $x_0 = x, \ldots, x_n = z$ we got

$$\begin{aligned} H_\epsilon[T(x), T(z)] &\leqslant \sum_{i=1}^{n} H_\epsilon[T(x_{i-1}), T(x_i)] \\ &\leqslant \sum_{i=1}^{n} H^p[T(x_{i-1}), T(x_i)] \\ &\leqslant \sum_{i=1}^{n} (\lambda d(x_{i-1}, x_i))^p \\ &\leqslant \lambda^p \sum_{i=1}^{n} d^p(x_{i-1}, x_i). \end{aligned}$$

Since the inequality between the first and last terms of the above inequalities holds for all ϵ-chains $x_0 = x, x_1, \ldots, x_n = z$, $n \in \mathbb{N}$, connecting x and z, it follows

$$H_\epsilon[T(x), T(z)] \leqslant \lambda^p d_\epsilon(x, z), \tag{16.10}$$

for all $x, z \in X$, $d(x, z) < \infty$, where $0 \leqslant \lambda^p < 1$. It means that T is a (λ^p, ϵ)-uniformly locally contractive multi-valued mapping with d_ϵ, H_ϵ and $\epsilon > 0$, $d(x_{n-1}, x_n) < \epsilon$. So the sequence $\{x_n\}$ starting from x_{n-1}, does not satisfy the condition (c) of Theorem 16.12, and consequently our statement (f) follows directly from (d) of Theorem 16.12. In other cases, the proof is similar. □

Remark 16.14. In the proofs of Theorem 16.12 and Theorem 16.13 we utilize some ideas contuined in [14].

Remark 16.15. If (X, d) is a metric space, then from Theorem 16.12 we got the famous Nadler's fixed point theorem for multi-valued contraction mappings (see also [12, 14, 15]).

Remark 16.16. If (X, d) is an ϵ-chainable metric space, one gets from Theorem 16.13, Corollary 13.4 of [14].

17. Strong b-metric spaces

The idea of strong b-metric and strong b-metric space come from the Kirk and Shahzad [35]. This is a mapping $d : X \times X \to [0, \infty)$, satisfying all the axioms of a metric and further the following condition

$$d(x, y) \leqslant d(x, z) + sd(z, y) \tag{17.1}$$

for some $s \geqslant 1$ and all $x, y, z \in X$. Looking at the "symmetry" of (17.1), this inequality is equivalent to

$$d(x, y) \leqslant \min\left[sd(x, z) + d(z, y); d(x, z) + sd(dz, y)\right] \quad (17.2)$$

for $s \geqslant 1$ and all $x, y, z \in X$.
Moreover, if d takes infinite values, i.e. $d(x, y) \in [0, \infty]$ for all $x, y \in X$, then d is called generalized strong b-metric and (X, d) a generalized strong b-metric space. It is not difficult to verify that

$$\left|d(x, y) - d(u, v)\right| \leqslant s[d(x, u) + d(y, v)]. \quad (17.3)$$

From (17.3) one gets: if $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$, then

$$d(x_n, y_n) \to d(x, y), \text{ as } n \to \infty.$$

which implies the continuity of the b-metric d at the point (x, y).
Following the ideas contained in [12], we consider the notion of generalized b-metric on a non-empty set X (so also generalized strong b-metric) as a mapping $d : X \times X \to [0, \infty)$ satisfying all the conditions of a b-metric (strong b-metric) for all $x, y \in X$.

Let's note the following idea of Jung [33].
By an isometric embedding of a b-metric space (X, d_1) into a b-metric space (X_2, d_2) we understand a mapping $f : X_1 \to X_2$ with

$$d_2[f(x), f(y)] = d_1(x, y),$$

for all $x, y \in X_1$. We say that two b-metric spaces (X_1, d_1), (X_2, d_2) (with s_1 and s_2, $s_i \geqslant 1$, $i = 1, 2$) are isometric iff there exists a surjective isometric embedding $f : X_1 \to X_2$. Let's note that the problem: every strong b-metric space has a completion, has been solved in [3]. The result reads as follows.

Theorem 17.1. *Let (X, d) be a strong b-metric space. Then*

1. *There exists a complete strong b-metric space (Y, d_1), which is a completion of the space (X, d).*
2. *The completion is unique up to an isometry, i.e. if (X_1, d_1) and (X_2, d_2) are two strong b-metric spaces which are completion of (X, d), then the b-metric spaces (X_1, d_1), (X_2, d_2) are isometric.*

Proof. The proof presented here, is similar to the proof for the metric spaces (see e.g. [12, 38]). The details come from Prof. S. Cobzas (see [12]). For more details see also [25].
Let's consider the space $C(X)$ of all Cauchy sequences in X. We cosider the equivalence relation

$$(x_n) \sim (y_n) \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0.$$

One considers the space (the quotient space)

$$Y := C(X)/_{\sim}.$$

We define in Y:

$$\rho(\xi, \eta) := \lim_{n\to\infty} d(x_n, y_n),$$

where $(x_n) \in \xi$, $(y_n) \in \eta$. We can show that (Y, ρ) is a complete strong b-metric space, which contains X isometrically as a dense subset. The details are similar to ones presented in e.g. [12, 25, 38]. □

Remark 17.2. The existance of a completion of an arbitrary b-metric space is an important open problem.
What concerns the completion of a b-metric space, we start with the following useful information.

Let (X, d) be a generalized b-metric space. By Jung [33] one has

$$x \sim y \Leftrightarrow d(x, y) < \infty, \quad x, y \in X. \quad (17.4)$$

So (17.4) is an equivalence relations an X. Let's denote by X_k, $k \in \mathbb{N}$, the equivalence classes for the equivalence relation "$\sim$" and denote

$$d_k(x, y) = d(x, y), \quad k \in \mathbb{N}, \quad x, y \in X_k.$$

Then (X_k, d_k) is a b-metric space, $k \in \mathbb{N}$. This means that X can be uniquely decomposed into equivalence classes X_k, $k \in \mathbb{N}$, called the canonical decomposition of X.

Now we can state the following result (see the paper Jung [33] and also [12]).

Theorem 17.3. *Let (X, d) be a generalized b-metric space and X_k, $k \in \mathbb{N}$ the canonical decomposition of X. Then*

1. *The space (X, d) is complete iff (X_k, d_k) are complete for every $k \in \mathbb{N}$.*
2. *For (Y_k, d_k), $k \in \mathbb{N}$, b-metric spaces with the same $s \geqslant 1$, and $Y_i \cap Y_j = \phi$, for all $i \neq j$ in $\mathbb{N}$, one has*

$$d(x,y) := \begin{cases} d_k(x,y), & \text{if } x,y \in Y_k, \text{ for some } k \in \mathbb{N}, \\ +\infty, & \text{if } x \in Y_k \text{ and } y \in Y_i \\ & \text{for some } k,i \in \mathbb{N} \text{ with } k \neq i. \end{cases} \tag{17.5}$$

Such d is a generalized b-metric on $Y = U_{k=1}^{\infty} Y_k$ and $\{Y_k : k \in \mathbb{N}\}$ is the family of equivalence classes corresponding to the equivalence relation (17.4).

Remark 17.4. If (X, d) is a generalized strong b-metric space, the thesis of Theorem 17.3 are also true.

In [12] there is also the following lemma.

Lemma 17.5. *Let (X, d) be a generalized b-metric space, (Z, D) a complete generalized b-metric space, with continuous generalized b-metrics d, D and Y a dense subset of X. Then for every isometric embedding $f : Y \to Z$ there exists a unique isometric embedding $F : X \to Z$ such that $F/_y = f$. If, in addition, X is complete and $f(Y)$ is dense in Z, then F is bijective (i.e F is an isometry of X onto Z).*
The proof of this lemma can be found e.g. in [12].
If necessary, put Y_i by $\overline{Y_i} := Y_i x\{i\}$, D_i by $\overline{D_i}[(x,i),(y,i)] = D_i(x,y)$ for any $x, y \in Y_i$ and $\overline{T_i}(x,i) := (T_i(x), i)$, $i \in \mathbb{N}$, $x \in Y_i$. Therefore, we may suppose, without restricting the generality, that

$$Y_i \cap Y_j = \phi \text{ for all } i,j \in \mathbb{N}, \text{ with } i \neq j.$$

Now, let $Y := \underset{k}{U} Y_k$, and define

$$D : Y \times Y \to [0, \infty]$$

by (17.5)*, as well as $T : X \to Y$ by*

$$T(x) := T_i(x), \quad i \in \mathbb{N},$$

where i is the unique element of $\mathbb{N}$ such that $x \in X_i$.

Now we are in a good position to present the result on a completion of a generalized strong b-metric space. Really, one has

Theorem 17.6 (*cf.* [12])**.** *Let (X, d) be a generalized strong b-metric space, and let (Y, D) be the generalized strong b-metric space defined above. Then one has*

(xiv) *(Y, D) is a complete generalized strong b-metric space;*

(xv) *$T : (X, d) \to (Y, D)$ is an isometric embedding with $T(X)$ D-dense in Y;*

(xvi) *any other complete generalized strong b-metric space (Z, q) that contains a q-dense isometric copy of (X, d), is isometric to (Y, D).*

Proof. Since, by assumptions, (Y_i, D_i) is a complete strong b-metric space, Theorem 17.3 implies that the generalized strong b-metric space (Y, D) is complete, too.

Let $x, y \in X$. For any $x, y \in X_i$ (for some $i \in \mathbb{N}$),

$$D[T(x), T(y)] = D_i[T_i(x), T_i(y)] = d_i(x,y) = d(x,y).$$

Moreover, for $x \in X_i$, $y \in X_j$ with $i \neq j$, one has

$$T(x) = T_i(x) \in Y_i \text{ and } T(y) = T_j(y) \in Y_j,$$

so

$$D[T(x), T(y)] = D[T_i(x), T_j(y)] = +\infty = d(x, y).$$

Further on, for $\xi \in Y$, there exists a unique $i \in \mathbb{N}$ such that $\xi \in Y_i$. Since $T_i(X_i)$ is dense in (Y_i, D_i), there exists a sequence (x_n) in X_i with

$$0 = \lim_{n\to\infty} D_i[T_i(x_n), \xi] = \lim_{n\to\infty} D[T(x_n), \xi].$$

This means that $T(X)$ is D - dense in (Y, D). Now we prove (xvi). Let $S : (X, d) \to (Z, q)$ be an isometric embedding and $S(X)$ be dense in Z. Put $R : T(X) \to X$ by the formula $R(T(x)) := x$, where $x \in X$. Therefore R is an isometry of $T(X)$ onto X and SoR is an isometric embedding of $T(x)$ into Z. Since $T(X)$ is dense in Y, and $S[R(T(X))] = S(X)$ is dense in Z, Lemma 17.5 yields the existence of an isometry U of Y onto Z. This complete the proof. □

18. Ulam-Hyers stability of nearly additive functions

This section is based on the book of Prof. M. Kuczma [37], professor working in the Institute of Mathematics of the Silesian University in Katowice, Poland.

Let X be a real vector space and Y a strong b-Banach space. A function $f : X \to Y$ is called ϵ-additive ($\epsilon > 0$), iff the inequality (see Kuczma [37])

$$\|f(x + y) - f(x) - f(y)\| \leqslant \epsilon, \tag{18.1}$$

holds true for all $x, y \in X$.

The main result an the stability of ϵ-additive functions is due to D.H. Hyers, as the answer to the problem stated by S. Ulam, in 1940. We present this result for b-metric (normed) spaces, following the method used by M. Kuczma in [37].

Let's start with the following result.

Theorem 18.1. *Let X be a vector space and Y a b-normed space. Then for any additive map $f : X \to Y$,*

$$f(\lambda x) = \lambda f(x) \tag{18.2}$$

for all $\lambda \in Q$ and $x \in X$.

Proof. The proof will be in a few steps. For $x = y = 0$, we have by

$$f(x + y) = f(x) + f(y), \quad x, y \in X \tag{18.3}$$

Now, setting in (18.2) $y = -x$, one gets

$$0 = f(0) = f(x - x) = f(x) + f(-x),$$

so

$$f(-x) = -f(x), \quad x \in X.$$

Farther, we got (by induction principle)

$$f(x_1 + x_2 + \ldots + x_n) = f(x_1) + f(x_2) + \ldots + f(x_n) \tag{18.4}$$

for all $x_1 \ldots, x_n \in X$, $n \in \mathbb{N}$.

Taking $x_1 = \ldots = x_n = x$, from (18.3), we got, for all $k \in \mathbb{N}$, $x \in X$

$$f(kx) = kf(x).$$

Let $\lambda \in Q$ be given by $\lambda = \frac{k}{m}$, $k \in Z$, $m \in \mathbb{N}$

$$f(kx) = kf(x) = f(m\lambda x) = mf(\lambda x),$$

i.e.

$$f(\lambda x) = \frac{k}{m} f(x) = \lambda f(x), \quad x \in X,$$

which means that (18.2) holds true. □

Theorem 18.2. *Let X be a vector space and let* $(Y, \|\cdot\|)$ *be a strong b- Banach space, with continuous* $\|\cdot\|$. *If* ϵ *so and*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \frac{\epsilon}{s}, \quad \epsilon \geqslant 0, \quad s \geqslant 1, \tag{18.5}$$

then there exists a unique additive mapping $g : X \to Y$, *such that*

$$\|f(x) - g(x)\| \leqslant \epsilon, \quad x \in X. \tag{18.6}$$

Proof. Take $x = y$, so by (15.5)

$$\|f(2x) - 2f(x)\| \leqslant \frac{\epsilon}{s}$$

i.e.

$$\|\frac{1}{2} f(2x) - f(x)\| \leqslant \frac{\epsilon}{2s}, \quad x \in X. \tag{18.7}$$

We prove, for $n \in \mathbb{N}$ and $x \in X$

$$\|2^{-n} f(z^n x) - f(x)\| \leqslant (1 - 2^{-n})\epsilon. \tag{18.8}$$

For $n+1$, by (15.7) and induction hypothesis,

$$\begin{aligned} \|2^{-(n+1)} f(2^{n+1}x) - f(x)\| &= \|2^{-(n+1)} f(2^{n+1}x) - 2^{-n} f(2^n x) + 2^{-n} f(2^n x) - f(x)\| \\ &\leqslant s\|2^{-(n+1)} f(2^{n+1}x) - 2^{-n} f(2^n x)\| + \|2^{-n} f(z^n x) - f(x)\| \\ &\leqslant s \cdot \frac{1}{2^n} \cdot \frac{\epsilon}{2s} + (1 - 2^{-n})\epsilon = \epsilon\Big[\frac{1}{2^{n+1}} + (1 - 2^{-n})\Big] \\ &= \epsilon(1 - 2^{-(n+1)}), \end{aligned}$$

i.e. (18.8) for $n+1$.
Let

$$g_n(x) := 2^{-n} f(2^n x), \quad x \in X, \quad n \in \mathbb{N}.$$

One has by (18.8) and $x \in X$,

$$\begin{aligned} \|g_{n+k}(x) - g_n(x)\| &= \|2^{-(n+k)} f(2^{n+k}x) - 2^{-n} f(2^n x)\| \\ &\leqslant 2^{-n}\|2^{-k} f(2^{n+k}x) - f(2^n x)\| \\ &\leqslant 2^{-n}(1 - 2^{-k})\epsilon \leqslant 2^{-n}\epsilon \to 0, \text{ as } n \to \infty. \end{aligned}$$

This means that $\{g_n(x)\}$ is a Cauchy sequence for all $x \in X$. Let

$$g(x) := \lim_{n\to\infty} g_n(x), \quad x \in X,$$

then

$$2^{-k}\|f(2^k(x+y)) - f(2^k x) - f(2^k y)\| \leqslant 2^{-k} \cdot \frac{\epsilon}{s}.$$

By the continuity of $\|\cdot\|$, on letting $k \to \infty$, one gets

$$g(x+y) - g(x) - g(y) = 0, \quad x \in X,$$

which means that g is an additive function.

For the uniqueness, let assume that $G : X \to Y$ is an additive function, such that

$$\|f(x) - G(x)\| \leqslant \epsilon, \quad x \in X.$$

Thus, by Theorem 18.1,

$$\|f(nx) - nG(x)\| \leqslant \epsilon, \quad x \in X,$$

and clearly,

$$\left\|\frac{1}{n} f(nx) - G(x)\right\| \leqslant \frac{\epsilon}{n}, \quad n \in \mathbb{N}, \quad x \in X.$$

Hence

$$G(x) = \lim_{n\to\infty} \left(\frac{1}{n} f(nx)\right).$$

This, of course, proves the uniqueness of g, so the proof is complete. □

Let's note the following

Lemma 18.3 (*cf.* [37]). *Let $f : X \to Y$, where X is a real vector space and Y is a strong b-Banach space, with continuous $\|\cdot\|$. If $g : X \to Y$ is additive and*

$$\|f(x) - g(x)\| \leqslant \epsilon, \quad \epsilon \geqslant 0,$$

for all $x \in X$, then g is given by

$$g(x) = \lim_{n\to\infty} \left(\frac{1}{n} f(nx)\right), \quad x \in X.$$

Proof. One has, by Theorem 17.1

$$\|f(nx) - ng(x)\| \leqslant \epsilon \Rightarrow \left\|\frac{1}{n} f(nx) - g(x)\right\| \leqslant \frac{\epsilon}{n}$$

for all $x \in X$, and hence

$$g(x) = \lim_{n\to\infty} \left(\frac{1}{n} f(nx)\right), \quad x \in X,$$

as claimed. □

Theorem 18.4. *Let $f : \underbrace{X \times \ldots \times X}_{k} \to Y$, where X is a real vector space and Y a strong b-Banach space, satisfies for every $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$, the system of inequalities*

$$\begin{aligned} &\|f(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_k) - f(x_1, \ldots, x_k) \\ &- f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_k)\| \leqslant \epsilon_i, \end{aligned} \tag{18.9}$$

$i = 1, \ldots, k$, where $\epsilon_i \geqslant 0$, $i = 1, \ldots, k$. Then there exists a unique k-additive function $g : \underbrace{X \times \ldots \times X}_{k} \to Y$, such that

$$\|f(x_1, \ldots, x_k) - g(x_1, \ldots, x_k)\| \leqslant \epsilon = \min(\epsilon_1, \ldots, \epsilon_k), \tag{18.10}$$

for all $x_1, \ldots, x_k \in X$.

Proof. By renumeration (for more details see [2, 37]) of the variables we achieve, without violating (18.9), that

$$\min(\epsilon_1, \ldots, \epsilon_k) = \epsilon_1.$$

In view of Theorem 18.2 and Lemma 18.3 there exists a k-additive function g and

$$g(x_1, \ldots, x_k) = \lim_{n\to\infty}\left[f(nx_1, \ldots, x_k)\frac{1}{n}\right],$$

for every $x_1, \ldots, x_k \in X$, such that $\|f(x_1, \ldots, x_k) - g(x_1, \ldots, x_k)\| \leqslant \epsilon_1$. For the uniqueness of g, let $g = g_1 - g_2$, so

$$\begin{aligned}\|g(x_1, \ldots, x_k)\| &= \|g_1(x_1, \ldots, x_k) - g_2(x_1, \ldots, x_k)\| \\ &\leqslant s\Big[\|g_1(x_1, \ldots, x_k) - f(x_1, \ldots, x_k)\| + \|g_2(x_1, \ldots, x_k) - f(x_1, \ldots, x_k)\|\Big] \\ &\leqslant 2s\epsilon_1.\end{aligned}$$

This means that g is bounded by $2\xi\epsilon_1$, and by Theorem 18.1

$$g(x_1, \ldots, x_k) = 0 \Leftrightarrow g_1 = g_2,$$

as claimed.
For more details, see [37] or [2]. □

19. Ulam-Hyers stability of quadratic functions in b-metric spaces

Drygas [30] considered the following functional equation (related to the problem of characterising quasi inner product spaces):

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in \mathbb{R}, \tag{19.1}$$

which is a generalization of quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in \mathbb{R}.$$

We shall present the following results.

Lemma 19.1. *Let X be a real vector space and Y a strong b-Banach space $(Y, \|\cdot\|)$. Let $f : X \to Y$ be a quadratic function, i.e.*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in X. \tag{19.2}$$

Then

$$f(2^n x) = 4^n f(x), \quad x \in X, \quad n \in \mathbb{N}. \tag{19.3}$$

Proof. We have

$$f(0) = 0$$

and for $x = y$

$$f(2x) + f(0) = 4f(x), \quad x \in X,$$

so

$$f(2x) = 4f(x), \quad x \in X.$$

Next

$$f(2^2 x) = 4f(2x) = 4^2 f(x), \quad x \in X \tag{19.4}$$

Clearly, by induction,

$$f(2^{n+1}x) = 4f(2^n x) = 4^{n+1} f(x), \quad x \in X,$$

i.e. (19.3) for $x \in X$ and $n+1$, $n \in \mathbb{N}$. □

Theorem 19.2. *Let $\epsilon \geqslant 0$ be fixed and let X be a real vector space and Y be a strong b-Banach space $(Y, \|\cdot\|)$ with continuous $\|\cdot\|$ and $f : X \to Y$ satisfy*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \frac{\epsilon}{s} \tag{19.5}$$

with

$$\|f(0)\| = 0, \tag{19.6}$$

for all $x, y \in X$. Then there exists exactly one mapping $g : X \to Y$, satisfying (19.2) *and such that*

$$\|f(x) - g(x)\| \leqslant \frac{\epsilon}{3} \tag{19.7}$$

for all $x \in X$.

Proof. $F_0\epsilon = 0$, the proof is trivial. So let's assume that $\epsilon > 0$. From (19.5) and (19.6), one gets for $x = y$

$$\|f(2x) + f(0) - 4f(x)\| \leqslant \frac{\epsilon}{s},$$

i.e.

$$\|f(2x) - 4f(x)\| \leqslant \frac{\epsilon}{s}, \quad x \in X.$$

Therefore,

$$\left\|\frac{1}{4}f(2x) - f(x)\right\| \leqslant \frac{\epsilon}{4s}, \quad x \in X, \quad s \geqslant 1, \quad \epsilon > 0. \tag{19.8}$$

For $n = 2$, we get

$$\begin{aligned}\left\|\frac{1}{4^2}f(2^2x) - f(x)\right\| &= \left\|\frac{1}{4}\left[\frac{1}{4}f(2^2x) - f(2x)\right] + \left[\frac{1}{4}f(2x) - f(x)\right]\right\| \\ &\leqslant s \cdot \frac{1}{4}\left\|\frac{1}{4}f(2 \cdot 2x)\right\| + \left\|\frac{1}{4}f(2x) - f(x)\right\| \\ &\leqslant s \cdot \frac{1}{4} \cdot \frac{\epsilon}{4s} + \frac{\epsilon}{4s} \\ &\leqslant \frac{\epsilon}{4^2} + \frac{\epsilon}{4} = \epsilon\left(\frac{1}{4} + \frac{1}{4^2}\right),\end{aligned}$$

so

$$\left\|\frac{1}{4^2}f(2 \cdot 2x) - f(x)\right\|| \leqslant \epsilon(\frac{1}{4} + \frac{1}{4^2}). \tag{19.9}$$

Similarly,

$$\left\|\frac{1}{4^3}f(2^3x) - f(x)\right\| \leqslant \epsilon\left[\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}\right], \quad x \in X.$$

We shall verify that for $x \in X$, $n \in \mathbb{N}$

$$\left\|\frac{1}{4^n}f(2^nx) - f(x)\right\| \leqslant \epsilon\left(\frac{1}{4} + \ldots + \frac{1}{4^n}\right), \quad x \in X, \quad n \in \mathbb{N}. \tag{19.10}$$

For $n = 1$, by (19.6) and from (19.8) one gets for $x \in X$,

$$\left\|\frac{1}{4}f(2x) - f(x)\right\| \leqslant \frac{\epsilon}{4s} \leqslant \frac{\epsilon}{4}.$$

For $n+1$, we obtain by (19.6) and induction condition (19.8),

$$\begin{aligned}\left\|\frac{1}{4^{n+1}}f(2^{n+1}x)-f(x)\right\| &= \left\|\frac{1}{4^n}\left[\frac{1}{4}f(2\cdot 2^n x)-f(2^n x)\right]+\left[\frac{1}{4^n}f(2^n x)-f(x)\right]\right\| \\ &\leqslant s\cdot\frac{1}{4^n}\cdot\frac{\epsilon}{4s}+\epsilon\left(\frac{1}{4}+\ldots+\frac{1}{4^n}\right) \\ &= \epsilon\left(\frac{1}{4}+\ldots+\frac{1}{4^{n+1}}\right),\end{aligned}$$

i.e. (19.10) for $n+1$. This completes the induction proof of (19.10).
Define

$$g_n(x) := 4^{-n}f(2^n x), \quad x\in X, \quad n\in\mathbb{N}. \tag{19.11}$$

We have for $m,n\in\mathbb{N}$

$$\begin{aligned}\|g_{n+m}(x)-g_n(x)\| &= \|4^{-(n+m)}f(2^{n+m}x)-4^{-n}f(2^n x)\| \\ &= \|4^{-n}[4^{-m}f(2^{n+m}x)-f(2^n x)]\| \\ &\leqslant 4^{-n}\epsilon\left(\frac{1}{4}+\ldots+\frac{1}{4^m}\right) \\ &\leqslant 4^{-n}\cdot\epsilon\cdot\frac{1}{3}\to 0, \text{ as } n\to\infty.\end{aligned}$$

This means that $\{g_n(x)\}$ is a Cauchy sequence of functions for all $x\in X$. Let (Y-complete)

$$g(x) := \lim_{n\to\infty} g_n(x), \quad x\in X. \tag{19.12}$$

One has

$$4^{-n}\|f(2^n(x+y))+f(2^n(x-y))-2f(2^n x)-2f(2^n y)\|\leqslant\frac{\epsilon}{s}\cdot 4^{-n},$$

so

$$\|g_n(x+y)+g_n(x-y)-2g_n(x)-2g_n(y)\|\leqslant\frac{\epsilon}{4^n s},$$

whence, on letting $n\to\infty$, we get

$$g(x+y)+g(x-y)-2g(x)-2g(y)=0, \quad x\in X,$$

and consequently, g is a quadratic function. Now we prove the uniqueness part of the theorem under consideration.
By Lemma 19.1 we have for a quadratic function G, satisfying (19.7) and (19.6):

$$G(2^n x)=4^n G(x), \quad g(2^n x)=4^n g(x),$$

for all $n\in\mathbb{N}$, $x\in X$. Since, $x_0\in X$,

$$\|G(2^n x_0)-g(2^n x_0)\| = 4^n\|G(x_0)-g(x_0)\|,$$

so

$$\|G(2^n x_0)-g(2^n x_0)\|\to\infty, \text{ as } n\to\infty.$$

But on the other side, by

$$\begin{aligned}\|G(2^n x_0)-g(2^n x_0)\| &= \|G(2^n x_0)-f(2^n x_0)+f(2^n x_0)-g(2^n x_0)\| \\ &\leqslant s\|G(2^n x_0)-f(2^n x_0)\|+\|f(2^n x_0)-g(2^n x_0)\| \\ &\leqslant SM+M=(s+1)M=M_1\end{aligned}$$

which means that

$$\|G(2^n x_0)-g(2^n x_0)\|\leqslant M_1,$$

which is a contradiction. Therefore quadratic function g, under presented assumptions, is a uniquely determined.
The condition (19.7) follows from Lemma 19.1 and 19.10 directly.
The proof is complete. □

For given $x, y, t \in X$, define

$$L_{x,y,t} := \{(x+y,t), (x+y,t), (x,y+t), (x,y+y), (y,t), (t,t)\}.$$

In this section we assume (see [47]), that

$$L_{x,y,z} \subset \Omega X^2. \tag{19.13}$$

Theorem 19.3. *Let $\epsilon \geqslant 0$ be fixed. Let $f : X \to Y$ where X is a real vector space and $(Y, \|\cdot\|)$ a strong b-Banach space, satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \frac{\epsilon}{s^2}, \quad s \geqslant 1, \tag{19.14}$$

for all $(x, y) \in \Omega$. Then there exists a unique mapping $g : X \to Y$, a solution of (19.1) *such that*

$$\|f(x) - g(x)\| \leqslant \epsilon.$$

Proof. For $f : X \to Y$ satisfying the inequality (19.14) we define $D : X \times X \to Y$, as

$$D(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y), \quad (x,y) \in \Omega.$$

Therefore, for $x, y \in X$ by Theorem 3 of [47], there exists $t \in X$, such that

$$\begin{aligned}
\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &= \| -\frac{1}{2}D(x+y,t) - \frac{1}{2}D(x-y,t) + \\
&+ \frac{1}{2}D(x,y+t) + \frac{1}{2}D(x,y+t) + \frac{1}{2}D(y,t) + \frac{1}{2}D(y,t)\| \\
&\leqslant s\| -\frac{1}{2}D(x+y,t)\| + s\| -\frac{1}{2}D(x+y,t) + \frac{1}{2}D(x,y+t) + \ldots + \frac{1}{2}D(y,t)\| \\
&\leqslant s(\| -\frac{1}{2}D(x+y,t)\| + \ldots + \|\frac{1}{2}D(y,t)\|) \cdot (\frac{1}{2}) \\
&\leqslant \frac{s}{2}\left(\frac{\epsilon}{s^2} + \ldots + \frac{\epsilon}{s^2}\right) = \frac{6\epsilon}{s} \cdot \left(\frac{1}{2}\right) = \frac{3\epsilon}{s} \leqslant 3\epsilon.
\end{aligned}$$

From Theorem 19.2, there exists a unique quadratic function $g : X \to Y$, such that

$$\|f(x) - g(x)\| \leqslant \frac{\epsilon}{3}, \quad x \in X.$$

This completes the proof. □

Remark 19.4. In the paper [47] there are examples of Ω of measure zero. Also instead of a quadratic equation, the Authors consider more general functional equation - the Drygas functional equation in metric spaces.
We also get a Corollary of Theorem 19.3 (for $\epsilon = 0$).

Corollary 19.5. *Let the assumptious of Theorem* 19.3 *be satisfied. If, moreover, $\epsilon = 0$, then there exists a unique quadratic function $F : X \to Y$, such that*

$$F(x+y) + F(x-y) = 2F(x) + 2F(y),$$

for all $x, y \in X$, and

$$F(x) = f(x),$$

for all $x, y \in \Omega$.

Let

$$\begin{aligned}
f &: X \to Y, \\
D &: X \times X \to Y, \\
\alpha &: X \to X, \quad \alpha(x+y) = \alpha(x) + \alpha(y), \\
& \quad x, y \in X; \quad \alpha(\alpha(x)) = x, \quad x \in X, \\
Q_{x,y,t} &:= \{(x+y, t), (x+\alpha(y), t), (x, y+t), \\
& \quad (x, y+\alpha(t)), (y, t), (\alpha(y), \alpha(t))\}, \\
D(x, y) &:= f(x+y) + f(x+\alpha(y)) - 2f(x) - f(y) - f(\alpha(y)),
\end{aligned} \tag{19.15}$$

where X is a complex normed space, Y is a b-normed space. In this section we assume that $Q(x, y; t) \subset \Omega CX^2$.

Theorem 19.6. *Let $\epsilon \geqslant 0$ be fixed. Assume that for all $x, y \in X$, there exists $t \in X$, such that*

$$Q(x, y; t) \subset \Omega CX^2. \tag{19.16}$$

Let, moreover,

$$\|f(x+y) + f(x+\alpha(y)) - 2f(x) - f(y) - f(\alpha(y))\| \leqslant \epsilon \tag{19.17}$$

for all $(x, y) \in \Omega CX^2$.
Then

1. *for given $x, y \in X$, there exists $t \in X$, such that*

$$\begin{aligned}
&\|D(x+y, t)\| \leqslant \epsilon, \quad \|D(x+\alpha(y), t)\| \leqslant \epsilon, \quad \|D(x, y+t)\| \leqslant \epsilon, \\
&\|D(x, y+\alpha(t)\| \leqslant \epsilon, \quad \|D(y, t)\| \leqslant \epsilon, \quad \|D(\alpha(y), \alpha(t))\| \leqslant \epsilon
\end{aligned}$$

2.

$$\begin{aligned}
\|f(x+y) + f(x+\alpha(y)) - 2f(x) - f(y) - f(\alpha(y))\| &= \frac{1}{2}\| - D(x+y, t) - D(x+\alpha(y), t) + D(x, y+t) \\
&+ D(x, y+\alpha(t)) + D(y, t) + D(\alpha(y), \alpha(t))\| \\
&\leqslant \frac{1}{2}\epsilon(s + \ldots + s^6).
\end{aligned}$$

Proof. For 1): it is enough to consider the definitions of D, Q and the conditions (19.12) and (19.13).
For 2): one has

$$\begin{aligned}
\|f(x+y) + f(x+\alpha(y)) - 2f(x) - f(y) - f(\alpha(y))\| &= \frac{1}{2}\|D(x+y, t) + D(x+\alpha(y), t) - D(x, y+t) \\
& -D(x, y+\alpha(t)) - D(y, t) - D(\alpha(y), \alpha(t))\| \\
&\leqslant \frac{1}{2}s\|D(x+y, t)\| + \frac{1}{2}s\|D(x+\alpha(y), t) - D(x, y+t)) \\
& -D(x, y+\alpha(t)) - D(y, t) - D(\alpha(y), \alpha(t))\| \\
&\leqslant \frac{1}{2}\epsilon(s + s^2 + \ldots + s^6).
\end{aligned}$$

□

Remark 19.7.

a) If there exists at least two points $t \in X$, what happens with the set Q?

b) In the proof of Theorem 3 in [47], the Authors use the result of Theorem 1 concerns the function $\alpha(x) = -x$, $x \in X$.

Remark 19.8. In the paper [47] there are the examples of Ω of measure zero. Also instead of a quadratic functional equation, the Authors consider more general functional equation - the Drygas functional equation in a metric space.

20. Meir-Keeler contraction mappings in generalized b-metric spaces

This section is based on the paper [34].

In [44] Meir-Keeler considered the very general result concerning the problem of existence and uniqueness of fixed points for special type of mappings in metric spaces. Such type of results are important in applications as well. Their result reads as follows.

Theorem 20.1 (*cf.* [44]). *Let (X, q) be a complete metric space, and let $T : X \to X$ satisfies the following condition:*

(α) *given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leqslant g(x, y) < \epsilon + \delta \Rightarrow g[T(x), T(y)] < \epsilon. \tag{20.1}$$

Then T has a unige fix point $\xi \in X$. Moreover, for any $x \in X$,

$$\lim_{n\to\infty} T^n(x) = \xi,$$

where $T^n(x)$ denotes the n-th iteration of T at a point $x \in X$.

We extend this interesting result, in the frame of generalized b-metric spaces. We shall use the following lemmas.

Lemma 20.2 (*cf.* [44]). *Let (X, d) be a generalized complete b-metric space, and let $T : X \to X$ satisfy* (20.1). *Let $\{T^n(x)\}$, $x \in X$, be a Cauchy sequence. Than T has a fixed point u, and*

$$d[T^n(x), u] \to 0 \text{ as } n \to \infty,$$

for $x \in X$. Moreover, T has at most one fixed point in $Y : \{t \in X : d[T^k(x), t] < \infty\}$, $k \in \mathbb{N}$.

Proof. The proof will follow the ideas presented in the paper [34]. Assume that $x \in X$ and there exists $k \in \mathbb{N}_0$ with

$$d[T^k(x), T^{k+1}(x)] < \infty.$$

For $x \neq y$ and $d(x, y) < \infty$, we got by (20.1),

$$d[t(x), T(y)] < d(x, y). \tag{20.2}$$

Therefore, T is a continuous map, but because X is a complete space, then there exists a $u \in X$ such that

$$d[T^n(x), u] \to 0 \text{ as } n \to \infty. \tag{20.3}$$

Also by (19.2) and the triangle inequality of d, one gets

$$\begin{aligned} d[T(u), u] &\leqslant s[d(T(u), T^{n+1}(x)] + d[T^{n+1}(x), u)] \\ &\leqslant s[d(u), T^n(x)) + d(T^{n+1}(x), u)] \to 0, \text{ as } n \to \infty. \end{aligned}$$

Consequently, wo conclude that $T(u) = u$ and

$$\lim_{n\to\infty} d[T^n(x), u] = 0,$$

(see (20.3)).

In the case $T^n(x) = u$ for some $n \in \mathbb{N}_0$, the proof of these part is almost trivial (see also [20]). It remains to verify that T has at most one fixed point in Y. For the contrary, assume that there exist two distinct fixed points of T in Y; namely $T(u_i) = u_i$, $i = 1, 2$, with $u_i \in Y$, $u_1 \neq u_2$. Taking into account these assumption, let's not that $d(u_1, u_2)$ is a finite positive real number, that is

$$d(u_1, u_2) \leqslant s[d(u_1, T^k(x)) + d(T^k(x), u_2)] < \infty.$$

On account of (20.2), we derive that

$$d(u_1, u_2) = d[T(u_1), T(u_2)] < d(u_1, u_2),$$

which is a contradiction. Since $d(u_1, u_2) = 0$, that is $u_1 = u_2$, we complete the proof of the Lemma. □

Lemma 20.3 (*cf.* [44])**.** *Let* (X, d) *be a generalized complete b-metric space, and let* $T : X \to X$ *be a mapping satisfying the condition* (19.1) *and the assumption of Theorem* 20.1. *Then*

$$\lim_{n\to\infty} d[T^n(x), T^{n+1}(x)] = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \quad x \in X,$$

where $x_n = T^n(x)$ *(n-th iteration of* T *at* x*).*
The proof of this Lemma can be done exactly us in the paper [34], *so we omitt it here.*

Now we can prove the following

Theorem 20.4. *Let* (X, d) *be a generalized complete b-metric space and let* $T : X \to X$ *satisfies: given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\epsilon \leqslant d(x, y) < \epsilon + \delta \text{ implies } d[T(x), T(y)] < \frac{\epsilon}{2s}, \quad x, y \in X. \tag{20.4}$$

Let $x \in X$. *Then one of the following alternative holds:*

(A) *for every* $n \in \mathbb{N}_0$,

$$d[T^n(x), T^{n+1}(x)] = \infty,$$

(B) *there exists* $k \in \mathbb{N}_0$ *such that*

$$d[T^k(x), T^{k+1}(x)] < \infty.$$

For the case (B), we assert the followings:

(i) *the sequence* $\{T^n(x)\}$ *is a Cauchy sequence in X;*

(ii) *there exists a point* $u \in X$ *such that*

$$T(u) = u \text{ and } \lim_{n\to\infty} d[T^n(x), u] = 0;$$

(iii) *u is the unique fixed point of* T *in*

$$Y = \{t \in X : d[T^k(x), t] < \infty\};$$

(iv) *for every* $t \in Y$,

$$\lim_{n\to\infty} d[T^n(t), u] = 0.$$

Proof. Let $x \in X$ and the case (A) does not hold. So there exists $k \in \mathbb{N}_0$ with

$$d[T^k(x), T^{k+1}(x)] < \infty.$$

We can prove that (Y, d) is a complete b-metric space. To do that, please consult [12] and [33]. We verify that $T(Y)CY$, that is, $T : Y \to Y$. Really, let $z \in Y$. By (20.3), for $\xi = d[T^k(x), z]$, there exists $\delta > 0$ such that

$$\xi = d[T^k(x), z] < \xi + \delta \text{ implies } d[T^{k+1}(x), T(z)] < \frac{\epsilon}{2s}.$$

Hence, we get

$$d[T^k(x), T(z)] \leqslant s\Big[d(T^k(x), T^{k+1}(x)) + d(T^{k+1}(x), T(z))\Big] \leqslant s\Big[\epsilon_1 + \frac{\epsilon}{2s}\Big] < \infty,$$

which shows that $T(z) \in Y$, and $\{T^n(z)\} \subset Y$. We verify that $\{T^n(z)\}$ is a Cauchy sequence. Let $\epsilon > 0$. By Lemma 20.3 for $y_m = T^m(z)$,

$$\lim_{n\to\infty} d(y_m, y_{m+1}) = 0.$$

Hence, there exists $m \in \mathbb{N}_0$ such that

$$d(y_m, y_{m+1}) < \frac{\epsilon}{2s}. \tag{20.5}$$

Define: $K(y_m, \epsilon) := \{y \in X : d(y_m, y) < \epsilon\}$. Clearly $T : K \to K$. In fact, for $y \in K$, there is $\delta > 0$ with

$$\xi = d(z, y_m) < \epsilon + \delta \Rightarrow d[T(z), T(y_m)] < \frac{\epsilon}{2s}. \tag{20.6}$$

Consequently, the conditions (20.5) and (20.6) yield

$$d(T(z), y_m) \leqslant s\Big[d(T(z), T(y_m)) + d(T(y_m), y_m)\Big] \leqslant s\Big[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\Big] = \epsilon,$$

i.e. $T(z) \in K$.
Finally, for $r, k > m$, one gets

$$d(y_r, y_k) \leqslant s[d(y_r, y_m) + d(y_m, y_k)] \leqslant s[\epsilon + \epsilon] = 2s\epsilon,$$

which shows that $\{y_n\} = \{T^n(z)\}$ is a Cauchy sequence. Eventually, using Lemma 20.2 we got our assertion, and complete the proof of the theorem. □

Remark 20.5. Unfortunately, for (X, d) to be a metric space, we do not obtain the result of Meir-Keeler, that is the Theorem 20.1 from the paper [44].

We present some ideas contained (and usefull) in [34].

Definition 20.6 (*cf.* [34])**.** Let $s \geqslant 1$ be a real number. A function $\phi : [0, \infty) \to [0, \infty)$ is called a (c)-comparison function if

1) ϕ is increasing,
2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k \cdot 1}\phi^{k+1}(t) \leqslant as^k\phi^k(t) + v_k$, for $k \geqslant k_0$ and any $t \geqslant 0$.

Denote by Ψ the set of all (C)-comparison functions. We shall use the following proporties.

Lemma 20.7 (*cf.* [28, 39, 49])**.** *For an (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ the following statements hold:*

3) *the series*
$$\sum_{k=0}^{\infty} s^k\varphi^k(t)$$
converges for any $t \in [0, \infty)$;
4) *the function $b_s : [0, \infty) \to [0, \infty)$, defined by*
$$b_s(t) := \sum_{k=0}^{\infty} s^k\varphi^k(t), \quad t \in [0, \infty)$$
is increasing and continuous at 0;
5) *each iterate φ^k, $k \geqslant 1$, is also a (c)-comparison function;*
6) *φ is continuous at* 0*;*
7) *$\varphi(t) < t$ for any $t > 0$.*

Following Popescu [49] (also [34]), we present.

Definition 20.8. Let $T : X \to X$ be a mapping and let $\alpha : X \times X \to [0, \infty]$ be a function. One says that T is a generalized α-orbital admissible if

$$\alpha[x, T(x)] \geqslant 1 \Rightarrow \alpha[T(x), T^2(x)] \geqslant 1;$$

and

$$\alpha[x, T(x)] < \infty \Rightarrow \alpha[T(x), T^2(x)] < \infty.$$

see that each α-orbital admissible mapping ([49]) is a generalized α-orbital admissible.

We also have

Definition 20.9. Let T be a self-mapping defined an a generalized b-metric space $(X, d; s)$, $s \geqslant 1$. Then T is called an (α, ψ) - Meir-Keeler contraction mapping if there exist auxiliary mappings: $\alpha : X \times X \to [0, \infty]$ and $\psi \in \Psi$ such that

$$\epsilon \leqslant \psi[d(x, y)] < \epsilon + \delta \Rightarrow \alpha(x, y)d[T(x), T(y)] < \epsilon, \tag{20.7}$$

for all $x, y \in X$ and $\epsilon, \delta > 0$ fixed.

Remark 20.10. If $x \neq y$, $d(x, y) < \infty$ with $\alpha(x, y) < \infty$, then by (20.7) one gets

$$\alpha(x, y)d[T(x), T(y)] < \psi[d(x, y)]. \tag{20.8}$$

We present the main result of this part.

Theorem 20.11 (*cf.* [30])**.** *Let $(X, d; s)$, $s \geqslant 1$, be a complete generalized b-metric space. Let a self-mapping $T : X \to X$ be an (α, ψ)-Meir-Keeler type contraction. Assume that*

8) T is a generalized α-orbital admissible;

9) there exists $x \in X$ such that $1 \leqslant \alpha(x, T(x)) < \infty$;

10) T is continuous.

Then for $x \in X$ one of the following statements holds:

(A) *for every $n \in \mathbb{N}_0$*

$$d[T^n(x), T^{n+1}(x)] = \infty \text{ or } \alpha[T^n(x), T^{n+1}(x)] = \infty;$$

(B) *there exists $k \in \mathbb{N}_0$ such that $d[T^k(x), T^{k+1}(x)] < \infty$ and $\alpha[T^k(x), T^{k+1}(x)] < \infty$. Moreover, there exists $u \in X$ such that $T(u) = u$.*

Proof. By (9), $\alpha(x, T(x)) \geqslant 1$ for some $x \in X$. Assume that (A) does not hold. Therefore, there is $k \in \mathbb{N}_0$ with $d[T^k(x), T^{k+1}(x)] < \infty$ and $\alpha(T^k(x), T^{k+1}(x)) < \infty$. In the case $T^k(x) = T^{k+1}(x)$, $u = T^k(x)$ and the proof is finished. So assume that $d(T^k(x), T^{k+1}(x)) > 0$. Hence the property of ψ and Remark 20.10 yields

$$\alpha[T^k(x), T^{k+1}(x)]d[T^{k+1}(x), T^{k+2}(x)] < \psi[d(T^k(x), T^{k+1}(x)] < d(T^k(x), T^{k+1}(x)) < \infty. \tag{20.9}$$

Since T is a generalized α-orbital admissible mapping, by (9), we get that

$$1 \leqslant \alpha(x, T(x)) < \infty \Rightarrow 1 \leqslant \alpha(T(x), T^2(x)) < \infty.$$

By the induction, one obtains

$$1 \leqslant \alpha[T^{k+1}(x), T^{k+n+1}(x)] < \infty \text{ for all } n \in \mathbb{N}_0. \tag{20.10}$$

Consequently, by (20.10) and (20.9), we got

$$d[T^{k+1}(x), T^{k+2}(x)] < \psi[d(T^k(x), T^{k+1}(x)] < \infty.$$

Thus

$$d[T^{k+n}(x), T^{k+n+1}(x)] < \infty \text{ for all } n \in \mathbb{N}_0. \tag{20.11}$$

Again, on account of (20.10), (20.11) and (20.9), by induction, one gets

$$d[T^{k+n}(x), T^{k+n+1}(x)] < \psi^n[d(T^k(x), T^{k+1}(x)]. \tag{20.12}$$

Therefore, for $n, v \in \mathbb{N}_0$, in view of (20.12), we have

$$\begin{aligned} d\Big[T^{k+n}(x), T^{k+n+v}(x)\Big] &\leqslant sd\Big[T^{k+n}(x), T^{k+n+1}(x)\Big] + \ldots + s^{n-1}d\Big[T^{k+n+v-2}(x), T^{k+n+v-1}(x)\Big] \\ &\quad + s^n d\Big[T^{k+n+v-1}(x), T^{k+n+v}(x)\Big] \\ &< s\psi^n\Big[d(T^k(x), T^{k+1}(x))\Big] + \ldots + s^n\psi^{n+v-1}\Big[d(T^k(x), T^{k+1}(x))\Big] \\ &\leqslant s\sum_{m=0}^{\infty} s^m\psi^{n+m}\Big[d(T^k(x), T^{k+1}(x))\Big] \\ &\leqslant s\sum_{m=0}^{\infty} s^m\psi^{m}\Big[d(T^k(x), T^{k+1}(x))\Big]. \end{aligned}$$

i.e.

$$d[T^{k+n}(x), T^{k+n+v}(x)] \leqslant s\sum_{m=0}^{\infty} s^m\psi^m\Big[d(T^k(x), T^{k+1}(x))\Big],$$

for all $m, v \in \mathbb{N}_0$. Hence by the fact that $\psi \in \Psi$, it follows that $\{T^n(x)\}$ is a Cauchy sequence of elements of X. But X is complete, so there exists a $u \in X$ with

$$\lim_{n\to\infty} d[T^n(x), u] = 0.$$

Since T is continuous with respect to d, therefore

$$u = \lim_{n\to\infty} T^{n+1}(x) = T\Big(\lim_{n\to\infty} T^n(x)\Big) = T(u),$$

so $u \in X$ is a fixed point of T, and the proof is complete. □

Let's note the next

Definition 20.12 (*cf.* [34])**.** Let $s \geqslant 1$ be a fixed. We say that a generalized b-metric space $(X, d; s)$ is regular, if $\{x_n\}$ is a sequence in X such that $1 \leqslant \alpha(x_n, x_{n+1}) < \infty$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $1 \leqslant \alpha(x_{n(k)}, x) < \infty$ and $0 < d(x_{n(k)}, x) < \infty$ for all $k \in \mathbb{N}_0$.

Than we have

Theorem 20.13 (*cf.* [34])**.** *Let $s \geqslant 1$ be fixed and let $(X, d; s)$ be a complete generalized b-metric space. Suppose that a self-mapping $T : X \to X$ is an (α, ψ)-Meir-Keeler type contraction. If, moreover,*

(11) T is a generalized α-orbital admissible mapping;

(12) there exists $x \in X$ with $1 \leqslant \alpha(x, T(x)) < \infty$

(13) $(X, d; s)$ is regular.

Then, one of the following statements holds:

(A) *for every $n \in \mathbb{N}_0$,*

$$d[T^n(x), T^{n+1}(x)] = \infty \text{ or } \alpha[T^n(x), T^{n+1}(x)] = \infty;$$

(B) *there exists $k \in \mathbb{N}_0$ such that $d[T^k(x), T^{k+1}(x)] < \infty$ and $\alpha[T^k(x), T^{k+1}(x)] < \infty$. Moreover, there exists $u \in X$ such that*

$$T(u) = u.$$

Proof. By the definition 20.9 and condition (12), there exists a subsequence $\{x_{n(k)}\}$ of the sequence $\{T^n(x)\}$ (converging to some $u \in X$)
such that

$$1 \leqslant \alpha[T^{n(k)}(x), T(u)] < \infty$$

and

$$0 < d[T^{n(k)}(x), u] < \infty$$

for all k.
Applying (20.8) for $k \in \mathbb{N}_0$, we got

$$\begin{aligned} d[T^{n(k)+1}(x), T(u)] &= d[T(T^{n(k)}(x)), T(u)] \\ &\leqslant \alpha[T^{n(k)}(x), u] \cdot d[T(T^{n(k)}(x)), t(u)] < \psi[d(T^{n(k)}(x), u)]. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, one gots

$$d(u, T(u)) = 0 \Leftrightarrow u = T(u).$$

For the uniqueness of fixed points of an (α, ψ) - Meir-Keeler contractions mappings T, in the space $Y := \{t \in X : d[T^k(x), t] < \infty\}$, we consider the following condition.

(C) For all $x, y \in Fix(T)$, we have $1 \leqslant \alpha(x, y) < \infty$. Clearly, $Fix(T)$ means here the set of fixed points of T.

We prove the following □

Theorem 20.14 (*cf.* [34])**.** *Let the assumption of Theorem* 20.11 *(respectively* 20.13*) be satisfied. Than T has at most one fixed point in Y.*

Proof. By Theorem 20.11 (respec. Theorem 20.13) T has a fixed point $u \in X$. Assume that there exist two distinct fixed points $u_1, u_2 \in Y$ of T, that is

$$d[T^k(x), u_i] < \infty \text{ for } i = 1, 2.$$

Now, we have

$$0 < d(u_1, u_2) \leqslant s[d(u_1, T^k(x)) + d(T^k(x), u_2)] < \infty.$$

In view of (C) and 20.8,

$$\begin{aligned} d(u_1, u_2) = d[T(u_1), T(u_2)] &\leqslant \alpha(u_1, u_2) d[T(u_1), T(u_2)] \\ &< \psi[d(u_1, u_2)] < d(u_1, u_2), \end{aligned}$$

so $u_1 = u_2$. This completes the proof. □

21. Aydi's results in generalized b-metric spaces

First we consider the following result ([7]).

Theorem 21.1. *Assume that $(X, d; s)$ is a complete generalized b-metric space. Let $T : X \to X$ be a continuous and satisfies the quasi-linear condition*

$$d[T(x), T^2(x)] \leqslant \alpha d[x, T(x)] \tag{21.1}$$

for all $x \in X$, such that $d[x, T(x)] < \infty$ and

$$\alpha s = q < 1. \tag{21.2}$$

Let $x \in X$ be arbitrarily fixed. Then the following alternative holds: either

(A) *for every $n \in \mathbb{N}_0$,*

$$d[T^n(x), T^{n+1}(x)] = \infty,$$

or

(B) *there exists a* $k \in \mathbb{N}_0$, *such that*

$$d[T^k(x), T^{n+1}(x)] < \infty.$$

In (B)

(i) *the sequence* $\{T^n(x)\}$ *is a Cauchy sequence in* X;

(ii) *there exists a point* $u \in X$ *with*

$$\lim_{m\to\infty} d[T^m(x), u] = 0$$

and

$$T(u) = u.$$

Proof. By the assumption (21.1) in case B one gets

$$d[T^{k+1}(x), T^{k+2}(x)] \leqslant \alpha d[T^k(x), T^{k+1}(x)] < \infty,$$

and consequently, by induction,

$$d[T^{k+n}(x), T^{k+n+1}(x)] \leqslant \alpha^n d[T^k(x), T^{k+1}(x)], \quad n = 1, 2, \dots \tag{21.3}$$

So, for all $n, v \in \mathbb{N}_0$, in virtue of (21.3),

$$\begin{aligned} d[T^{k+n}(x), T^{k+n+v}(x)] &\leqslant sd[T^{k+n}(x), T^{k+n+1}(x)] + \dots + s^n d[T^{k+n+v-1}(x), T^{k+n+v}(x)] \\ &\leqslant s\alpha^n d[T^k(x), T^{k+1}(x)] + \dots + s^n \alpha^{n+v-1} d[T^k(x), T^{k+1}(x)] \\ &\leqslant s\alpha^n [1 + (s\alpha) + \dots + (s\alpha)^{v-1}] d[T^k(x), T^{k+1}(x)] \\ &\leqslant s\alpha^n \sum_{m=1}^{\infty} (s\alpha)^m d[T^k(x), T^{k+1}(x)] \\ &\leqslant \frac{s\alpha^n}{1 - s\alpha} d[T^k(x), T^{k+1}(x)], \end{aligned}$$

If means that

$$d[T^{k+n}(x), T^{k+n+v}(x)] \leqslant \alpha\alpha^n [1 - s\alpha]^{-1} d[T^k(x), T^{k+1}(x)], \tag{21.4}$$

for all $n, v \in \mathbb{N}_0$. This means that $\{T^n(x)\}$ is clearly a Cauchy sequence of elements of X. But X beeing complete, so there exists a $u \in X$ such that

$$\lim_{n\to\infty} d[T^n(x), u] = 0.$$

By the assumption (T is continuous with respect to d), therefore

$$u = \lim_{n\to\infty} T^{n+1}(x) = T(\lim_{n\to\infty} T^n(x)) = T(u),$$

which and the proof of theorem. □

Remark 21.2. T may have more than one fixed point.

Remark 21.3. If, additionally, d is a continuous function, we have the following estimation

$$d[T^{k+n}(x), u] \leqslant \frac{s\alpha^n}{1 - s\alpha} d[T^k(x), T^{k+1}(x)], \quad x \in X.$$

Remark 21.4. A function d may not be continuous; see [49].

Remark 21.5. Operator T, satisfying (21.1), may not be continuous. For this, consider $T : X \to X$ with

$$d[T(x), T(y)] \leqslant \frac{1}{2}\alpha\Big[d(x, T(x)) + d(y, T(y))\Big], \quad x, y \in X. \tag{21.5}$$

It is clear that if T satisfies (21.5), then T satisfies (21.1) as well.

Remark 21.6. Theorem 20.1 generalizes other theorems (Diaz - Margolis [28], Luxemberg [39], Banach [10] and others: ([1, 5, 6, 8]). Now we state another result contained in [7] (for nonlinear contractions).

Theorem 21.7. *Let $(X, d; s)$ be a complete generalized b-metric space and let $T : X \to X$ satisfies the condition*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)], \tag{21.6}$$

for all $x, y \in X$ such that $d(x, y) < \infty$, where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing and

$$\lim_{n\to\infty} \varphi^n(z) = 0, \quad z > 0. \tag{21.7}$$

Let $x \in X$ be arbitrarily fixed. Then the following alternative holds: either

(C) *for every $n \in \mathbb{N}_0$,*

$$d[T^n(x), T^{n+1}(x)] = \infty,$$

or

(D) *there exists an $k \in \mathbb{N}_0$, such that*

$$d[T^k(x), T^{k+1}(x)] < \infty.$$

In the case (D)

(iii) *the sequence $\{T^n(x)\}$ is a Cauchy sequence in X;*

(iv) *there exists a point $u \in X$, such that*

$$\lim_{n\to\infty} d[T^n(x), u] = 0, \text{ and } T(u) = u.$$

(v) *u is the unique fixed point of T in*

$$B := \{t \in X : d(T^k(x), t) < \infty\};$$

(vi) *for all $t \in B$,*

$$\lim_{n\to\infty} d[T^n(t), u] = 0$$

If moreover, d is continuous (with respect to one variable) and

$$\sum_{n=1}^{\infty} s^n \varphi^n(t) < \infty, \text{ for } t > 0,$$

then

$$d[T^m(t), u] \leqslant \sum_{n=0}^{\infty} s^{n+t} \varphi^{m+n}[d(t, T(t))], \quad m \in \mathbb{N}_0, \text{ for } t \in B. \tag{21.8}$$

Proof.

1. Let's take $x \in X$, $\epsilon > 0$. Consider $n \in \mathbb{N}$ such that

$$\varphi^n(\epsilon) < \frac{\epsilon}{2s},$$

Put

$$F = T^n, \quad \alpha = \varphi^n, \quad x_m = F^m(x), \text{ for } m \in \mathbb{N}.$$

Than, for all $x, y \in X$ with $d(x, y) < \infty$, one gets

$$d[F(x), F(y)] \leqslant \varphi^n[d(x, y)] = \alpha[d(x, y)]. \tag{21.9}$$

2. One can prove that (B, d) is a complete b-metric space. Clearly, $T^k(x), T^{k+1}(x) \in B$.

3. Note that $T : B \to B$. For it, if $t \in B$, i.e. $B(T^k(x), t) < \infty$, then

$$\begin{aligned} d[T^k(x), T(t)] &\leqslant s\Big[d(T^k(x), T^{k+1}(x)) + d(t^{k+1}(x), T(t))\Big] \\ &\leqslant s\Big[\epsilon_1 + \varphi[d(T^k(x), t)]\Big] \\ &\leqslant s[\epsilon_1 + \epsilon_2] < \infty, \end{aligned}$$

where ϵ_1, ϵ_2 are some positive numbers, and hence $F : B \to B$.

4. Fix $t \in B$, so $\{F^m(t)\} \subset B$ for all $m \in \mathbb{N}_0$. Now we show that $\{F^m(t)\}$ is a Cauchy sequence. Really, put $y_m = F^m(t)$, $m \in \mathbb{N}_0$, so

$$d[F(x), F^2(x)] \leqslant \alpha[d(t), F(t)].$$

By induction,

$$d[F^m(t), F^{m+1}(t)] \leqslant \alpha^m[d(t), F(t)],$$

i.e.

$$d(y_m, y_{m+1}) \leqslant \alpha^m[d(t, F(t))],$$

whence

$$d(y_m, y_{m+1}) \to 0, \text{ as } m \to \infty.$$

Take m such that

$$d(y_m, y_{m+1}) < \frac{\epsilon}{2s}.$$

For every $z \in K(y_m, \epsilon) := \{y \in X : d(y_m, y) \leqslant \epsilon\}$, we got

$$d[F(z), F(y_m)] \leqslant \alpha[d(z, y_m)] \leqslant \alpha(\epsilon) = \varphi^n(\epsilon) < \frac{\epsilon}{2s},$$

and

$$d[F(y_m), y_m] < \frac{\epsilon}{2s}.$$

Therefore,

$$\begin{aligned} d[F(z), y_m] &\leqslant s\Big[d(F(z), F(y_m)) + d(F(y_m), y_m)\Big] \\ &\leqslant s[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}] = \epsilon, \end{aligned}$$

that is F maps $K(y_m, \epsilon)$ into itself. Therefore,

$$d(y_r, y_l) \leqslant 2s\epsilon, \quad r, l \geqslant m,$$

so the sequence $\{y_r\} = \{F^r(t)\}$ is a Cauchy sequence.

5. Since B is complete, there exists $u \in BCX$ with

$$y_r \to u \text{ as } r \to \infty.$$

Furthermore, by the continuity of F (see (21.9))

$$F(u) = \lim_{r\to\infty} F(y_r) = \lim_{n\to\infty} y_{r+1} = u,$$

which means that u is a fixed point of F. Taking into consideration the fact, that $\alpha(t) = \varphi^n(t) < t$ for all $t > 0$, it is cleare that F has only one fixed point in the space B.
Also, by the assumption (21.5), T is a continous in B, which leads to

$$T(u) = \lim_{r\to\infty}(T(F^r(t))) = \lim_{r\to\infty} F^r(T(t)) = u,$$

whence we infor that u is a fixed point of T as well. Clearly, by (21.5), such point is only one in the space B. Eventually, since for every $t \in B$ and every $r = 0, 1, \ldots, n-1$,

$$T^m(x) = T^{nl+r}(t) = F^l[T^r(t)] \to u, \text{ as } l \to \infty,$$

so

$$d[T^m(t), u] \to 0, \text{ as } m \to \infty,$$

for all $t \in B$, which shows (vi).

6. Next, for any $t \in B$ and $m, n \in \mathbb{N}_0$,

$$\begin{aligned} d[T^m(t), T^{m+n}(t)] &\leqslant s\Big[d[T^m(t), T^{m+n}(t)] + d[T^{m+n}(t), T^{m+n}(t)]\Big] \\ &\leqslant sd[T^m(t), T^{m+1}(t)] + \ldots + s^n d[T^{m+n-1}(t), T^{m+n}(t)] \\ &\leqslant s\varphi^m[d(t), T(t)] + \ldots + s^n\varphi^{m+n-1}[d(t), T(t)] \\ &\leqslant \sum_{r=0}^{\infty} s^{r+1}\varphi^{m+r}[d(t, T(t)]. \end{aligned}$$

Therefore,

$$d[T^m(t), T^{m+n}(t)] \leqslant \sum_{r=0}^{\infty} s^{r+1}\varphi^{m+r}[d(t, T(t)].$$

Consequently, if $n \to \infty$ and d is continuous (for one variable only), so for $t \in B$ and $m \in \mathbb{N}_0$, one gets

$$d[T^m(t), u] \leqslant \sum_{r=0}^{\infty} s^{r+1}\varphi^{m+r}[d(t, T(t))],$$

which concludes the proof of the theorem.

□

Please, note also the following. For X as a b-metric, so $B = X$ and one gets from Theorem 21.7:

Corollary 21.8 (*cf.* [7])**.** *Let (X, d) be a complete b-metric space and let $T : X \to X$ satisfy*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)], \quad x, y \in X,$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing function, such that

$$\lim_{n\to\infty} \varphi^n(t) = 0, \quad t > 0.$$

Then T has exactly one fixed point $u \in X$, and

$$\lim_{n\to\infty} d[T^n(x), u] = 0$$

for each $x \in X$.
If, moreover, d is continuous (with respect to one variable) and the series of iterates

$$\sum_{k=1}^{\infty} s^k\varphi^k(t) < \infty \textit{ for } t > 0,$$

Then for all $z \in X$ and all $m \in \mathbb{N}_0$,

$$d[T^m(z), u] \leqslant \sum_{k=0}^{\infty} s^{k+t}\varphi^{m+k}[d(z, T(z))]. \tag{21.10}$$

Remark 21.9 (*cf.* [7]). It is interesting to consider a converqense of a series of iterates

$$\sum_{n=1}^{\infty} \varphi^n(t).$$

For example, one of the sufficient conditions is the following

$$\lim_{n\to\infty} \sup \frac{\varphi^{n+1}(t)}{\varphi^n(t)} = g(t) < 1, \quad t > 0. \tag{21.11}$$

Let's also note another one such conditions:

$$\lim_{n\to\infty} \inf \Big[-\frac{\ln \varphi^n(t)}{\ln n} \Big] = \alpha(t) > 1, \quad t > 0. \tag{21.12}$$

More details the Reader may find in [23].

Now we shall present two results for local transformations in generalized b-metric spaces (see also [29]).

Theorem 21.10. *Let $(X, d; s)$ be a complete generalized b-metric space, with $s \geqslant 1$. Let $\varphi : \langle 0, \infty) \to \langle 0, \infty)$ be increasing in $(0, \infty)$ and let $\varphi(t) < \frac{t}{s}$ for $t > 0$. Let $T : X \to X$ and let there exists a constant $c > 0$, such that*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)] \, for \; x, y \in X, \quad d(x, y) < C. \tag{21.13}$$

Let $x \in X$. Then the following alternative holds: either

(A) *$d[T^n(x), T^{n+1}(x)] \geqslant C$ for all $n \in \mathbb{N}_0$*
or

(B) *there exists a $k \in \mathbb{N}_0$ such that*

$$d[T^k(x), T^{k+1}(x)] < C.$$

In (B)

- **(i)** *the sequence $\{T^n(x)\}$ is a Cauchy sequence in X;*
- **(ii)** *there exists a point $u \in X$, such that $T(u) = u$ and*

$$\lim_{n\to\infty} d[T^n(x), u] = 0.$$

Proof. Take $x \in X$. In (B), in view of (21.13) one has

$$d[T^{k+1}(x), T^{k+2}(x)] \leqslant \varphi[d(T^k(x), T^{k+1}(x))],$$

and by the induction principle

$$d[T^{k+n}(x), T^{k+n+1}(x)] \leqslant \varphi^n[d(T^k(x), T^{k+1}(x)], \quad n \in \mathbb{N}. \tag{21.14}$$

Using the properties on φ, we got for $t > 0$

$$\varphi^2(t) \leqslant \varphi\left(\frac{t}{s}\right) \leqslant \frac{t}{s^2},$$

and similarly,

$$\varphi^n(t) \leqslant \frac{t}{s^n}, \quad n \in \mathbb{N},$$

so hence,

$$\lim_{n\to\infty} \varphi^n(t) = 0, \text{ for } t > 0.$$

For $x_n = T^n(x)$, $n \in \mathbb{N}_0$, by (21.14),

$$\lim_{n\to\infty} d(x_{k+m}, x_{k+n+1}) = 0.$$

Let $0 < \epsilon < C$ be given. There exists an $n \in \mathbb{N}_0$ with

$$d(x_{k+n}, x_{k+n+1}) < \frac{\epsilon}{s} - \varphi(\epsilon). \tag{21.15}$$

Define

$$B(T^{k+n}(x), \epsilon) := \{y \in X : d(T^{n+k}(x), y) < \epsilon\}.$$

One guess that $T : B \to B$. In fact, for $y \in B$, by the condition (21.14)

$$\begin{aligned} d[T^{k+n}(x), T(y)] &\leqslant s[d(T(u), T^{k+n+1}(x)) + d(T^{k+n+1}(x), T^{k+n}(x))] \\ &\leqslant s\Big[\varphi[d(y, T^{k+n}(x)) + d(x_{k+n+1}, x_{k+n})]\Big] \\ &< s[\varphi(\epsilon) + \frac{\epsilon}{s} - \varphi(\epsilon)] = \epsilon, \end{aligned}$$

i.e. $T(y) \in B$. Clearly, $T^m(x) \in B$ for $m \geqslant k + n$.
Let's verify now that $\{T^m(x)\}$ is a Cauchy sequence. For if $m, l \geqslant k + n$, then

$$\begin{aligned} d[T^m(x), T^l(x)] &\leqslant s\Big[d(T^m(x), T^{k+n}(x)) + d(T^{k+n}(x), T^l(x))\Big] \\ &\leqslant s[\epsilon + \epsilon] = 2s\epsilon. \end{aligned}$$

Since the last inequality holds for arbitrary $\epsilon > 0$ and X is complete, it means that there exists $u \in X$ such that

$$\lim_{n\to\infty} d[T^m(x), u] = 0.$$

Cosequently, for m sufficiently large, we have

$$d[T^{m+1}(x), T(u)] \leqslant \varphi[d(T^m(x)), u] < d[T^m(x), u]$$

which means that $T^m(x) \to T(u)$, as $m \to \infty$, and by the uniqueness of the limit point, we get

$$T(u) = u.$$

Therefore, this completes the proof. □

Let's note the following remarks

Remark 21.11 (*cf.* [29]). If $\epsilon < \frac{c}{2}$, in B the map T may have at most one fixed point. In fact, if $T(u_i) = u_i$, $i = 1, 2$, $d(u_1, u_2) > 0$, $u_1, u_2 \in B$, then

$$d(u_1, u_2) = d[T(u_1), T(u_2)] \leqslant \varphi[d(u_1, u_2)] < d(u_1, u_2),$$

i.e. $d(u_1, u_2) = 0$, what is impossible.

Remark 21.12 (*cf.* [29]). If X is a metric space, i.e. $s = 1$, in Theorem 21.10, we can assume only that $\varphi : \langle 0, \infty) \to \langle 0, \infty)$ is increasing and

$$\lim_{n\to\infty} \varphi^n(t) = 0, t > 0.$$

The proof (in such case) runs very similarly. By an increasing function, we understand here a nondecreasing one. We have also

Theorem 21.13 (*cf.* [29]). *Let $(X, d; s)$ be a complete generalized b-metric space. Let $T : X \to X$ be continuous and satisfies the conditions:*
there exists a constant $c > 0$ such that

$$d[T(x), T^2(x)] \leqslant \alpha d(x, T(x)) \, \textit{for} \; d(x, T(x)) < C, \quad x \in X, \tag{21.16}$$

and

$$\alpha s = q < 1. \tag{21.17}$$

Let $x \in X$. Then the following alternative holds: either

(C) $d[T^n(x), T^{n+1}(x)] \geqslant C$ *for* $n \in \mathbb{N}_0$
or

(D) *there exists a* $k \in \mathbb{N}_0$ *such that*

$$d[T^k(x), T^{k+1}(x)] < C.$$

In (D)

(iii) *the sequence* $\{T^n(x)\}$ *is a Cauchy sequence;*

(iv) *there exists a point* $u \in X$ *such that* $T(u) = u$*, and*

$$\lim_{n\to\infty} d[T^n(x), u] = 0.$$

Proof. Let $x \in X$ and assume that the case (C) does not hold. So there exists $k \in \mathbb{N}_0$ such that,

$$d[T^k(x), T^{k+1}(x)] < C.$$

So we verify, by (21.15),

$$d[T^n(x), T^{n+1}(x)] < C \text{ for } n \geqslant k.$$

and

$$d[T^{k+n}(x), T^{k+n+1}(x)] \leqslant \alpha^n d[T^k(x), T^{k+1}(x)], \tag{21.18}$$

for $n \in \mathbb{N}_0$. Thus, for $n, v \in \mathbb{N}_0$, one gets (applying (21.18) and assumptions)

$$\begin{aligned} d[T^{k+n}(x), T^{k+n+v}(x)] &\leqslant s\Big[d(T^{k+n}(x), T^{k+n+1}(x)) + d(T^{k+n+1}(x), T^{k+n+v}(x))\Big] \\ &\leqslant sd(T^{k+n}(x), T^{k+n+1}(x)) + \ldots + s^n d(T^{k+n+v-1}(x), T^{k+n+v}(x)) \\ &\leqslant s\alpha^n d[T^k(x), T^{k+1}(x)] + \ldots + s^n \alpha^{n+v-1} d[T^k(x), T^{k+1}(x)] \\ &\leqslant s\alpha^n \sum_{n=0}^{\infty} (s\alpha)^n d[T^k(x), T^{k+1}(x)] \\ &\leqslant s\alpha^n (1 - s\alpha)^{-1} d[T^k(x), T^{k+1}(x)]. \end{aligned}$$

Therefore, we get the estimation

$$d[T^{k+n}(x), T^{k+n+v}(x)] \leqslant s\alpha^n (1 - s\alpha)^{-1} d[T^k(x), T^{k+1}(x)], \tag{21.19}$$

for $n, v \in \mathbb{N}_0$.
It means that $\{T^n(x)\}$ is a Cauchy sequence but in view of the assumption that X is a generalized complete b-metric space, $T^n(x) \to u \in X$ as $n \to \infty$:

$$\lim_{n\to\infty} d(T^n(x), u) = 0.$$

By the continuity of T with respect to d, we have

$$u = \lim_{n\to\infty} T^{n+1}(x) = T\Big(\lim_{n\to\infty} T^n(x)\Big) = T(u),$$

i.e. $T(u) = u$ and the proof is complete. □

What concerns b-metric spaces, one has

Theorem 21.14 (*cf.* [29])**.** *Let* $(X, d; s)$*,* $s > 1$*, be a complete b-metric space and let* $T : X \to X$ *be continuous. Let* $x_0 \in X$ *and*

$$B := B(T^k(x_0), r) = \{x \in X : d(T^k(x_0), x) < r\}, \quad r > 0. \tag{21.20}$$

and

$$d[T^k(x_0), T^{k+1}(x_0)] < \frac{r}{s} - \varphi r.$$

where $r > 0$ and $k \in \mathbb{N}_0$, and φ is a function as in Theorem 21.10*. Let T satisfies*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)], \quad x, y \in B. \tag{21.21}$$

Then for any $x \in B$ there exists a $u \in X$ such that

(v) *$T(u) = u$ and $\lim_{n\to\infty} d[T^n(x), u] = 0$;*

(vi) *T has in B at most one fixed point;*

(vii) *$u \in K[T^k(x_0), sr] := \Big\{t \in X : d[T^k(x_0), t] \leqslant sr\Big\}$.*

Proof. We show that $T : B \to B$. For if $x \in B$, then by (21.19), (21.20),

$$\begin{aligned} d[T^k(x_0), T(x)] &\leqslant s\Big[d(T^k(x_0), T^{k+1}(x_0)) + d(T^{k+1}(x_0), T(x))\Big] \\ &\leqslant s\Big[\frac{r}{s} - \varphi(x) + \varphi(r)\Big] = r. \end{aligned}$$

We can also (as in the proof of Theorem 21.10) verify that for every $x \in B$, the sequence $\{T^n(x)\}$ is a Cauchy sequence. Therefore,

$$\lim_{n\to\infty} d[T^n(x), u] = 0, \text{ for } u \in X.$$

Since T is continuous, for $x \in B$,

$$u = \lim_{n\to\infty} T^{n+1}(x) = T\Big(\lim_{n\to\infty} T^n(x)\Big) = T(u),$$

i.e. $T(u) = u$, $u \in B$.
In the sequel, to check up (vi), assume that $u_i \in B$, $T(u_i) = u_i$, $i = 1, 2$, $u_1 \neq u_2$. So one has

$$d(u_1, u_2) = d[T(u_1), T(u_2)] \leqslant \varphi[d(u_1, u_2)] < d(u_1, u_2),$$

what is impossible.
Eventually, for the sequence $\{T^n(x)\}$, $x \in B$, $T^n(x) \to u$, as $n \to \infty$, for $\epsilon > 0$ and n sufficiently large,

$$\begin{aligned} d[u, T^k(x_0)] &\leqslant s[d(u, T^n(x)) + d(T^n(x), T^k(x_0))] \\ &\leqslant s[\epsilon + r] \to sr, \text{ as } \epsilon \to 0. \end{aligned}$$

Hence

$$d(u, T^k(x_0)) \leqslant sr,$$

so $u \in K[T^k(x_0), sr]$, which complete the proof. □

Remark 21.15. The idea of considering for a fixed points of a map defined in a generalized b-metric comes from Luxemberg [39] (see also [28]).

22. Luxemberg's results on differential equations

Luxemberg observed the successive approximations for ordinary differential equations and noticed that the uniqueness of solutions depends an a metric considered. So we proposed the following generalization.

Let X be an abstract set, and let $d : X \times X \to [0, \infty]$ be defined and satisfy the following conditions:

(1) $d(x, y) = 0$ iff $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, y) \leqslant d(x, z) + d(z, y)$; $x, y, z \in X$

(4) every d - Cauchy sequence in X is d - converdent, i.e. $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ for a sequence $x_n \in X$, $n \in \mathbb{N}$ implies the existence of an element $x \in X$ with $\lim_{n\to\infty} d(x, x_n) = 0$ (x is unique by (01) and (03)).

Such space X with d satisfying the conditions (01) - (04) is called a generalized complete metric space.

The difference beetwen the usual concept of a complete metric space is the fact that not every two points in X have necessarily a finite distance.
In this context, he considers the map $T : X \to X$ satisfying the following conditions:

(C1) there exist a constant $0 < g < 1$ such that

$$d[T(x), T(y)] \leqslant qd(x, y),$$

for all $x, y \in X$ with $d(x, y) < \infty$;

(C2) for every sequence of successive approximations $x_n := T(x_{n-1})$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element belonging to X, there exists an index $N(x_0)$ such that $d(x_N, x_{N+k}) < \infty$ for all $k = 1, 2, \ldots$.

(C3) If x and y, $x, y \in X$, are fix points of T, i.e. $T(x) = x$, $T(y) = y$, then $d(x, y) < \infty$.

Remark 22.1 (*cf.* [36]). Luxemberg presents the example showing that if (C3) is not satisified, then T may not have only one fix point. The important is the assumption that d is a generalized metric.
H6 [36] presents the following theorem.

Theorem 22.2. *Let the assumptions (C1)-(C3) be satisfied. The equation $T(x) = x$ has one and only one solution, and every sequence of successive approximations $x_n = T(x_{x-1})$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, is convergent in distance to this unique solution.*
Very general result concernig the existance of fixed points in a spaces with generalized metric are presented by Diaz and Margolis in [28].

23. Some generalizations of Banach principle

Theorem 23.1 (*cf.* [42])**.** *Let $(X, d; s)$ be a complete b-metric space and $T : X \to X$ a map. Assume*

(a) *for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $d(x, T(x)) < \delta(\epsilon)$, then $T[B(x, \epsilon)] \subset B(x, \epsilon)$, $(x \in X)$.*

Then, if $d(T^n(x), T^{n+1}(x)) \to 0$ for $x \in X$, as $n \to \infty$, the sequence $\{T^n(x)\}$ is convergent to a unique fixed point for $T \in X$.

Proof. Let $T^n x = x_n$, $x \in X$, $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence. For a given $\epsilon > 0$, take n_0 such that $d(x_n, x_{n+1}) < \delta(\epsilon)$, where $\delta(\epsilon)$ is as in (a), for $n \geqslant n_0$. By (a) one has

$$T[B(x_{n_0}, \epsilon)] \subset B(x_{n_0}, \epsilon) = \{x \in X : d(x_{n_0}, x) < \epsilon\}.$$

By the induction principle,

$$T^n x_{n_0} = x_{n+n_0} \in B(x_{n_0}, \epsilon) \text{ for } n \geqslant 1, \quad n \in \mathbb{N}_0.$$

Thus

$$d(x_k, x_l) \leqslant s[d(x_k, x_{n_0}) + d(x_{n_0}, x_l)] \leqslant 2s\epsilon,$$

for all $k, l \geqslant n_0$, so $\{x_k\}$ is a Cauchy sequence and $x_n \to u \in X$ as $n \to \infty$ with respect to d. Now we show that $u = T(u)$. Assume (for the Contrary) that

$$d(u, T(u)) = a > 0.$$

Take $x_n \in B[u, \frac{a}{3s}]$, such that

$$d(x_n, x_{n+1}) < \delta(\frac{a}{3s}).$$

By (a) let $T[B(x_n, \frac{a}{3s})] \subset B(x_n, \frac{a}{3s})$. Therefore, we have

$$Tu \in B(x_n, \frac{a}{3s}), \quad u \in B(x_n, \frac{a}{3s}).$$

But

$$a = d(T(u), u) \leqslant s\big[d(T(u), x_n) + d(x_n, u)\big] \leqslant s\Big[\frac{a}{3s} + \frac{a}{3s}\Big] = s2\frac{a}{3s} = \frac{2}{3}a,$$

which is impossible and hence

$$u = T(u), \text{ and } d[u, T^n(r)] \to 0, \quad n \to \infty, \quad x \in X.$$

For see also [4, 22, 42].
Using Theorem 23.1 we can get simpler proofs of many theorems (see remark 4.5 in [22]).
Define the class of function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with

(a) φ is nondecreasing,

(b)

$$\lim_{n\to\infty} \varphi^n t = 0, \quad t > 0, \tag{23.1}$$

(c) $\varphi(t) < \frac{t}{s}$; $t > 0$.

□

Theorem 23.2 (*cf.* [42])**.** *Let $(X, d; s)$ be a complete b-metric space and let*

$$d[T(x), T(y)] \leqslant \varphi[d(x, y)], \quad x, y \in X, \quad d(x, y) > 0, \tag{23.2}$$

where φ is as in (23.1)*. Then T has exactly one fixed point $u \in X$ and*

$$d[T^n(x), u] \to 0, \text{ as } n \to \infty,$$

for each $x \in X$.

Proof. We apply Theorem 23.1. Let $\epsilon > 0$ be given. Take

$$\delta(\epsilon) = \frac{\epsilon}{s} - \varphi(\epsilon) > 0.$$

For $z \in B(x, \epsilon)$ and properties of φ, one has

$$\begin{aligned} d(x, T(x)) &< \delta(\epsilon), \\ d[T(z), x] &\leqslant s\big[d(T(z), T(x)) + d(T(x), x)\big] \\ &\leqslant s\big[\varphi(d(z, x)) + d(T(x), x)\big] \\ &\leqslant s\big[\varphi(\epsilon) + \frac{\epsilon}{s} - \varphi(\epsilon)\big] = \epsilon \end{aligned}$$

and $T(z) \in B(x, \epsilon)$.
So by Theorem 23.1 we get our thesis, and the proof is complete ($d[T^n(x), T^{n+1}(x)] \to 0$, as $n \to \infty$ for all $x \in X$, is obvious by (23.1), (23.2)). □

Remark 23.3. For another proof, see [4] (for metric spaces, of course).

24. Domain invariance for contractive fields

Theorem 24.1. *Let*

1. (Y, d) *b-metric complete,* $d(x, \cdot)$ *continuous*
2. $B = B(y_0, r) = \{y \in Y : d(y, y_0) < r\} \subset Y$, $y_0 \in Y$,
3. $F : B \to Y$ *such that*

$$d[F(x), F(y)] \leqslant \alpha d(x, y), \quad x, y \in B, \quad \alpha \leqslant \alpha s < 1, \quad s \geqslant 1,$$

4. $d[F(y_0), y_0] < r(\frac{1}{s} - \alpha)$, $\alpha s < 1$.
   Then
5. F *has a fixed point* $u_0 \in \overline{D} C B C Y$ *(*u_0 *only in* B*).*

Proof.

6. $\epsilon < r$ s.t. $d[F(y_0), y_0] \leqslant \epsilon(\frac{1}{s} - \alpha) r(\frac{1}{s} - \alpha)$, $\epsilon < r$.
7. $\overline{D} = \{y \in Y : d(y, y_0) \leqslant \epsilon\}$, $y_0 \in Y$,
8. $F : \overline{D} \to \overline{D}$:

$$\begin{aligned} y \in \overline{D}, x_0 \quad d[F(y), y_0] &\leqslant s\Big[d(F(y), F(y_0)) + d(F(y_0), y_0)\Big] \\ &\leqslant s[\alpha d(y, y_0) + d(F(y_0), y_0)] \\ &\leqslant s[\alpha\epsilon + \frac{\epsilon}{s} - \epsilon\alpha] = s \cdot \frac{\epsilon}{s} = \epsilon. \end{aligned}$$

9. Cobzas + Czerwik [12] $\Rightarrow$

   a) $\exists, g \leqslant d^p \leqslant 4g$, $(2s)^p = 2$, $0 < p \leqslant 1$,
   b) g - metric complete with $\alpha^p < 1$; $F : \overline{D} \to \overline{D}$
   c) $\overline{D}$ - complete, b-metric
   d) $g[d(F(x), F(y)] \leqslant \alpha^p p(x, y)$, $x, y \in \overline{D}$, $F : \overline{D} \to \overline{D}$.

   So by Banach ts $\Rightarrow$ F has a fixed point $u_0 \in \overline{D} C Y$, $F(u_0) = u_0$.

□

Theorem 24.2. *Let*

1. *E b - Banach space, UCE*
2. $F : U \to E$ *and*
$$\|F(x) - F(y)\| \leqslant \alpha \|x - y\|, \quad x, y \in U, \quad \alpha s < 1, \quad s \geqslant 1,$$
3. $|F(y_0) - y_0| < r(\frac{1}{s} - \alpha), \quad r > 0, \quad \alpha s < 1,$
4. $f(x) = x - F(X), \quad f : U \to E, \quad x \in U.$
   Then
5. $f : U \to E$ *open mapping,* $f(U)$ *Open,*
6. $f : U \to f(u)$ *homeomorphism.*

Proof. We show that:

7. Let $u \in U$, $B(u,r) \in U$, than $B[f(a), r(\frac{1}{s} - \alpha)] \subset f[B(u,r)]$: see also [42], Theorem 2.1)
   Let $y_0 \in B[f(u), r(\frac{1}{s} - \alpha)]$ and $G : B \to Y$ by $G(y) = y_0 + F(y)$, $y \in B$
   Then

8. $\|G(y_1) - G(y_2)\| \leqslant |y_0 + F(y_1) - (y_0 + F(y_2))| = |F(y_1) - F(y_2)| \leqslant \alpha|y_1 - y_2|, \quad y_1, y_2 \in B$
   and

9. $\|G(u) - u\| = |y_0 + F(u) - u| = |y_0 - f(u)| < r(\frac{1}{s} - \alpha)$
   So by Theorem 24.1, by 8), 9) $\Rightarrow$

$$\exists u_0 \in B(u,r) \text{ s.t. } u_0 = G(u_0) \Rightarrow u_0 = y_0 + F(u_0) \Rightarrow y_0 = f(u_0) \Rightarrow \tag{24.1}$$

   = 7) and proofs 5).

Thus f open and since continuous (by assumption) $\Rightarrow$

a) f open mapping,

b) $f(u)$ open, ϵ 5)

but since c) f is onto $\Rightarrow$ f homeomorphism) i.e. 6). □

Theorem 24.3. *Let E be a b-Banach space and let $F : E \to E$ be contractive, i.e.*

$$\|F(x) - F(y)\| \leqslant \alpha|x - y|, \quad x, y \in E, \quad \alpha \leqslant \alpha s < 1. \tag{24.2}$$

Then the corresponding field $f = I - F$, $f(x) = x - F(x)$, $x \in E$ is a homeomorphism of E onto itself.

Proof. In view of Theorem 24.2 we show only that $f(E) = E$. For $y_0 \in E$ take

$$G : E \to E \text{ by } G(x) = y_0 + F(x), \quad x \in E.$$

So G is contractive:

$$|G(x_1) - G(x_2)| = |y_0 + F(x_1) - y_0 - F(x_2)| = |F(x_1) - F(x_2)| \leqslant \alpha|x_1 - x_2|,$$

for $x_1, x_2 \in E$.
By Theorem 4.6 of [22], G has a fixed point $u_0 = y_0 + F(u_0)$, $u_0 \in E$, that is $y_0 = f(u_0)$, so $u_0 \in E \leftrightarrow f(u_1) = y_0 \in f(E)$, and the proof is complete. □

Corollary 24.4. *If $(X, d; s)$ is a b-normed space, than*

$$|x - y| \geqslant \frac{1}{s}|x| - |y|, \quad x, y \in X. \tag{24.3}$$

Proof.

$$|x| = |y + (x - y)| \leqslant s|y| + s|y - x|,$$

i.e.

$$s|x - y| \geqslant |x| - s|y|,$$
$$|x - y| \geqslant \frac{1}{s}|x| - |y|.$$

□

Corollary 24.5. *If* $(X, |\cdot|; s)$ *is a b-normed space and*

$$G : X \to X, \quad G(x) = x - F(x),$$

and $F : X \to X$ *is contractive with* $0 \leqslant \alpha < 1$ *i.e. for* $x, y \in X$

$$|F(x) - F(y)| \leqslant \alpha|x - y|, \tag{24.4}$$

then

$$|G(x) - G(y)| \geqslant (\frac{1}{s} - \alpha)|x - y|, \quad x, y \in X. \tag{24.5}$$

Proof. By Corollary 24.4, one has for $x, y \in X$,

$$\|G(x) - G(y)\| = |x - F(x) - (y - F(y))| =$$

$$\|(x - y) + (F(y) - F(x))\| \geqslant \frac{1}{s}|x - y| - |F(x) - F(y)|$$

$$\geqslant \frac{1}{s}|x - y| - \alpha|x - y| \geqslant (\frac{1}{s} - \alpha)|x - y|.$$

□

Corollary 24.6. *Under the assumptions of Corollary* 24.5*,* $\alpha s < 1$*,* G *is injective.*

Proof. For $x, y \in X$, $x \neq y$, let $G(x) = G(y)$,

$$|G(x) - G(y)| \geqslant (\frac{1}{s} - \alpha)|x - y|$$

whence $x = y$, a contradiction, so

$$x \neq y \Rightarrow G(x) \neq G(y),$$

as claimed. □

25. Conclusion

In the survey article we consider b-metric spaces and some mappings and discuss the problem of existence and uniqueness of fixed points. This survey article is useful for students and mathematicians working in function analysis and their applications. B-metric and b-metric spaces are very useful in all areas of mathematics and applications.

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