Montes Taurus J. Pure Appl. Math. 5 (1), 1–21, 2023

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Families of unified and modified presentation of Fubini numbers and polynomials

Neslihan Kilar [a], Yilmaz Simsek [b]

[a]Department of Computer Technologies, Bor Vocational School, Niğde Ömer Halisdemir University, TR-51700 Niğde, Turkey
[b]Department of Mathematics, Faculty of Science, University of Akdeniz, TR-07058 Antalya, Turkey

Abstract

The goal of this paper is to define new families of unified and modified presentation of the Fubini numbers and polynomials with their generating functions. Using generating functions and their functional equations, many properties of these polynomials and numbers are presented. Relations among unified and modified presentation of the Fubini numbers and polynomials, Stirling type numbers, combinatorial type polynomials, and unified presentation of the generalized Bernoulli, Euler and Genocchi polynomials are given. Many novel identities and relations including these polynomials and numbers are also given. Moreover, new Hurwitz-Lerch type zeta functions, which interpolate unified and modified presentation of the Fubini numbers and polynomials at negative integers, are defined. Furthermore, suitable links of identities and relations, which are found in this paper, with those in earlier and future studies are indicated.

Keywords: Apostol type numbers and polynomials, combinatorial type numbers, Fubini type numbers and polynomials, generalized Apostol type numbers and polynomials, Stirling type numbers, special numbers, generating function

2010 MSC: 05A15, 05A19, 11B68, 11B73, 11S40, 33E20, 26C05, 30B40

1. Introduction

Special functions, polynomials, and numbers have many useful applications in diverse areas of pure and applied sciences. Because these functions, polynomials, and numbers have also been used to construct mathematical models, they have been employed to solve real world problems and other mathematical problems. These functions are widely used to solve computational science and engineering problems (*cf.* [1]-[57]).

The main motivation of this paper is to construct generating functions for certain classes of unified and modified presentation of the Fubini numbers and polynomials. These generating functions give us to find many new and useful formulas and relations involving combinatorial sums. These generating functions are also related to many generating functions for certain classes of special polynomials and numbers. Using these new generating functions, we derive some novel formulas and relations involving the unified and modified presentation of the Fubini numbers and polynomials, unified presentation of the generalized Bernoulli, Euler and Genocchi polynomials, and the classical Bernoulli, Euler and Genocchi numbers and polynomials, the Stirling type numbers, the combinatorial type numbers and polynomials, the generalized Eulerian type polynomials, and the Fubini type polynomials and numbers.

†*Article ID:* MTJPAM-D-22-00025

Email addresses: neslihankilar@ohu.edu.tr; neslihankilar@gmail.com (Neslihan Kilar), ysimsek@akdeniz.edu.tr (Yilmaz Simsek)

Received:15 August 2022, Accepted:14 November 2022, Published:17 December 2022

⋆*Corresponding Author:* Neslihan Kilar

Let us recall some known definitions and results in special numbers and polynomials. Let

$$\mathbb{N}=\{1,2,3,...\},\ \mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mathbb{Z}=\{\ldots,-2,-1,0,1,2,...\},$$

and $\mathbb{R}$, $\mathbb{R}^{+}$ and $\mathbb{C}$ denote the sets of real numbers, positive real numbers and complex numbers, respectively. Also, for $\mathrm{t}\in\mathbb{C}$, $r\in\mathbb{N}$,

$$\binom{\mathrm{t}}{r}r!=\{t\}_r=\mathrm{t}\,(\mathrm{t}-1)\,(\mathrm{t}-2)\ldots(\mathrm{t}-r+1)\ \text{ and }\ \{t\}_0=1$$

and also

$$0^r=\begin{cases}1 & r=0\\ 0 & r\in\mathbb{N}\end{cases}.$$

We assume that $\ln z$ denotes the principal branch of the many-valued function with the imaginary part $\mathrm{Im}\,(\ln z)$ constrained by

$$-\pi<\mathrm{Im}\,(\ln z)\leq\pi$$

(*cf.* [1]-[57]).

Let $z\in\mathbb{C}$ (or $\mathbb{R}$). The Apostol-Bernoulli polynomials $\mathcal{B}_k^{(z)}(x;\beta)$ of order z are defined by

$$\mathcal{M}_b\,(\omega,z,x;\beta)=\left(\frac{\omega}{\beta e^{\omega}-1}\right)^z e^{\omega x}=\sum_{k=0}^{\infty}\mathcal{B}_k^{(z)}(x;\beta)\,\frac{\omega^k}{k!},\tag{1.1}$$

where $|\omega|<2\pi$ when $\beta=1$; $|\omega|<|\ln\beta|$ when $\beta\neq1$; $1^z:=1$ (see, for detail, [33, Eq. (9)] and also [44, 48, 49]). When $x=0$ in (1.1), we have

$$\mathcal{B}_k^{(z)}(0;\beta)=\mathcal{B}_k^{(z)}(\beta),$$

which $\mathcal{B}_k^{(z)}(\beta)$ denote the Apostol-Bernoulli numbers of order z (*cf.* [33, 44, 48, 49]).

Let $z\in\mathbb{C}$ (or $\mathbb{R}$). The Apostol-Euler polynomials $\mathcal{E}_k^{(z)}(x;\beta)$ of order z are defined by

$$\mathcal{M}_e\,(\omega,z,x;\beta)=\left(\frac{2}{\beta e^{\omega}+1}\right)^z e^{\omega x}=\sum_{k=0}^{\infty}\mathcal{E}_k^{(z)}(x;\beta)\,\frac{\omega^k}{k!},\tag{1.2}$$

where $|\omega|<\pi$ when $\beta=1$; $|\omega|<|\ln(-\beta)|$ when $\beta\neq1$; $1^z:=1$ (*cf.* [32, Eq. (1)]; and also [33, 44, 48, 49]). Putting $x=0$ in (1.2), we get

$$\mathcal{E}_k^{(z)}(0;\beta)=\mathcal{E}_k^{(z)}(\beta),$$

which $\mathcal{E}_k^{(z)}(\beta)$ denote the Apostol-Euler numbers of order z (*cf.* [32, 33, 44, 48, 49]).

Let $z\in\mathbb{C}$ (or $\mathbb{R}$). The Apostol-Genocchi polynomials $\mathcal{G}_k^{(z)}(x;\beta)$ of order z are defined by

$$\mathcal{M}_g\,(\omega,z,x;\beta)=\left(\frac{2\omega}{\beta e^{\omega}+1}\right)^z e^{\omega x}=\sum_{k=0}^{\infty}\mathcal{G}_k^{(z)}(x;\beta)\,\frac{\omega^k}{k!},\tag{1.3}$$

where $|\omega|<\pi$ when $\beta=1$; $|\omega|<|\ln(-\beta)|$ when $\beta\neq1$; $1^z:=1$ (*cf.* [49, P. 99, Eq. (58)]; and also [44, 48]). When $x=0$ in (1.3), we get

$$\mathcal{G}_k^{(z)}(0;\beta)=\mathcal{G}_k^{(z)}(\beta),$$

which $\mathcal{G}_k^{(z)}(\beta)$ denote the Apostol-Genocchi numbers of order z (*cf.* [44, 48, 49]).

Let $a,b,c\in\mathbb{R}^{+}$ with $a\neq b$, $x\in\mathbb{R}$ and $z\in\mathbb{C}$. The generalized Bernoulli polynomials $\mathfrak{B}_k^{(z)}(x;\beta;a,b,c)$ of order z are defined by

$$\left(\frac{\omega}{\beta b^{\omega}-a^{\omega}}\right)^z c^{\omega x}=\sum_{k=0}^{\infty}\mathfrak{B}_k^{(z)}(x;\beta;a,b,c)\,\frac{\omega^k}{k!},\tag{1.4}$$

where $\left|\omega\ln\left(\frac{b}{a}\right)+\ln\beta\right|<2\pi$; $1^z:=1$ (*cf.* [50, Eq. (20)]; see also [48]).

When $a = 1$ and $b = c = e$ in (1.4), we have

$$\mathfrak{B}_k^{(z)}(x;\beta;1,e,e) = \mathcal{B}_k^{(z)}(x;\beta)$$

(*cf.* [48, 50]).

Let $a, b, c \in \mathbb{R}^+$ with $a \neq b$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$. The generalized Euler polynomials $\mathfrak{E}_k^{(z)}(x;\beta;a,b,c)$ of order z are defined by

$$\left(\frac{2}{\beta b^\omega + a^\omega}\right)^z c^{\omega x} = \sum_{k=0}^{\infty} \mathfrak{E}_k^{(z)}(x;\beta;a,b,c)\frac{\omega^k}{k!}, \tag{1.5}$$

where $\left|\omega \ln\left(\frac{b}{a}\right) + \ln\beta\right| < \pi$; $1^z := 1$ (*cf.* [51, Eq. (23)]; see also [48]).

Substituting $a = 1$ and $b = c = e$ into (1.5), we have

$$\mathfrak{E}_k^{(z)}(x;\beta;1,e,e) = \mathcal{E}_k^{(z)}(x;\beta)$$

(*cf.* [48, 50]).

Let $a, b, c \in \mathbb{R}^+$ with $a \neq b$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$. The generalized Genocchi polynomials $\mathfrak{G}_k^{(z)}(x;\beta;a,b,c)$ of order z are defined by

$$\left(\frac{2\omega}{\beta b^\omega + a^\omega}\right)^z c^{\omega x} = \sum_{k=0}^{\infty} \mathfrak{G}_k^{(z)}(x;\beta;a,b,c)\frac{\omega^k}{k!}, \tag{1.6}$$

where $\left|\omega \ln\left(\frac{b}{a}\right) + \ln\beta\right| < \pi$; $1^z := 1$ (*cf.* [51, Eq. (70)]; see also [48]).

Putting $a = 1$ and $b = c = e$ in (1.6), we obtain

$$\mathfrak{G}_k^{(z)}(x;\beta;1,e,e) = \mathcal{G}_k^{(z)}(x;\beta)$$

(*cf.* [48, 50]).

Let $m \in \mathbb{N}_0$, $x \in \mathbb{R}$, $a, d \in \mathbb{R}^+$, $\beta \in \mathbb{C}$ and $v \in \mathbb{N}$. Ozden [36] defined the polynomials $\mathcal{Y}_{k,\beta}^{(v)}(x;m,a,d)$ as follows:

$$\left(\frac{2^{1-m}\omega^m}{\beta^d e^\omega - a^d}\right)^v e^{\omega x} = \sum_{k=0}^{\infty} \mathcal{Y}_{k,\beta}^{(v)}(x;m,a,d)\frac{\omega^k}{k!}, \tag{1.7}$$

where $\left|\omega + d\ln\left(\frac{\beta}{a}\right)\right| < 2\pi$.

The polynomials $\mathcal{Y}_{k,\beta}^{(v)}(x;m,a,d)$ are so-called unification of the Bernoulli, Euler and Genocchi polynomials of higher order (see also [34]-[36]).

Substituting $x = 0$ into (1.7), one has

$$\mathcal{Y}_{k,\beta}^{(v)}(0;m,a,d) = \mathcal{Y}_{k,\beta}^{(v)}(m,a,d),$$

which are called unification of the Bernoulli, Euler and Genocchi numbers of higher order v (*cf.* [36]).

Let $a, d \in \mathbb{R}^+$, $x \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The second author [40] defined the generalized array type polynomials, $\mathcal{S}_v^k(x;a,d;\beta)$, and the numbers $\mathcal{S}(k,v;a,d;\beta)$, respectively:

$$\frac{(\beta d^\omega - a^\omega)^v}{v!} d^{\omega x} = \sum_{k=0}^{\infty} \mathcal{S}_v^k(x;a,d;\beta)\frac{\omega^k}{k!} \tag{1.8}$$

and

$$\frac{(\beta d^\omega - a^\omega)^v}{v!} = \sum_{k=0}^{\infty} \mathcal{S}(k,v;a,d;\beta)\frac{\omega^k}{k!}, \tag{1.9}$$

where $\omega \in \mathbb{C}$ (*cf.* [40, Eqs. (8) and (1)]; see also [6, 39]).

Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$, $x \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $u \in \mathbb{C} \setminus \{\beta\}$. The generalized Eulerian type polynomials of order v are defined by

$$\left(\frac{a^{\omega} - u}{\beta b^{\omega} - u}\right)^{v} c^{x\omega} = \sum_{k=0}^{\infty} \mathcal{H}_k^{(v)}(x; u; a, b, c; \beta) \frac{\omega^k}{k!}, \tag{1.10}$$

where $|\omega| < \frac{2\pi}{|\ln b|}$ when $\beta = u$; $\left|\omega \ln b + \ln\left(\frac{\beta}{u}\right)\right| < 2\pi$ when $\beta \neq u$ (*cf.* [40]; see also [39]).

Setting $x = 0$ in (1.10), we have

$$\mathcal{H}_k^{(v)}(0; u; a, b, c; \beta) = \mathcal{H}_k^{(v)}(u; a, b, c; \beta),$$

which $\mathcal{H}_k^{(v)}(u; a, b, c; \beta)$ denote the generalized Eulerian type numbers of order v (*cf.* [39, 40]).

When $a = \beta = 1$ and $b = c = e$ in (1.10), we obtain

$$\left(\frac{1 - u}{e^{\omega} - u}\right)^{v} e^{x\omega} = \sum_{k=0}^{\infty} H_k^{(v)}(x; u) \frac{\omega^k}{k!}, \tag{1.11}$$

which $H_k^{(v)}(x; u)$ denote the Frobenius-Euler polynomials of order v, $v \in \mathbb{N}_0$ (*cf.* [5, 21, 22, 29, 39, 41]).

The polynomials $W_k^{(v)}(x; \beta)$ are defined by

$$\frac{e^{\omega x}}{(\beta e^{\omega} + \beta^{-1} e^{-\omega} + 2)^{v}} = \sum_{k=0}^{\infty} W_k^{(v)}(x; \beta) \frac{\omega^k}{k!}, \tag{1.12}$$

where $|\omega| < \pi$ when $\beta = 1$; $|\omega| < |\ln(-\beta)|$ when $\beta \neq 1$. Putting $x = 0$ in (1.12), we get

$$W_k^{(v)}(0; \beta) = W_k^{(v)}(\beta)$$

(*cf.* [41, Eqs. (23) and (25)]; see also [27, 28, 30, 46]).

Let $b \in \mathbb{R}^+$ with $b \geq 1$, $r \in \mathbb{Z}$ and $u \in \mathbb{C} \setminus \{1\}$. The polynomials $Y_k^{(r)}(x, u; b)$ are defined by

$$\left(\frac{1}{b^{\omega} - u}\right)^{r} b^{x\omega} = \sum_{k=0}^{\infty} Y_k^{(r)}(x, u; b) \frac{\omega^k}{k!} \tag{1.13}$$

(*cf.* [42]; see also [39, 40]).

Setting $x = 0$ in (1.13), we get

$$Y_k^{(r)}(0, u; b) = Y_k^{(r)}(u; b)$$

(*cf.* [39, 40, 42]).

In [15], Kilar and Simsek defined the Fubini type polynomials $a_k^{(v)}(x)$ of order v:

$$\frac{2^{v}}{(2 - e^{\omega})^{2v}} e^{\omega x} = \sum_{k=0}^{\infty} a_k^{(v)}(x) \frac{\omega^k}{k!}, \tag{1.14}$$

where $|\omega| < \ln 2$. When $x = 0$ in (1.14), we have

$$a_k^{(v)}(0) = a_k^{(v)},$$

which $a_k^{(v)}$ denote the Fubini type numbers of order v (see also [10, 12, 13]).

Let $a, d, c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $\beta, z \in \mathbb{C}$. The unified form of generalized polynomials $F_{k,m}^{(z)}(x, y; a, d, c; \beta)$ are defined by

$$\left(\frac{a^{-\omega} \omega^{m}}{1 - y\left(\beta\left(\frac{d}{a}\right)^{\omega} - 1\right)}\right)^{z} c^{\omega x} = \sum_{k=0}^{\infty} F_{k,m}^{(z)}(x, y; a, d, c; \beta) \frac{\omega^k}{k!} \tag{1.15}$$

where $\left|\omega \ln\left(\frac{d}{a}\right) + \ln\left(\frac{\beta y}{y+1}\right)\right| < 2\pi$; $1^z := 1$ (*cf.* [1, Eq. (3)]).

When $a = 1$, $d = c = e$, $m = 0$ in (1.15), we get the bivariate Apostol-Fubini polynomials of order z:

$$\left(\frac{1}{1 - y\left(\beta e^{\omega} - 1\right)}\right)^z e^{\omega x} = \sum_{k=0}^{\infty} F_k^{(z)}(x, y; \beta) \frac{\omega^k}{k!} \tag{1.16}$$

(*cf.* [1, Eq. (5)]). For $\beta = 1$, equation (1.16) is reduced to the work of [24], see also [23].

The polynomials $\mathcal{F}_k^{(v)}(x, y; a, d; \beta, m)$ are defined by

$$\left(\frac{\omega^m}{1 - y\left(\beta^d e^{\omega} - a^d\right)}\right)^v e^{\omega x} = \sum_{k=0}^{\infty} \mathcal{F}_k^{(v)}(x, y; a, d; \beta, m) \frac{\omega^k}{k!}, \tag{1.17}$$

where $\left|\omega + \ln\left(\frac{y\beta^d}{1+ya^d}\right)\right| < 2\pi$, $a, d \in \mathbb{R}^+$, $x, y \in \mathbb{R}$ and $m, v \in \mathbb{N}_0$ (*cf.* [31, Eq. (2.1)]). The polynomials $\mathcal{F}_k^{(v)}(x, y; a, d; \beta, m)$ are called the unified Apostol-type Bernoulli, Euler, Genocchi, and Fubini polynomials of order v.

The numbers $y_{9,k}(\beta; a)$ are defined by

$$\frac{2}{a^{\omega} + \beta} = \sum_{k=0}^{\infty} y_{9,k}(\beta; a) \frac{\omega^k}{k!},$$

where $\left|\omega \ln a + \ln\left(\frac{1}{\beta}\right)\right| < \pi$ (*cf.* [45, Eq. (5.1)]).

Recently, Gun et al. [8] introduced the higher order power of numbers $y_{9,k}(\beta; a)$ as follows:

$$\left(\frac{2}{a^{\omega} + \beta}\right)^z = \sum_{k=0}^{\infty} y_{9,k}^{(z)}(\beta; a) \frac{\omega^k}{k!}, \tag{1.18}$$

where $\left|\omega \ln a + \ln\left(\frac{1}{\beta}\right)\right| < \pi$, $a \in \mathbb{R}^+$, $x \in \mathbb{R}$ and $\beta, z \in \mathbb{C}$.

We now give brief summary of this paper as follows:

In Section 2, we construct generating functions for new classes of special polynomials and numbers, which are so-called unified and modified presentation of the Fubini numbers and polynomials. We give some properties of these polynomials and numbers. By using these generating functions, we obtain some identities and formulas related to these numbers and polynomials, and some certain classes of special polynomials and numbers. Using these formulas, we give some numerical values of these numbers and polynomials.

In Section 3, we introduce Hurwitz-Lerch type zeta functions, which interpolate unified and modified presentation of the Fubini numbers and polynomials. We give also some properties of these functions.

In Section 4, using generating functions for the unified and modified presentation of the Fubini numbers and polynomials, and the Apostol-Fubini type polynomials and numbers, we give many identities involving the Stirling type numbers, the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, and the generalized Bernoulli, Euler and Genocchi polynomials and numbers. We also present some special cases of our results.

Finally, we give conclusion section.

2. New classes of unified and modified presentation of the Fubini numbers and polynomials

In this section, we construct new generating functions for unified and modified presentation of the Fubini numbers and polynomials. Using these generating functions and their functional equations, we present some properties of these numbers and polynomials. We also define generating functions for the Apostol-Fubini type polynomials and numbers. We find a derivative formula of these polynomials. We also give some identities involving these numbers and polynomials, the generalized Eulerian type polynomials, the Frobenius-Euler polynomials, and also the polynomials $\mathcal{Y}_{k,\beta}^{(v)}(x; m, a, d)$, the numbers $y_{9,k}^{(v)}(\beta; a)$, the polynomials $\mathcal{F}_k^{(v)}(x, y; a, d; \beta, m)$.

Let $b, c \in \mathbb{R}^{+}$ with $b, c \geq 1$, $\vartheta, \mu, z \in \mathbb{C}$ and $\mu \neq \vartheta$. We define the following generating functions for unified and modified presentation of Fubini numbers and polynomials of order z, respectively:

$$\mathcal{N}_f\left(\omega, z, \mu; \vartheta, b\right) = \frac{2^z}{\left(\mu b^{\omega} - \vartheta\right)^{2z}} = \sum_{k=0}^{\infty} \mathfrak{a}_k^{(z)}\left(\mu; \vartheta, b\right) \frac{\omega^k}{k!} \tag{2.1}$$

and

$$\mathcal{N}_p\left(\omega, z, x, \mu; \vartheta, b, c\right) = \frac{2^z}{\left(\mu b^{\omega} - \vartheta\right)^{2z}} c^{\omega x} = \sum_{k=0}^{\infty} \mathfrak{a}_k^{(z)}\left(x, \mu; \vartheta, b, c\right) \frac{\omega^k}{k!}, \tag{2.2}$$

where $|\omega| < \frac{2\pi}{|\ln b|}$ when $\mu = \vartheta$; $\left|\omega \ln b + \ln\left(\frac{\mu}{\vartheta}\right)\right| < 2\pi$ when $\mu \neq \vartheta$; $1^z := 1$. Here we note that the principal branch of the logarithm is assumed to be taken.

By using equation (2.2), we also define parametric type of the poynomials $\mathfrak{a}_k^{(z)}\left(x, \mu; \vartheta, b, c\right)$ as follows.

Substituting $x = t + iu$ into (2.2), we get

$$\frac{2^z e^{\omega(t+iu)\ln c}}{\left(\mu b^{\omega} - \vartheta\right)^{2z}} = \sum_{k=0}^{\infty} \mathfrak{a}_k^{(z)}\left(t + iu, \mu; \vartheta, b, c\right) \frac{\omega^k}{k!}.$$

Combining the above equation with the following well-known the Euler's formula

$$e^{ai} = \cos a + i \sin a,$$

we define

$$\frac{2^z e^{\omega t \ln c}}{\left(\mu b^{\omega} - \vartheta\right)^{2z}} \left(\cos\left(\omega u \ln c\right) + i \sin\left(\omega u \ln c\right)\right) = \sum_{k=0}^{\infty} \mathfrak{a}_k^{(z)}\left(t + iu, \mu; \vartheta, b, c\right) \frac{\omega^k}{k!}.$$

Thus

$$\frac{2^z e^{\omega t \ln c}}{\left(\mu b^{\omega} - \vartheta\right)^{2z}} \left(\cos\left(\omega u \ln c\right) + i \sin\left(\omega u \ln c\right)\right) = \sum_{k=0}^{\infty} \left(\mathfrak{a}_k^{(z,C)}\left(t, u, \mu; \vartheta, b, c\right) + i\mathfrak{a}_k^{(z,S)}\left(t, u, \mu; \vartheta, b, c\right)\right) \frac{\omega^k}{k!},$$

where

$$\mathfrak{a}_k^{(z,C)}\left(t, u, \mu; \vartheta, b, c\right) = \mathrm{Re}\left\{\mathfrak{a}_k^{(z)}\left(t + iu, \mu; \vartheta, b, c\right)\right\} \text{ and } \mathfrak{a}_k^{(z,S)}\left(t, u, \mu; \vartheta, b, c\right) = \mathrm{Im}\left\{\mathfrak{a}_k^{(z)}\left(t + iu, \mu; \vartheta, b, c\right)\right\}.$$

There are many properties of the polynomials $\mathfrak{a}_k^{(z,C)}\left(t, u, \mu; \vartheta, b, c\right)$ and $\mathfrak{a}_k^{(z,S)}\left(t, u, \mu; \vartheta, b, c\right)$. Here we note that it is left to researchers to investigate these interesting properties deeply with their applications. Consequently, by the help of the method, given in [18, 19, 20], many new and interesting identities and relations with their applications of these polynomials may be investigated in detail.

For $z \in \mathbb{C}$, motivations of the generating functions (2.2) and (2.1) are given as follows:

$$\vartheta^{2z} \left(\ln b\right)^{2z} \sum_{k=0}^{\infty} \mathfrak{a}_k^{(z)}\left(x, \mu; \vartheta, b, c\right) \frac{\omega^{k+2z}}{2^z k!} = \sum_{k=0}^{\infty} (\ln b)^k \mathcal{B}_k^{(2z)}\left(\frac{\ln c}{\ln b} x; \frac{\mu}{\vartheta}\right) \frac{\omega^k}{k!}.$$

Further coments and observation about the above equation. This equation needs to investigate main properties of the analytic continuation with the principal branch of the logarithm for w^{2z} in order to compare to the coefficients $\frac{\omega^k}{k!}$ both sides. Although this technical and detailed subject has been dealt with in some studies in recent years, we have not yet come across whether there has been almost any study on "*analytical continuation and its applications for* $f(w; z) = w^z$ *with* $w, z \in \mathbb{C}$" in this context. Recently, there are some studies on this case, for instance, see also (*cf.* [4, 5, 7, 9, 46, 47, 49, 56, 57]). It should be noted here that this technical and detailed issue is not covered within the scope of this article.

We now set

$$\sum_{k=0}^{\infty} \mathfrak{a}_k^{(z)}\left(x, \mu; -\vartheta, b, c\right) \frac{\omega^k}{k!} = \frac{1}{\vartheta^{2z} 2^z} \sum_{k=0}^{\infty} (\ln b)^k \mathcal{E}_k^{(2z)}\left(\frac{\ln c}{\ln b} x; \frac{\mu}{\vartheta}\right) \frac{\omega^k}{k!}.$$

Therefore,

$$\mathfrak{a}_k^{(z)}(x,\mu;-\vartheta,b,c)=\frac{(\ln b)^k}{\vartheta^{2z}2^z}\mathcal{E}_k^{(2z)}\left(\frac{\ln c}{\ln b}x;\frac{\mu}{\vartheta}\right).$$

Substituting $x=0$ into (2.2), we obtain

$$\mathfrak{a}_k^{(z)}(\mu;\vartheta,b):=\mathfrak{a}_k^{(z)}(0,\mu;\vartheta,b,c).$$

Setting $z=0$ in (2.2), we have

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(0)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!}=\sum_{k=0}^{\infty}x^k(\ln c)^k\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the above equation, we get

$$\mathfrak{a}_k^{(0)}(x,\mu;\vartheta,b,c)=x^k(\ln c)^k.$$

Putting $\omega=0$ in (2.2) and (2.1), we have

$$\begin{aligned}\frac{2^z}{(\mu-\vartheta)^{2z}} &= \mathfrak{a}_0^{(z)}(x,\mu;\vartheta,b,c)\\ &= \mathfrak{a}_0^{(z)}(\mu;\vartheta,b),\end{aligned}\tag{2.3}$$

where $\mu\neq\vartheta$.

By putting $b=c=e$ and $\vartheta=2$ in equations (2.2) and (2.1) respectively, we get

$$\mathfrak{a}_k^{(z)}(x;\mu):=\mathfrak{a}_k^{(z)}(x,\mu;2,e,e)$$

and

$$\mathfrak{a}_k^{(z)}(\mu):=\mathfrak{a}_k^{(z)}(\mu;2,e).$$

The Apostol-Fubini type polynomials and numbers of order z are, respectively, given by

$$\frac{2^z}{(\mu e^{\omega}-2)^{2z}}e^{x\omega}=\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(x;\mu)\frac{\omega^k}{k!}\tag{2.4}$$

and

$$\frac{2^z}{(\mu e^{\omega}-2)^{2z}}=\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(\mu)\frac{\omega^k}{k!},\tag{2.5}$$

where $\left|\omega+\ln\left(\frac{\mu}{2}\right)\right|<2\pi$; $1^z:=1$.

Theorem 2.1. *Let $k\in\mathbb{N}_0$ and $v\in\mathbb{N}$. Then we have*

$$\mathcal{Y}_{k,\mu}^{(2v)}(x;0,\vartheta,1)=2^v\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,e,e).$$

Proof. Setting $d=1$ and $m=0$ in (1.7), we get

$$\left(\frac{2}{\mu e^{\omega}-\vartheta}\right)^{2v}e^{\omega x}=\sum_{k=0}^{\infty}\mathcal{Y}_{k,\mu}^{(2v)}(x;0,\vartheta,1)\frac{\omega^k}{k!}.\tag{2.6}$$

Substituting $z=v$, $v\in\mathbb{N}$ and $b=c=e$ into (2.2), we obtain

$$\frac{2^v}{(\mu e^{\omega}-\vartheta)^{2v}}e^{\omega x}=\sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,e,e)\frac{\omega^k}{k!}.\tag{2.7}$$

Combining (2.6) with (2.7), we obtain

$$2^{v}\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(v)}\left(x,\mu;\vartheta,e,e\right)\frac{\omega^{k}}{k!}=\sum_{k=0}^{\infty}\mathcal{Y}_{k,\mu}^{(2v)}\left(x;0,\vartheta,1\right)\frac{\omega^{k}}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we get the desired result. □

Now we give some special cases of the unified and modified presentation of the Fubini numbers and polynomials. Setting $b=c$ and $z=r$, $r\in\mathbb{Z}$ in (2.2), we have

$$\frac{2^{r}b^{\omega x}}{\mu^{2r}\left(b^{\omega}-\frac{\vartheta}{\mu}\right)^{2r}}=\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(r)}\left(x,\mu;\vartheta,b,b\right)\frac{\omega^{k}}{k!}.$$

Combining the above equation with (1.13), we get

$$\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(r)}\left(x,\mu;\vartheta,b,b\right)\frac{\omega^{k}}{k!}=\frac{2^{r}}{\mu^{2r}}\sum_{k=0}^{\infty}Y_{k}^{(2r)}\left(x,\frac{\vartheta}{\mu};b\right)\frac{\omega^{k}}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the above equation, we get the following corollary:

Corollary 2.2. *For $k\in\mathbb{N}_0$ and $r\in\mathbb{Z}$, we have*

$$\mathfrak{a}_{k}^{(r)}\left(x,\mu;\vartheta,b,b\right)=\frac{2^{r}Y_{k}^{(2r)}\left(x,\frac{\vartheta}{\mu};b\right)}{\mu^{2r}}.\tag{2.8}$$

Remark 2.3. Taking $\mu=1$ in (2.8), we obtain the following relation:

$$\mathfrak{a}_{k}^{(r)}\left(x,1;\vartheta,b,b\right)=2^{r}Y_{k}^{(2r)}\left(x,\vartheta;b\right)$$

(*cf.* [14]).

Remark 2.4. Putting $y=1$ in (1.16) and using (2.4), we get

$$(-1)^{2z}2^{z}F_{k}^{(2z)}\left(x,1;\mu\right)=\mathfrak{a}_{k}^{(z)}\left(x;\mu\right).$$

Remark 2.5. Putting $b=c=e$, $\mu=1$, $\vartheta=2$ and $z=v$, $v\in\mathbb{N}_0$ in (2.2) and (2.1), we have

$$\mathfrak{a}_{n}^{(v)}\left(x,1;2,e,e\right)=a_{n}^{(v)}\left(x\right)$$

and

$$\mathfrak{a}_{n}^{(v)}\left(1;2,e\right)=a_{n}^{(v)}.$$

When $\mu=1$ and $z=v$, $v\in\mathbb{N}_0$ in (2.4) and (2.5), we have

$$\mathfrak{a}_{n}^{(v)}\left(x;1\right)=a_{n}^{(v)}\left(x\right)$$

and

$$\mathfrak{a}_{n}^{(v)}\left(1\right)=a_{n}^{(v)}.$$

By using (1.10), (1.13) and (2.2), we get

$$\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(v)}\left(x,\mu;\vartheta,b,c\right)\frac{\omega^{k}}{k!}=2^{v}\sum_{k=0}^{\infty}Y_{k}^{(2v)}\left(\vartheta;a\right)\frac{\omega^{k}}{k!}\sum_{k=0}^{\infty}\mathcal{H}_{k}^{(2v)}\left(x;\vartheta;a,b,c;\mu\right)\frac{\omega^{k}}{k!}.$$

Thus,

$$\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(v)}\left(x,\mu;\vartheta,b,c\right)\frac{\omega^{k}}{k!}=2^{v}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}Y_{j}^{(2v)}\left(\vartheta;a\right)\mathcal{H}_{k-j}^{(2v)}\left(x;\vartheta;a,b,c;\mu\right)\frac{\omega^{k}}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the above equation, we get the following corollary:

Corollary 2.6. *We have*

$$\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)=2^v\sum_{j=0}^{k}\binom{k}{j}Y_j^{(2v)}(\vartheta;a)\,\mathcal{H}_{k-j}^{(2v)}(x;\vartheta;a,b,c;\mu). \tag{2.9}$$

Corollary 2.7. *By substituting $a=\mu=1$ and $b=c=e$ into* (2.9)*, we obtain*

$$\mathfrak{a}_k^{(v)}(x,1;\vartheta,e,e)=2^v\sum_{j=0}^{k}\binom{k}{j}Y_j^{(2v)}(\vartheta;1)\,H_{k-j}^{(2v)}(x;\vartheta).$$

Setting $\mu=1$ in (2.1), we have

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(1;\vartheta,b)\frac{\omega^k}{k!}=\frac{2^z}{(b^\omega-\vartheta)^{2z}}.$$

Combining the above equation with (1.18), we obtain

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(1;\vartheta,b)\frac{\omega^k}{k!}=\frac{1}{2^z}\sum_{k=0}^{\infty}y_{9,k}^{(2z)}(-\vartheta;b)\frac{\omega^k}{k!}.$$

Hence, we get the following corollary:

Corollary 2.8. *For $k\in\mathbb{N}_0$, we have*

$$\mathfrak{a}_k^{(z)}(1;\vartheta,b)=\frac{y_{9,k}^{(2z)}(-\vartheta;b)}{2^z}.$$

Putting $a=1$ and $m=0$ in (1.15), we obtain

$$\left(\frac{1}{1-y\mu d^\omega+y}\right)^{2z}c^{\omega x}=\sum_{k=0}^{\infty}F_{k,0}^{(2z)}(x,y;1,d,c;\mu)\frac{\omega^k}{k!}.$$

Combining the above equation with (2.2), one has

$$\frac{1}{(-1)^{2z}2^z}\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(x,y\mu;y+1,d,c)\frac{\omega^k}{k!}=\sum_{k=0}^{\infty}F_{k,0}^{(2z)}(x,y;1,d,c;\mu)\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we get the following corollary:

Corollary 2.9. *We have*

$$\mathfrak{a}_k^{(z)}(x,y\mu;y+1,d,c)=(-1)^{2z}2^zF_{k,0}^{(2z)}(x,y;1,d,c;\mu).$$

Using (1.17), for $m=0$ and $d=1$, we obtain

$$\left(\frac{1}{1-y\mu e^\omega+y\vartheta}\right)^{2v}e^{\omega x}=\sum_{k=0}^{\infty}\mathcal{F}_k^{(2v)}(x,y;\vartheta,1;\mu,0)\frac{\omega^k}{k!}.$$

Combining the above equation with (2.2), we have

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,y\mu;y\vartheta+1,e,e)\frac{\omega^k}{k!}=2^v\sum_{k=0}^{\infty}\mathcal{F}_k^{(2v)}(x,y;\vartheta,1;\mu,0)\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the following corollary:

Corollary 2.10. *For $k,v\in\mathbb{N}_0$, we have*

$$\mathfrak{a}_k^{(v)}(x,y\mu;y\vartheta+1,e,e)=2^v\mathcal{F}_k^{(2v)}(x,y;\vartheta,1;\mu,0).$$

Theorem 2.11. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\mathfrak{a}_{k-2v}^{(v)}(x,\mu;\vartheta,b,c) = \frac{2^v(\ln b)^{k-2v}}{\vartheta^v \{k\}_{2v}} \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_j^{(v)}\left(\frac{\mu}{\vartheta}\right) \mathcal{Y}_{k-j,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right).$$

Proof. Combining (2.2) with (1.7) and (1.1), we get

$$\frac{2^v}{(\ln b)^{2v}} \sum_{k=0}^{\infty} \mathcal{B}_k^{(v)}\left(\frac{\mu}{\vartheta}\right) \frac{(\ln b)^k \omega^k}{k!} \sum_{k=0}^{\infty} \mathcal{Y}_{k,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right) \frac{(\ln b)^k \omega^k}{k!} = \vartheta^v \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c) \frac{\omega^{k+2v}}{k!}.$$

Therefore

$$\begin{aligned} &\frac{2^v}{(\ln b)^{2v}} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \mathcal{B}_j^{(v)}\left(\frac{\mu}{\vartheta}\right) (\ln b)^j \mathcal{Y}_{k-j,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right) \frac{(\ln b)^{k-j}\omega^k}{k!} \\ = \;& \vartheta^v \sum_{k=0}^{\infty} \{k\}_{2v}\, \mathfrak{a}_{k-2v}^{(v)}(x,\mu;\vartheta,b,c) \frac{\omega^k}{k!}. \end{aligned}$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

Theorem 2.12. *Let $k, v \in \mathbb{N}_0$ with $k \geq v$. Then we have*

$$\mathfrak{a}_{k-v}^{(v)}(x,\mu;\vartheta,b,c) = \frac{(\ln b)^{k-v}}{(-\vartheta)^v \{k\}_v} \sum_{j=0}^{k} \binom{k}{j} \mathcal{E}_j^{(v)}\left(-\frac{\mu}{\vartheta}\right) \mathcal{Y}_{k-j,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right).$$

Proof. Combining (2.2) with (1.7) and (1.2), we obtain

$$\sum_{k=0}^{\infty} \mathcal{E}_k^{(v)}\left(-\frac{\mu}{\vartheta}\right) \frac{(\ln b)^k \omega^k}{k!} \sum_{k=0}^{\infty} \mathcal{Y}_{k,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right) \frac{(\ln b)^{k-v} \omega^k}{k!} = (-\vartheta)^v \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c) \frac{\omega^{k+v}}{k!}.$$

Therefore

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \mathcal{E}_j^{(v)}\left(-\frac{\mu}{\vartheta}\right) (\ln b)^j \mathcal{Y}_{k-j,\mu}^{(v)}\left(\frac{x\ln c}{\ln b};1,\vartheta,1\right) \frac{(\ln b)^{k-j-v}\omega^k}{k!} = (-\vartheta)^v \sum_{k=0}^{\infty} \{k\}_v\, \mathfrak{a}_{k-v}^{(v)}(x,\mu;\vartheta,b,c) \frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.13. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\mathfrak{a}_{k-2v}^{(v)}(x,\mu;\vartheta,b,c) = \frac{(\ln b)^{k-2v}}{(-\vartheta)^v \{k\}_{2v}} \sum_{j=0}^{k} \binom{k}{j} \mathcal{G}_j^{(v)}\left(-\frac{\mu}{\vartheta}\right) \mathcal{Y}_{k-j,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right).$$

Proof. Combining (2.2) with (1.7) and (1.3), we get

$$\begin{aligned} &\frac{1}{(\ln b)^{2v}} \sum_{k=0}^{\infty} \mathcal{G}_k^{(v)}\left(-\frac{\mu}{\vartheta}\right) \frac{(\ln b)^k \omega^k}{k!} \sum_{k=0}^{\infty} \mathcal{Y}_{k,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right) \frac{(\ln b)^k \omega^k}{k!} \\ = \;& (-\vartheta)^v \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c) \frac{\omega^{k+2v}}{k!}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (\ln b)^{k-2v} \mathcal{G}_j^{(v)}\left(-\frac{\mu}{\vartheta}\right) \mathcal{Y}_{k-j,\mu}^{(v)}\left(\frac{\ln c}{\ln b}x;1,\vartheta,1\right) \frac{\omega^k}{k!} \\ = \;& (-\vartheta)^v \sum_{k=0}^{\infty} \{k\}_{2v}\, \mathfrak{a}_{k-2v}^{(v)}(x,\mu;\vartheta,b,c) \frac{\omega^k}{k!}. \end{aligned}$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

Theorem 2.14. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$y_{9,k}^{(v)}\left(-\frac{\vartheta}{\mu};b\right)=\sum_{d=0}^{v}(-1)^{v-d}\binom{v}{d}\mu^{d+v}\vartheta^{v-d}\sum_{j=0}^{k}\binom{k}{j}\mathfrak{a}_j^{(v)}(\mu;\vartheta,b)(d\ln b)^{k-j}.$$

Proof. Combining (1.18) with (2.1), we get

$$\sum_{k=0}^{\infty}y_{9,k}^{(v)}\left(-\frac{\vartheta}{\mu};b\right)\frac{\omega^k}{k!}=\mu^v\sum_{d=0}^{v}(-1)^{v-d}\binom{v}{d}\mu^d\vartheta^{v-d}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\mathfrak{a}_j^{(v)}(\mu;\vartheta,b)d^{k-j}(\ln b)^{k-j}\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

Theorem 2.15. *Let $b \neq 1$. Let $k \in \mathbb{N}$ and $v \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{2v}\sum_{r=0}^{k}(-1)^j\binom{2v}{j}\binom{k}{r}\vartheta^{2v-j}j^r\mu^j(\ln b)^r\,\mathfrak{a}_{k-r}^{(v)}(\mu;\vartheta,b)=0. \tag{2.10}$$

Proof. Using binomial theorem in (2.1), we get

$$2^v=\sum_{j=0}^{2v}(-1)^j\binom{2v}{j}\vartheta^{2v-j}\mu^j\sum_{k=0}^{\infty}(j\ln b)^k\frac{\omega^k}{k!}\sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Therefore,

$$2^v=\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{2v}(-1)^j\binom{2v}{j}\binom{k}{r}\vartheta^{2v-j}\mu^j j^r(\ln b)^r\,\mathfrak{a}_{k-r}^{(v)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Equating the coefficients of the previous equation, we can easily arrive at the desired result. □

Substituting $b = e$ into (2.10), we also get the following result:

Corollary 2.16. *Let $k \in \mathbb{N}$ and $v \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{2v}\sum_{r=0}^{k}(-1)^j\binom{2v}{j}\binom{k}{r}\vartheta^{2v-j}j^r\mu^j\mathfrak{a}_{k-r}^{(v)}(\mu;\vartheta,e)=0.$$

Theorem 2.17. *Let $k \in \mathbb{N}_0$ and $z_1, z_2 \in \mathbb{C}$. Then we have*

$$\mathfrak{a}_k^{(z_1+z_2)}(\mu;\vartheta,b)=\sum_{r=0}^{k}\binom{k}{r}\mathfrak{a}_{k-r}^{(z_1)}(\mu;\vartheta,b)\,\mathfrak{a}_r^{(z_2)}(\mu;\vartheta,b). \tag{2.11}$$

Proof. Using (2.1), we get

$$\mathcal{N}_f(\omega,z_1,\mu;\vartheta,b)\,\mathcal{N}_f(\omega,z_2,\mu;\vartheta,b)=\mathcal{N}_f(\omega,z_1+z_2,\mu;\vartheta,b).$$

From the above equation, we have

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z_1)}(\mu;\vartheta,b)\frac{\omega^k}{k!}\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z_2)}(\mu;\vartheta,b)\frac{\omega^k}{k!}=\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z_1+z_2)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Hence

$$\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\mathfrak{a}_{k-r}^{(z_1)}(\mu;\vartheta,b)\,\mathfrak{a}_r^{(z_2)}(\mu;\vartheta,b)\frac{\omega^k}{k!}=\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z_1+z_2)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

Using (2.10) (or (2.11)), for $v = 1$, few values of the numbers $\mathfrak{a}_n^{(1)}(\mu;\vartheta,b)$ are given by

$$\begin{aligned}
\mathfrak{a}_0^{(1)}(\mu;\vartheta,b) &= \frac{2}{(\mu-\vartheta)^2},\\
\mathfrak{a}_1^{(1)}(\mu;\vartheta,b) &= -\frac{4\mu\ln b}{(\mu-\vartheta)^3},\\
\mathfrak{a}_2^{(1)}(\mu;\vartheta,b) &= \frac{4\mu(\ln b)^2(\vartheta+2\mu)}{(\mu-\vartheta)^4},\\
\mathfrak{a}_3^{(1)}(\mu;\vartheta,b) &= -\frac{4\mu(\ln b)^3\left(4\mu^2+7\mu\vartheta+\vartheta^2\right)}{(\mu-\vartheta)^5},
\end{aligned}$$

and so on.

Using (2.10), for $v = 2$, few values of the numbers $\mathfrak{a}_n^{(2)}(\mu;\vartheta,b)$ are given as follows:

$$\begin{aligned}
\mathfrak{a}_0^{(2)}(\mu;\vartheta,b) &= \frac{4}{(\mu-\vartheta)^4},\\
\mathfrak{a}_1^{(2)}(\mu;\vartheta,b) &= -\frac{16\mu\ln b}{(\mu-\vartheta)^5},\\
\mathfrak{a}_2^{(2)}(\mu;\vartheta,b) &= \frac{16\mu(\ln b)^2(4\mu+\vartheta)}{(\mu-\vartheta)^6},
\end{aligned}$$

and so on.

Theorem 2.18. *Let $k \in \mathbb{N}_0$ and $z \in \mathbb{C}$. Then we have*

$$\mathfrak{a}_k^{(z)}(x,\mu;\vartheta,b,c) = \sum_{r=0}^{k}\binom{k}{r}x^{k-r}(\ln c)^{k-r}\,\mathfrak{a}_r^{(z)}(\mu;\vartheta,b). \tag{2.12}$$

Proof. Using (2.1) and (2.2), we get

$$\mathcal{N}_p(\omega,z,x,\mu;\vartheta,b,c) = \mathcal{N}_f(\omega,z,\mu;\vartheta,b)\,c^{\omega x}.$$

From the above functional equation, we obtain

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!} = \sum_{k=0}^{\infty}x^k(\ln c)^k\frac{\omega^k}{k!}\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Thus,

$$\sum_{k=0}^{\infty}\mathfrak{a}_k^{(z)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!} = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}x^{k-r}(\ln c)^{k-r}\,\mathfrak{a}_r^{(z)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we get the equation (2.12). □

Using (2.12), for $z = 1$, few values of the polynomials $\mathfrak{a}_n^{(1)}(x,\mu;\vartheta,b,c)$ are given as follows:

$$\begin{aligned}
\mathfrak{a}_0^{(1)}(x,\mu;\vartheta,b,c) &= \frac{2}{(\mu-\vartheta)^2},\\
\mathfrak{a}_1^{(1)}(x,\mu;\vartheta,b,c) &= -\frac{4\mu\ln b}{(\mu-\vartheta)^3}+\frac{2x\ln c}{(\mu-\vartheta)^2},\\
\mathfrak{a}_2^{(1)}(x,\mu;\vartheta,b,c) &= \frac{4\mu(\ln b)^2(\vartheta+2\mu)}{(\mu-\vartheta)^4}-\frac{8x\mu(\ln c)(\ln b)}{(\mu-\vartheta)^3}+\frac{2x^2(\ln c)^2}{(\mu-\vartheta)^2},
\end{aligned}$$

and so on.

Using (2.12), for $z = 2$, few values of the numbers $\mathfrak{a}_n^{(2)}(x,\mu;\vartheta,b,c)$ are given as follows:

$$\begin{aligned}
\mathfrak{a}_0^{(2)}(x,\mu;\vartheta,b,c) &= \frac{4}{(\mu-\vartheta)^4},\\
\mathfrak{a}_1^{(2)}(x,\mu;\vartheta,b,c) &= -\frac{16\mu\ln b}{(\mu-\vartheta)^5} + \frac{4x\ln c}{(\mu-\vartheta)^4},\\
\mathfrak{a}_2^{(2)}(x,\mu;\vartheta,b,c) &= \frac{16\mu(\ln b)^2(4\mu+\vartheta)}{(\mu-\vartheta)^6} - \frac{32x\mu(\ln c)(\ln b)}{(\mu-\vartheta)^5} + \frac{4x^2(\ln c)^2}{(\mu-\vartheta)^4},
\end{aligned}$$

and so on.

Theorem 2.19. *Let $k, m \in \mathbb{N}$ with $k \geq m$ and $v \in \mathbb{N}_0$. Then we have*

$$\frac{\partial^m}{\partial x^m}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c) = (\ln c)^m \{k\}_m \mathfrak{a}_{k-m}^{(v)}(x,\mu;\vartheta,b,c). \tag{2.13}$$

Proof. Using (2.2), we get

$$\sum_{k=0}^{\infty}\frac{\partial^m}{\partial x^m}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!} = (\ln c)^m\sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^{k+m}}{k!}.$$

After some calculations, we obtain

$$\sum_{k=0}^{\infty}\frac{\partial^m}{\partial x^m}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!} = (\ln c)^m\sum_{k=0}^{\infty}\{k\}_m\mathfrak{a}_{k-m}^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we obtain (2.13). □

Series representation for the polynomials $\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)$ is given by the following theorem:

Theorem 2.20. *Let $\left|\frac{\mu}{\vartheta}b^\omega\right| < 1$. For $k, v \in \mathbb{N}_0$, we have*

$$\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c) = 2^v\vartheta^{-2v}\sum_{m=0}^{\infty}\binom{2v+m-1}{m}\left(\frac{\mu}{\vartheta}\right)^m\sum_{j=0}^{k}\binom{k}{j}(\ln c)^j(\ln b)^{k-j}x^j m^{k-j}. \tag{2.14}$$

Proof. From (2.2), we get

$$\frac{2^v c^{\omega x}}{(\vartheta)^{2v}\left(-\frac{\mu}{\vartheta}b^\omega+1\right)^{2v}} = \sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!}.$$

Assuming that $\left|\frac{\mu}{\vartheta}b^\omega\right| < 1$, applying binomial series to the previous equation, we obtain

$$2^v\vartheta^{-2v}\sum_{m=0}^{\infty}\binom{2v+m-1}{m}\left(\frac{\mu}{\vartheta}\right)^m c^{\omega x}b^{m\omega} = \sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!}.$$

Therefore,

$$2^v\vartheta^{-2v}\sum_{m=0}^{\infty}\binom{2v+m-1}{m}\left(\frac{\mu}{\vartheta}\right)^m\sum_{k=0}^{\infty}x^k(\ln c)^k\frac{\omega^k}{k!}\sum_{k=0}^{\infty}m^k(\ln b)^k\frac{\omega^k}{k!} = \sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!}.$$

After some elementary calculations in the above equation, we get

$$2^v\vartheta^{-2v}\sum_{m=0}^{\infty}\binom{2v+m-1}{m}\left(\frac{\mu}{\vartheta}\right)^m\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(\ln c)^j(\ln b)^{k-j}x^j m^{k-j}\frac{\omega^k}{k!} = \sum_{k=0}^{\infty}\mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^k}{k!}.$$

Using the above equation, we arrive at the desired result. □

Substituting $x = 0$ into (2.14), we have the following corollary:

Corollary 2.21. *Let* $\left|\frac{\mu}{\vartheta}b^{\omega}\right| < 1$. *For* $k, v \in \mathbb{N}_0$, *we have*

$$\mathfrak{a}_k^{(v)}(\mu;\vartheta,b) = \frac{2^v (\ln b)^k}{\vartheta^{2v}} \sum_{m=0}^{\infty} \binom{2v+m-1}{m} \left(\frac{\mu}{\vartheta}\right)^m m^k. \tag{2.15}$$

We now give the following special cases for the equations (2.14) and (2.15), respectively:
Substitute $b = e$, $c = 1$ and $k = 0$ into (2.14), we arrive at the equation (2.3). That is, for $\left|\frac{\mu}{\vartheta}\right| < 1$, we also have

$$\begin{aligned}\mathfrak{a}_0^{(v)}(x,\mu;\vartheta,e,1) &= 2^v\vartheta^{-2v} \sum_{m=0}^{\infty} \binom{2v+m-1}{m} \left(\frac{\mu}{\vartheta}\right)^m \\ &= \frac{2^v}{(\mu-\vartheta)^{2v}}.\end{aligned}$$

Setting $b = 1$ and $c = e$ in (2.14), for $\left|\frac{\mu}{\vartheta}\right| < 1$, we have

$$\frac{2^v x^k}{(\mu-\vartheta)^{2v}} = \mathfrak{a}_k^{(v)}(x,\mu;\vartheta,1,e).$$

Putting $b = c = e$, $\mu = 1$ and $\vartheta = 2$ in (2.14), we obtain

$$a_k^{(v)}(x) = 2^{-v} \sum_{m=0}^{\infty} \binom{2v+m-1}{m} \left(\frac{1}{2}\right)^m (x+m)^k. \tag{2.16}$$

3. Hurwitz-Lerch type zeta functions

In this section, we define Hurwitz-Lerch type zeta functions, which interpolate the unified and modified of the Fubini numbers and polynomials.

Using (2.14), interpolation function for unified and modified of the Fubini polynomials is given by the following definition:

Definition 3.1. Let $\left|\frac{\mu}{\vartheta}b^{\omega}\right| < 1$ with $b, c \neq \{0, 1, -1, -2, -3, \cdots\}$. Let $v \in \mathbb{N}$. We define

$$\mathfrak{Z}_v(s;x,\mu;\vartheta,b,c) := 2^v\vartheta^{-2v} \sum_{m=0}^{\infty} \binom{2v+m-1}{m} \left(\frac{\mu}{\vartheta}\right)^m \sum_{j=0}^{\infty} \binom{-s}{j} \frac{(\ln c)^j (\ln b)^{-s-j} x^j}{m^{s+j}}, \tag{3.1}$$

where $s \in \mathbb{C}$ when $\left|\frac{\mu}{\vartheta}\right| < 1$; $\mathrm{Re}(s) > 1$ and $\left|\frac{\mu}{\vartheta}\right| = 1$.

Substituting $s = -k$, $k \in \mathbb{N}$, into (3.1), and using (2.14), we arrive at the following theorem:

Theorem 3.2. *Let* $k \in \mathbb{N}$ *and* $\left|\frac{\mu}{\vartheta}b^{\omega}\right| < 1$ *with* $b, c \neq \{0, 1, -1, -2, -3, \cdots\}$. *For* $v \in \mathbb{N}$, *we have*

$$\mathfrak{Z}_v(-k;x,\mu;\vartheta,b,c) = \mathfrak{a}_k^{(v)}(x,\mu;\vartheta,b,c).$$

By using (2.15), interpolation function for unified and modified of the Fubini numbers is given by the following definition:

Definition 3.3. Let $\left|\frac{\mu}{\vartheta}b^{\omega}\right| < 1$ with $b \neq \{0, 1, -1, -2, -3, \cdots\}$. Let $v \in \mathbb{N}$. We define

$$\mathfrak{Z}_v(s;\mu;\vartheta,b) := 2^v\vartheta^{-2v}(\ln b)^{-s} \sum_{m=0}^{\infty} \binom{2v+m-1}{m} \left(\frac{\mu}{\vartheta}\right)^m \frac{1}{m^s}, \tag{3.2}$$

where $s \in \mathbb{C}$ when $\left|\frac{\mu}{\vartheta}\right| < 1$; $\mathrm{Re}(s) > 1$ and $\left|\frac{\mu}{\vartheta}\right| = 1$.

Substituting $s = -k$, $k \in \mathbb{N}$ into (3.2), and using (2.15), we arrive at the following theorem:

Theorem 3.4. *Let $k \in \mathbb{N}$ and $\left|\frac{\mu}{\vartheta}b^{\omega}\right| < 1$ with $b \neq \{0, 1, -1, -2, -3, \cdots\}$. For $v \in \mathbb{N}$, we have*

$$\mathfrak{Z}_v(-k; \mu; \vartheta, b) := \mathfrak{a}_k^{(v)}(\mu; \vartheta, b).$$

In [30, Eq. (17)], Kucukoglu et al. introduced the Lerch-type zeta function as follows:

$$\zeta_w(s, x, v; \mu) = \sum_{m=0}^{\infty} (-1)^m \binom{2v + m - 1}{m} \frac{\mu^{m+v}}{(x + m + v)^s}, \tag{3.3}$$

where $\mu \in \mathbb{C}$ ($|\mu| < 1$), $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$.

Substituting $b = c = e$ and $\vartheta = 2$ into (3.1), we obtain

$$\mathfrak{Z}_v(s; x, \mu; 2, e, e) = 2^{-v} \sum_{m=0}^{\infty} \binom{2v + m - 1}{m} \left(\frac{\mu}{2}\right)^m \sum_{j=0}^{\infty} \binom{-s}{j} \frac{x^j}{m^{s+j}}.$$

By using

$$\binom{-s}{j} = (-1)^j \binom{j + s - 1}{j},$$

and binomial series in the above equation, we have the following result:

$$\mathfrak{Z}_v(s; x, \mu; 2, e, e) = 2^{-v} \sum_{m=0}^{\infty} \binom{2v + m - 1}{m} \left(\frac{\mu}{2}\right)^m \frac{1}{(m + x)^s}, \tag{3.4}$$

where $x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $s \in \mathbb{C}$ when $\left|\frac{\mu}{2}\right| < 1$; $\mathrm{Re}(s) > 1$ and $\left|\frac{\mu}{2}\right| = 1$.

From (3.3), we get

$$\zeta_w(s, x - v, v; -\mu) = (-1)^v \mu^v \sum_{m=0}^{\infty} \binom{2v + m - 1}{m} \frac{\mu^m}{(x + m)^s}, \tag{3.5}$$

where $x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $s \in \mathbb{C}$ when $|\mu| < 1$; $\mathrm{Re}(s) > 1$ and $|\mu| = 1$.

Combining (3.4) with (3.5), we obtain the following corollary:

Corollary 3.5.

$$\mathfrak{Z}_v(s; x, 2\mu; 2, e, e) = (-2\mu)^{-v} \zeta_w(s, x - v, v; -\mu).$$

Remark 3.6. Substituting $v = 1$ into (3.5), we get

$$\zeta_w(s, x - 1, 1; -\mu) = -\mu \sum_{m=0}^{\infty} \frac{(m + 1)\mu^m}{(x + m)^s}.$$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $|\mu| < 1$, the series $\sum_{m=0}^{\infty} \frac{(m+1)\mu^m}{(x+m)^s}$ is an absolutely convergent, thus we have

$$\begin{aligned} \zeta_w(s, x - 1, 1; -\mu) &= -\mu \frac{\partial}{\partial \mu} \left\{ \sum_{m=0}^{\infty} \frac{\mu^{m+1}}{(x + m)^s} \right\} \\ &= -\mu \frac{\partial}{\partial \mu} \{\mu \Phi(\mu, s, x)\}, \end{aligned}$$

where $\Phi(\mu, s, x)$ denotes the Hurwitz-Lerch zeta function. This function is an interpolation function of the Apostol-Bernoulli polynomials. This function is defined as follows:

$$\Phi(\mu, s, x) = \sum_{m=0}^{\infty} \frac{\mu^m}{(x + m)^s},$$

where $x \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$, $s \in \mathbb{C}$ when $|\mu| < 1$; $\mathrm{Re}(s) > 1$ when $|\mu| = 1$, and furthermore, $\zeta(s,x) := \Phi(1,s,x)$ denotes the Hurwitz zeta function, which is interpolation function for the Bernoulli polynomials, $\zeta(s) := \Phi(1,s,1)$ denotes the Riemann zeta function, which is interpolation function for the Bernoulli numbers (*cf.* [49, p. 201]; see also [55]).

Remark 3.7. Substituting $s = -k$, $k \in \mathbb{N}$ into (3.3), Kucukoglu et al. [30, Eq. (16)] gave the following Lerch-type zeta function, interpolation function of the polynomials $W_k^{(v)}(x;\mu)$:

$$W_k^{(v)}(x;\mu) = \sum_{m=0}^{\infty} (-1)^m \binom{2v+m-1}{m} \mu^{m+v} (x+m+v)^k. \tag{3.6}$$

A relation between the polynomials $W_k^{(v)}(x;\mu)$ and the polynomials $a_n^{(v)}(x)$ is given by

$$a_k^{(v)}(x) = (-1)^v\, W_k^{(v)}\left(x - v; -\frac{1}{2}\right) \tag{3.7}$$

(*cf.* [17]). By combining (3.6) with (3.7), we also arrive at the equation (2.16).

Remark 3.8. In [3], Aygunes and Simsek also studied on the unification of the multiple Lerch zeta type functions, which are interpolated the unification of the Bernoulli, Euler and Genocchi numbers of higher order.

4. Relations among unified and modified presentation of the Fubini polynomials, Apostol type polynomials and Stirling type numbers

In this section, using generating functions for the unified and modified presentation of the Fubini numbers and polynomials, we give some formulas related to these polynomials, the Apostol-Fubini type polynomials and numbers, the Stirling type numbers, the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, and the generalized Bernoulli, Euler and Genocchi polynomials. We also give some remarks on these formulas.

Theorem 4.1. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\sum_{r=0}^{k} \binom{k}{r} \mathcal{B}_{k-r}^{(2v)}\left(\sqrt{\frac{\mu}{\vartheta}}\right) \mathcal{E}_r^{(2v)}\left(\sqrt{\frac{\mu}{\vartheta}}\right) = 2^{k-v}\vartheta^{2v} (\ln b)^{2v-k} \{k\}_{2v}\, \mathfrak{a}_{k-2v}^{(v)}(\mu;\vartheta,b), \tag{4.1}$$

where $\frac{\mu}{\vartheta}$ is any nonzero complex number, $\sqrt{\frac{\mu}{\vartheta}} = e^{\frac{1}{2}\ln\left(\frac{\mu}{\vartheta}\right)}$, and the principal value of the $\arg\left(\frac{\mu}{\vartheta}\right)$ is given in $0 < \arg\left(\frac{\mu}{\vartheta}\right) < \pi$.

Proof. By (1.1), (1.2) and (2.2), we have

$$\mathcal{M}_b\left(\frac{\omega}{2}\ln b, 2v, 0; \sqrt{\frac{\mu}{\vartheta}}\right) \mathcal{M}_e\left(\frac{\omega}{2}\ln b, 2v, 0; \sqrt{\frac{\mu}{\vartheta}}\right) = 2^{-v}\vartheta^{2v}(\ln b)^{2v}\,\omega^{2v}\mathcal{N}_f(\omega, v, \mu; \vartheta, b).$$

From the above functional equation, we get

$$\sum_{k=0}^{\infty} \mathcal{B}_k^{(2v)}\left(\sqrt{\frac{\mu}{\vartheta}}\right)\left(\frac{\ln b}{2}\right)^k \frac{\omega^k}{k!} \sum_{k=0}^{\infty} \mathcal{E}_k^{(2v)}\left(\sqrt{\frac{\mu}{\vartheta}}\right)\left(\frac{\ln b}{2}\right)^k \frac{\omega^k}{k!} = 2^{-v}\vartheta^{2v}(\ln b)^{2v} \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(\mu;\vartheta,b)\frac{\omega^{k+2v}}{k!}.$$

Hence,

$$\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\left(\frac{\ln b}{2}\right)^k \mathcal{B}_{k-r}^{(2v)}\left(\sqrt{\frac{\mu}{\vartheta}}\right)\mathcal{E}_r^{(2v)}\left(\sqrt{\frac{\mu}{\vartheta}}\right)\frac{\omega^k}{k!} = 2^{-v}\vartheta^{2v}(\ln b)^{2v}\sum_{k=0}^{\infty}\{k\}_{2v}\,\mathfrak{a}_{k-2v}^{(v)}(\mu;\vartheta,b)\frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the (4.1). □

Theorem 4.2. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\mathfrak{a}_{k-2v}^{(v)}(x,\mu;\vartheta,b,c) = \frac{2^{v}\vartheta^{-2v}\mathfrak{B}_{k}^{(2v)}\left(x;\frac{\mu}{\vartheta};1,b,c\right)}{\{k\}_{2v}}. \tag{4.2}$$

Proof. When $a = 1$ in (1.4) and using (2.2), we have

$$2^{-v}\vartheta^{2v}\omega^{2v}\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^{k}}{k!} = \sum_{k=0}^{\infty}\mathfrak{B}_{k}^{(2v)}\left(x;\frac{\mu}{\vartheta};1,b,c\right)\frac{\omega^{k}}{k!}.$$

Hence,

$$2^{-v}\vartheta^{2v}\sum_{k=0}^{\infty}\{k\}_{2v}\,\mathfrak{a}_{k-2v}^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^{k}}{k!} = \sum_{k=0}^{\infty}\mathfrak{B}_{k}^{(2v)}\left(x;\frac{\mu}{\vartheta};1,b,c\right)\frac{\omega^{k}}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

When $\vartheta = 1$ in (4.2), we have the following corollary:

Corollary 4.3. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\mathfrak{a}_{k-2v}^{(v)}(x,\mu;1,b,c) = \frac{2^{v}\mathfrak{B}_{k}^{(2v)}(x;\mu;1,b,c)}{\{k\}_{2v}}.$$

Taking $\vartheta = 2$ and $b = c = e$ in (4.2), we have the following result:

Corollary 4.4. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$\mathfrak{a}_{k}^{(v)}(x;\mu) = \frac{\mathcal{B}_{k+2v}^{(2v)}\left(x;\frac{\mu}{2}\right)}{2^{v}\{k+2v\}_{2v}}. \tag{4.3}$$

Remark 4.5 (*cf.* [10, 12]). Substituting $\mu = 1$ into (4.3), we have

$$a_{k}^{(v)}(x) = \frac{1}{2^{v}\{k+2v\}_{2v}}\mathcal{B}_{k+2v}^{(2v)}\left(x;\frac{1}{2}\right). \tag{4.4}$$

Theorem 4.6. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$\mathfrak{a}_{k}^{(v)}(x,\mu;\vartheta,b,c) = 2^{-v}\vartheta^{-2v}\mathfrak{E}_{k}^{(2v)}\left(x;-\frac{\mu}{\vartheta};1,b,c\right). \tag{4.5}$$

Proof. From (1.5) and (2.2), we obtain

$$2^{v}\vartheta^{2v}\sum_{k=0}^{\infty}\mathfrak{a}_{k}^{(v)}(x,\mu;\vartheta,b,c)\frac{\omega^{k}}{k!} = \sum_{k=0}^{\infty}\mathfrak{E}_{k}^{(v)}\left(x;-\frac{\mu}{\vartheta};1,b,c\right)\frac{\omega^{k}}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

Taking $\vartheta = -1$ in (4.5), we have the following corollary:

Corollary 4.7. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$\mathfrak{a}_{k}^{(v)}(x,\mu;-1,b,c) = 2^{-v}\mathfrak{E}_{k}^{(2v)}(x;\mu;1,b,c).$$

Substituting $\vartheta = 2$ and $b = c = e$ into (4.5), we have the following result:

Corollary 4.8. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$\mathfrak{a}_{k}^{(v)}(x;\mu) = 2^{-3v}\mathcal{E}_{k}^{(2v)}\left(x;-\frac{\mu}{2}\right). \tag{4.6}$$

Remark 4.9 (*cf.* [11, Eq. (10)]). When $\mu = 1$ in (4.6), we have

$$a_k^{(v)}(x) = 2^{-3v}\mathcal{E}_k^{(2v)}\left(x; -\frac{1}{2}\right). \tag{4.7}$$

Theorem 4.10. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\mathfrak{a}_{k-2v}^{(v)}(x, \mu; \vartheta, b, c) = \frac{\mathfrak{G}_k^{(2v)}\left(x; -\frac{\mu}{\vartheta}; 1, b, c\right)}{2^v \vartheta^{2v}\{k\}_{2v}}. \tag{4.8}$$

Proof. Using (1.6) and (2.2), we get

$$2^v \vartheta^{2v} \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(x, \mu; \vartheta, b, c) \frac{\omega^{k+2v}}{k!} = \sum_{k=0}^{\infty} \mathfrak{G}_k^{(v)}\left(x; -\frac{\mu}{\vartheta}; 1, b, c\right) \frac{\omega^k}{k!}.$$

After some calculations, comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the final equation, we have the desired result. □

When $\vartheta = -1$ in (4.8), we obtain the following result:

Corollary 4.11. *Let $k, v \in \mathbb{N}_0$ with $k \geq 2v$. Then we have*

$$\mathfrak{a}_{k-2v}^{(v)}(x, \mu; -1, b, c) = \frac{2^{-v}\mathfrak{G}_k^{(2v)}(x; \mu; 1, b, c)}{\{k\}_{2v}}.$$

Putting $\vartheta = 2$ and $b = c = e$ in (4.8), we have the following result:

Corollary 4.12. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$\mathcal{G}_{k+2v}^{(2v)}\left(x; -\frac{\mu}{2}\right) = 2^{3v}\{k+2v\}_{2v}\, \mathfrak{a}_k^{(v)}(x; \mu). \tag{4.9}$$

Remark 4.13 (*cf.* [10, 12]). Substituting $\mu = 1$ into (4.9), we get

$$a_k^{(v)}(x) = \frac{\mathcal{G}_{k+2v}^{(2v)}\left(x; -\frac{1}{2}\right)}{2^{3v}\{k+2v\}_{2v}}. \tag{4.10}$$

Remark 4.14. Combining (4.4), (4.7) and (4.10) with (3.7), respectively, we obtain the special case of the results in Kucukoglu and Simsek, see for detail [27, Eqs. (27), (31) and (35)].

Theorem 4.15. *Let $k, v \in \mathbb{N}_0$. Then we have*

$$\sum_{r=0}^{k} \binom{k}{r} \mathfrak{a}_{k-r}^{(v)}(x, \mu; 1, b, c)\, \mathcal{S}(r, 2v; 1, b; \mu) = \frac{2^v x^k (\ln c)^k}{(2v)!}. \tag{4.11}$$

Proof. Substituting $\vartheta = 1$ into (2.2), we get

$$\frac{2^v c^{\omega x}}{(2v)!} = \frac{(\mu b^{\omega} - 1)^{2v}}{(2v)!} \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(x, \mu; 1, b, c) \frac{\omega^k}{k!}.$$

Combining the above equation with (1.9), we find that

$$\frac{2^v}{(2v)!} \sum_{k=0}^{\infty} (\ln c)^k x^k \frac{\omega^k}{k!} = \sum_{k=0}^{\infty} \mathcal{S}(k, 2v; 1, b; \mu) \frac{\omega^k}{k!} \sum_{k=0}^{\infty} \mathfrak{a}_k^{(v)}(x, \mu; 1, b, c) \frac{\omega^k}{k!}.$$

Thus,

$$\frac{2^v}{(2v)!} \sum_{k=0}^{\infty} (\ln c)^k x^k \frac{\omega^k}{k!} = \sum_{k=0}^{\infty} \sum_{r=0}^{k} \binom{k}{r} \mathcal{S}(r, 2v; 1, b; \mu)\, \mathfrak{a}_{k-r}^{(v)}(x, \mu; 1, b, c) \frac{\omega^k}{k!}.$$

Comparing the coefficients of $\frac{\omega^k}{k!}$ on both sides of the previous equation, we arrive at the desired result. □

Setting $x = 0$ in (4.11), we get the following corollary:

Corollary 4.16. *Let $k \in \mathbb{N}$. Then we have*

$$\sum_{r=0}^{k} \binom{k}{r} \mathfrak{a}_{k-r}^{(v)}(\mu; 1, b)\, \mathcal{S}(r, 2v; 1, b; \mu) = 0. \tag{4.12}$$

Remark 4.17. Putting $\vartheta = 2$, $b = c = e$ and $\mu = 1$ in (2.2) and using (1.9) for $d = e$, $a = 1$ and $\beta = \frac{1}{2}$, we obtain the Proposition 4.2 in [10, P. 28]. Further, when $x = 0$ in the this final equation, we also obtain the Corollary 4.3 in [10, P. 29], see also [12].

5. Conclusions

We constructed generating functions for new families of the unified and modified presentation of the Fubini numbers and polynomials. By using these generating functions and their derivative and functional equations, we not only investigated the properties of these polynomials and numbers, but also we gave derivative formula and many new identities and relations including the Stirling type numbers, the Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi numbers and polynomials, the combinatorial type numbers, the Fubini type polynomials and numbers, the generalized Eulerian type polynomials, the Frobenius-Euler polynomials and also the generalized Bernoulli, Euler and Genocchi polynomials. Moreover, we calculated few numerical values for these numbers and polynomials. Additionally, we introduced Hurwitz-Lerch type zeta functions, which interpolate the unified and modified presentation of the Fubini numbers and polynomials.

Consequently, the results of this paper not only may have the potential to be used (effectively) in many different areas of mathematics and other applied sciences, but also may use in combinatorics and in analytic number theory.

Acknowledgments

The authors would like to thank to all referees for their valuable comments.

Author Contributions: All authors contribute equally to this paper.

Conflict of Interest: The authors declare no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] N. G. Acala, *A unification of the generalized multiparameter Apostol-type Bernoulli, Euler, Fubini, and Genocchi polynomials of higher order*, Eur. J. Pure Appl. Math. **13 (3)**, 587–607, 2020.

[2] M. Alkan and Y. Simsek, *Generating function for q-Eulerian polynomials and their decomposition and applications*, Fixed Point Theory Appl. **2013**, 2013; Article ID: 72.

[3] A. A. Aygunes and Y. Simsek, *Unification of multiple Lerch-zeta type function*s, Adv. Stud. Contemp. Math. **21 (4)**, 367–373, 2011.

[4] A. Bayad and J. Chikhi, *Reduction and duality of the generalized Hurwitz-Lerch zetas*, Fixed Point Theory Appl. **2023**, 2013; Article ID: 82.

[5] A. Bayad and T. Kim, *Identities for Apostol-type Frobenius-Euler polynomials resulting from the study of a nonlinear operator*, Russ. J. Math. Phys. **23 (2)**, 164–171, 2016.

[6] A. Bayad, Y. Simsek and H. M. Srivastava, *Some array type polynomials associated with special numbers and polynomials*, Appl. Math. Comput. **244**, 149–157, 2014.

[7] M. A. Evgrafov, *Analytic functions*, W. B. Saunders Company, Philadelphia and London, 1966.

[8] D. Gun, A. Bayad and Y. Simsek, *Identities derived from generating functions for generalized special combinatorial numbers and polynomials*, Preprint.

[9] D. Gun and Y. Simsek, *Modification exponential Euler type spline derive from Apostol-Euler numbers and polynomials of complex order*, Preprint.

[10] N. Kilar, *Fubini type numbers and their generating functions*, Akdeniz University, MSc Thesis in Mathematics, Antalya, 2017.

[11] N. Kilar, *Remarks and some formulas on Fubini type numbers and polynomials,* In: Proceedings Book of MICOPAM 2020-2021 (Ed. by Y. Simsek, M. Alkan, I. Kucukoglu and O. Öneş), The 3rd & 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021), Antalya, Tukey, 45–49, 2021.

[12] N. Kilar, *Combinatorial sums and identities associated with functional equations of generating functions for Fubini type polynomials*, GUJ Sci **36 (2)**, 2023.
[13] N. Kilar, *Formulas for Fubini type numbers and polynomials of negative higher order*, Montes Taurus J. Pure Appl. Math. **5 (3)**, 23–36, 2023.
[14] N. Kilar, *Notes on Fubini type numbers and their properties*, Accepted for the publication in AIP Conf. Proc. 2023.
[15] N. Kilar and Y. Simsek, *A new family of Fubini numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials*, J. Korean Math. Soc. **54 (5)**, 1605–1621, 2017.
[16] N. Kilar and Y. Simsek, *Identities and relations for Fubini type numbers and polynomials via generating functions and p-adic integral approach*, Publ. Inst. Math., Nouv. Sér. **106 (120)**, 113–123, 2019.
[17] N. Kilar and Y. Simsek, *Some relationships between Fubini type polynomials and other special numbers and polynomials*, AIP Conf. Proc. **2116**, 2019; Article ID: 100017.
[18] N. Kilar and Y. Simsek, *Computational formulas and identities for new classes of Hermite-based Milne–Thomson type polynomials: Analysis of generating functions with Euler's formula*, Math. Methods Appl. Sci. **44**, 11245–11268, 2021.
[19] N. Kilar and Y. Simsek, *Combinatorial sums involving Fubini type numbers and other special numbers and polynomials: Approach trigonometric functions and p-adic integrals*, Adv. Stud. Contemp. Math. **31 (1)**, 75–87, 2021.
[20] N. Kilar and Y. Simsek, *Identities and relations for Hermite-based Milne-Thomson polynomials associated with Fibonacci and Chebyshev polynomials*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **115**, 2021; Article ID: 28.
[21] T. Kim, *The modified q-Euler numbers and polynomials*, Adv. Stud. Contemp. Math. **16**, 161–170, 2008.
[22] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory **132 (12)**, 2854–2865, 2012.
[23] T. Kim, D. S. Kim and G.-W. Jang, *A note on degenerate Fubini polynomials*, Proc. Jangjeon Math. Soc. **20 (4)**, 521–531, 2017.
[24] D. S. Kim, T. Kim, H.-I. Kwon and J.-W. Park, *Two variable higher-order Fubini polynomials*, J. Korean Math. Soc. **55 (4)**, 975–986, 2018.
[25] D. Kim, H. Ozden Ayna, Y. Simsek and A. Yardimci, *New families of special numbers and polynomials arising from applications of p-adic q-integrals*, Adv. Difference Equ. **2017**, 2017; Article ID: 207.
[26] I. Kucukoglu and Y. Simsek, *Identities and derivative formulas for the combinatorial and Apostol-Euler type numbers by their generating functions*, Filomat **32 (20)**, 6879–6891, 2018.
[27] I. Kucukoglu and Y. Simsek, *Remarks on recurrence formulas for the Apostol-type numbers and polynomials*, Adv. Stud. Contemp. Math. **28 (4)**, 643–657, 2018.
[28] I. Kucukoglu and Y. Simsek, *Numerical evaluation of special power series including the numbers of Lyndon words: An approach to interpolation functions for Apostol-type numbers and polynomials*, Electron. Trans. Numer. Anal. **50**, 98–108, 2018.
[29] I. Kucukoglu and Y. Simsek, *Identities and relations on the q-Apostol type Frobenius-Euler numbers and polynomials*, J. Korean Math. Soc. **56 (1)**, 265–284, 2019.
[30] I. Kucukoglu, Y. Simsek and H. M. Srivastava, *A new family of Lerch-type zeta functions interpolating a certain class of higher-order Apostol-type numbers and Apostol-type polynomials*, Quaest. Math. **42 (4)**, 465–478, 2019.
[31] B. Kurt, *Unified degenerate Apostol-type Bernoulli, Euler, Genocchi, and Fubini polynomials*, J. Math. Computer Sci. **25 (3)**, 259–268, 2022.
[32] Q.-M. Luo, *Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions*, Taiwanese J. Math. **10 (4)**, 917–925, 2006.
[33] Q.-M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl. **308**, 290–302, 2005.
[34] M. A. Özarslan, *Unified Apostol–Bernoulli, Euler and Genocchi polynomials*, Comput. Math. Appl. **62 (6)**, 2452–2462, 2011.
[35] H. Ozden, *Unification of generating function of the Bernoulli, Euler and Genocchi numbers and polynomials*, AIP Conf. Proc. **1281**, 2010; Article ID: 1125.
[36] H. Ozden, *Generating functions of the unified representation of the Bernoulli, Euler and Genocchi polynomials of higher order*, AIP Conf. Proc. **1389**, 2011; Article ID: 349.
[37] H. Ozden, Y. Simsek and H. M. Srivastava, *A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials*, Comput. Math. Appl. **60 (10)**, 2779–2787, 2010.
[38] Y. Simsek, *Generating functions for q-Apostol type Frobenius-Euler numbers and polynomials*, Axioms **1 (3)**, 395–403, 2012; doi:10.3390/axioms1030395.
[39] Y. Simsek, *Identities associated with generalized Stirling type numbers and Eulerian type polynomials*, Math. Comput. Appl. **18 (3)**, 251–263, 2013.
[40] Y. Simsek, *Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Theory Appl. **2013**, 2013; Article ID: 87.
[41] Y. Simsek, *Computation methods for combinatorial sums and Euler-type numbers related to new families of numbers*, Math. Methods Appl. Sci. **40 (7)**, 2347–2361, 2017.
[42] Y. Simsek, *On generating functions for the special polynomials*, Filomat **31 (1)**, 9–16, 2017.
[43] Y. Simsek, *Interpolation function for the families of numbers related to the Apostol-type numbers*, AIP Conf. Proc. **1978**, 2018; Article ID: 040012.
[44] Y. Simsek, *Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.
[45] Y. Simsek, *Interpolation functions for new classes special numbers and polynomials via applications of p-adic integrals and derivative operator*, Montes Taurus J. Pure Appl. Math. **3 (1)**, 38–61, 2021.
[46] Y. Simsek, *Applications of Apostol-type numbers and polynomials: Approach to techniques of computation algorithms in approximation and interpolation functions*, In: Approximation and Computation in Science and Engineering (Ed. by N. J. Daras and Th. M. Rassias), Springer Optimization and Its Applications **180**, 783–860, 2022.
[47] H. M. Srivastava, *Some families of generating functions associated with orthogonal polynomials and other higher transcendental functions*, Mathematics **10 (20)**, 2022; Article ID: 3730.
[48] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomial*s, Appl. Math. Inf. Sci.

5 (3), 390–444, 2011.

[49] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier Science Publishers, Amsterdam, 2012.

[50] H. M. Srivastava, M. Garg and S. Choudhary, *A new generalization of the Bernoulli and related polynomials*, Russian J. Math. Phys. **17**, 251–261, 2010.

[51] H. M. Srivastava, M. Garg and S. Choudhary, *Some new families of generalized Euler and Genocchi polynomials*, Taiwanese J. Math. Taiwanese **15 (1)**, 283–305, 2011.

[52] H. M. Srivastava, S. Khan, S. Araci, M. Acikgoz and M. Riyasat, *A general class of the three-variable unified Apostol-type q-polynomials and multiple power q-sums*, Bull. Iranian Math. Soc. **46**, 519–542, 2020.

[53] H. M. Srivastava and C. Kızılateş, *A parametric kind of the Fubini-type polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**, 3253–3267, 2019.

[54] H. M. Srivastava, R. Srivastava, A. Muhyi, G. Yasmin, H. Islahi and S. Araci, *Construction of a new family of Fubini-type polynomials and its applications*, Adv. Difference Equ. **2021**, 2021; Article ID: 36.

[55] H. M. Srivastava, T. Kim and Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys. **12**, 241–268, 2005.

[56] H. M. Srivastava and P. G. Todorov, *An explicit formula for the generalized Bernoulli polynomials*, J. Math. Anal. Appl. **130**, 509–513, 1988.

[57] P. G. Todorov, *Une formule simple explicite des nombres de Bernoulli generalises*, C. R. Acad. Sci. Paris Sér. I Math. **301**, 665–666, 1985.