Montes Taurus J. Pure Appl. Math. 5 (1), 22–33, 2023

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Solutions of high-order linear Volterra integro-differential equations via Lucas polynomials

Deniz Elmacı [a], Nurcan Baykuş Savaşaneril [b]

[a]*Dokuz Eylul University, Bergama Vocational School, Izmir, Turkey*
[b]*Dokuz Eylul University, Izmir Vocational School, Izmir, Turkey*

Abstract

It is often not possible to find the analytical solution of every type of equation encountered in physics and engineering applications. For this reason, approximate solution methods are needed and errors occur in the solutions obtained by such methods. Therefore, besides being practical and useful, solution methods that give the best approach are sought. For this purpose a matrix method called the Lucas collocation method is presented for numerically solving high-order linear Volterra integro-differential equations under mixed conditions in this paper. Numerical results were compared and interpreted with tables and graphs and the solution was shown to be consistent. The outcomes demonstrate the effectiveness and precision of the current work. On the computer, a MATLAB program was used to run all of the numerical calculations.

Keywords: 47G20, 11B39, 40C05, 65L60

2010 MSC: Volterra integro-differential equations, Lucas series and polynomials, Lucas matrix method, collocation points

1. Introduction

In general, it is possible to transform some fundamental problems in engineering and science into integral and integro-differential equations. Integro-differential equations have attracted a lot of interest, and solving them has been one of mathematicians' most intriguing missions [4, 6], [17]-[19], [25]. For the numerical solution of these equations, several strategies have been developed [5], [7]-[10], [21, 23, 24].
In this research, a method based on Lucas polynomials to solve Volterra integro-differential equations (VIDE) is presented. The equation that we are going to investigate is

$$\sum_{k=0}^{m} P_k(\xi)y^{(k)}(\xi) = g(\xi) + \int_a^{\xi} K_v(\xi,\varsigma)\, y(\varsigma)\, d\varsigma \quad a \le \xi, \varsigma \le b, \tag{1.1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}\left(a_{sk}y^{(k)}(a) + b_{sk}y^{(k)}(b) + c_{sk}y^{(k)}(c)\right) = \lambda_s, \quad s = 0, 1, ..., m-1, \tag{1.2}$$

†*Article ID:* MTJPAM-D-22-00026

Email addresses: deniz.elmaci@deu.edu.tr (Deniz Elmacı), nurcan.savasaneril@deu.edu.tr (Nurcan Baykuş Savaşaneril)
Received:15 August 2022, Accepted:29 November 2022, Published:28 December 2022
⋆*Corresponding Author:* Deniz Elmacı

This work is licensed under a Creative Commons Attribution 4.0 International License.

where $P_k(\xi)$, $g(\xi)$ and $K_v(\xi,\varsigma)$ are functions defined on the interval $a \le \xi, \varsigma \le b$; a_{sk}, b_{sk}, c_{sk} and λ_s are appropriate constants; $y(\xi)$ is an unknown solution function to be determined.
We presume that the problem Eq. (1.1) - Eq. (1.2) has been approximately solved in the form of truncated Lucas polynomials for our purposes as follows:

$$y(\xi) \cong y_N(\xi) = \sum_{n=0}^{N} a_n \mathrm{L}_n(\xi), \qquad a \leqslant \xi \leqslant b, \tag{1.3}$$

where $a_n, n = 0, 1, 2, ..., N$ are unknown coefficients to be determined and $L_n(\xi)$ indicates the Lucas polynomials which are originally studied in 1970 by Bicknell [3]. Lucas polynomials are defined recursively as follows [2, 11, 12]:

$$L_{n+1}(\xi) = \xi L_n(\xi) + L_{n-1}(\xi), \quad n \geqslant 1, \quad L_0(\xi) = 2, \quad L_1(\xi) = \xi. \tag{1.4}$$

The Binet's formula for Lucas polynomials is

$$L_n(\xi) = \frac{(\xi + \sqrt{\xi^2+4})^n + (\xi - \sqrt{\xi^2+4})^n}{2^n}, \quad n \geq 0. \tag{1.5}$$

Their explicit form for $n \geqslant 1$ is

$$L_n(\xi) = \sum_{n=0}^{\frac{n}{2}} \frac{n}{n-k} \binom{n-k}{k} \xi^{n-2k}, \tag{1.6}$$

where ξ is the largest integer smaller than or equal to ξ.
By using equations in (1.4)-(1.6) the first Lucas polynomials respectively are given by

$$L_0(\xi) = 2, \quad L_1(\xi) = \xi, \quad L_2(\xi) = \xi^2 + 2, \quad L_3(\xi) = \xi^3 + 3\xi,$$

$$L_4(\xi) = \xi^4 + 4\xi^2 + 2, \quad L_5(\xi) = \xi^5 + 5\xi^3 + 5\xi, \quad L_6(\xi) = \xi^6 + 6\xi^4 + 9\xi^2 + 2.$$

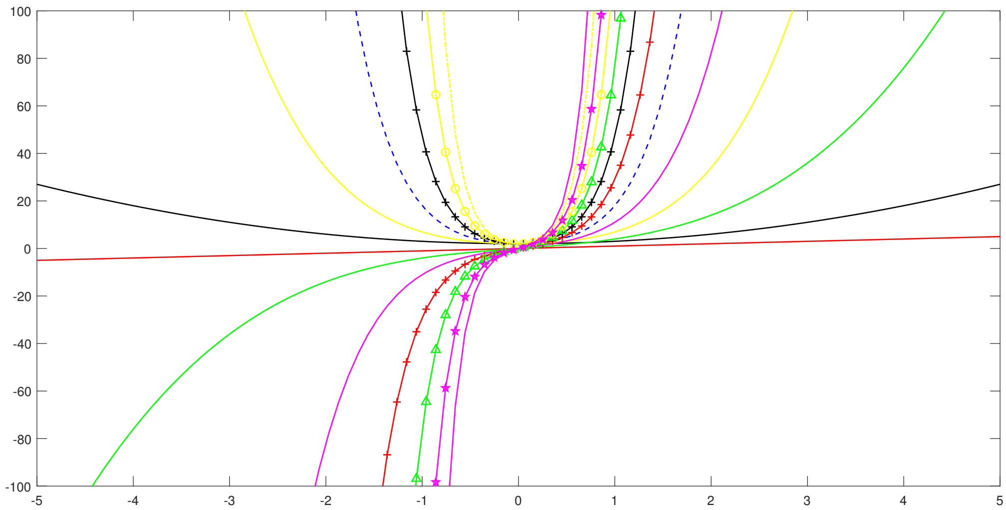

Figure 1. Graph of the Lucas polynomials

2. Main results

2.1. Matrix relations

The following process is used in this section to convert the expressions defined in Eq. (1.1) and Eq. (1.2) into matrix forms. First, the function $y(\xi)$ defined by Eq. (1.3) can be expressed in matrix form.

$$y(\xi) \cong y_N(\xi) = \mathbf{L}(\xi)\,\mathbf{A} \quad \mathbf{L}(\xi) = \mathbf{T}(\xi)\,\mathbf{D}^T, \tag{2.1}$$

where

$$\mathbf{L}(\xi) = \left[L_0(\xi)\ L_1(\xi)\ \cdots\ L_N(\xi)\right], \quad \mathbf{A} = [a_0\ a_1\ \cdots\ a_N]^T, \ \ \mathbf{T}(\xi) = \left[1\ \xi\ \xi^2\ \cdots\ \xi^N\right].$$

If N is odd,

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & \\ \frac{N-1}{(\frac{N-1}{2})}\binom{\frac{N-1}{2}}{\frac{N-1}{2}} & 0 & \frac{N-1}{(\frac{N+1}{2})}\binom{\frac{N+1}{2}}{\frac{N-3}{2}} & 0 & \cdots & \cdots & \frac{N-1}{(\frac{2N-2}{2})}\binom{\frac{2N-1}{2}}{0} & 0 \\ 0 & \frac{N}{(\frac{N+1}{2})}\binom{\frac{N+1}{2}}{\frac{N-1}{2}} & 0 & \frac{N}{(\frac{N+3}{2})}\binom{\frac{N+3}{2}}{\frac{N-3}{2}} & \cdots & \cdots & 0 & \frac{N}{N}\binom{N}{0} \end{bmatrix}$$

and if N is even,

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & \\ 0 & \frac{N-1}{(\frac{N}{2})}\binom{\frac{N}{2}}{\frac{N-2}{2}} & 0 & \frac{N-1}{(\frac{N+2}{2})}\binom{\frac{N+2}{2}}{\frac{N-4}{2}} & \cdots & \cdots & 0 & \frac{N-1}{N-1}\binom{N-1}{0} \\ \frac{N}{(\frac{N}{2})}\binom{\frac{N}{2}}{\frac{N}{2}} & 0 & \frac{N}{(\frac{N+2}{2})}\binom{\frac{N+2}{2}}{\frac{N-2}{2}} & 0 & \cdots & \cdots & \frac{N}{(\frac{2N}{2})}\binom{\frac{2N}{2}}{0} & 0 \end{bmatrix}.$$

From the matrix relations Eq. (2.1), it follows that

$$y_N(\xi) = \mathbf{T}(\xi)\,\mathbf{D}^T\mathbf{A}. \tag{2.2}$$

Besides, the relation between the matrix $\mathbf{T}(\xi)$ and its derivatives are

$$\mathbf{T}^{(k)}(\xi) = \mathbf{T}(\xi)\,\mathbf{B}^k,$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

By using Eq. (2.1) and Eq. (2.2), we have the matrix relation

$$y_N^{(k)}(\xi) = \mathbf{T}(\xi)\,\mathbf{B}^k\mathbf{D}^T\mathbf{A}, \quad k = 0, 1, 2, \ldots. \tag{2.3}$$

Additionally, the kernel function $K_v(\xi, \varsigma)$ can be approximated by the truncated Maclaurin series [13]

$$K_v(\xi, \varsigma) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn}\xi^m\varsigma^n,$$

where $\mathbf{K}_v = \mathrm{K} = [k_{mn}], \quad m,n = 0,1,\ldots,N$

$$k_{mn} = \frac{1}{m!n!}.\frac{\partial^{m+n}\mathrm{K}(0,0)}{\partial\xi^m\partial\varsigma^n}.$$

The expression's matrix form is obtained as follows:

$$K_v(\xi,\varsigma) = \mathbf{T}(\xi)\mathbf{K}_v\mathbf{T}(\varsigma)^T, \tag{2.4}$$

$$\int_a^{\xi} \mathrm{K}_v(\xi,\varsigma)\,y(\varsigma)\,d\varsigma = \mathbf{T}(\xi)\mathbf{K}_v\mathbf{Q}_v\mathbf{D}^T\mathbf{A}, \tag{2.5}$$

where

$$\mathbf{Q}_v = [q^v{}_{mn}] = \int_a^{\xi} \mathbf{T}^T(\varsigma)\mathbf{T}(\varsigma)d\varsigma,$$

$$q^v{}_{mn} = \frac{\xi^{m+n+1} - a^{m+n+1}}{m+n+1}, \quad m,n = 0,1,\ldots,N.$$

We first compute the Lucas coefficients using the collocation points provided by Eq. (1.3) to get the Lucas polynomials solution of Eq. (1.1) as follows:

$$\xi_i = a + \frac{b-a}{N}i, \quad i = 0,1,\ldots,N.$$

The following steps are taken to obtain the matrix equation system:

$$\sum_{k=0}^{m} P_k(\xi_i)\,y^{(k)}(\xi_i) = g(\xi_i) + \int_a^{\xi} K_v(\xi_i,\varsigma_i)\,y(\varsigma_i)\,d\varsigma. \tag{2.6}$$

The fundamental matrix equation corresponding to the VIDEs is constructed by substituting the matrix relations Eqs. (2.3)-(2.5) into Eq. (1.1):

$$\sum_{k=0}^{m} P_k(\xi_i)\,\mathbf{T}(\xi_i)\,\mathbf{B}^k\mathbf{D}^T\mathbf{A} = g(\xi_i) + \overline{\mathbf{T}}(\xi_i)\overline{\mathbf{K}}_v\overline{\mathbf{Q}}_v\mathbf{D}^T\mathbf{A} \tag{2.7}$$

or briefly,

$$\sum_{k=0}^{m} \mathbf{P}_k\mathbf{T}\mathbf{B}^k\mathbf{D}^T\mathbf{A} - \overline{\mathbf{T}\mathbf{K}_v}\overline{\mathbf{Q}}_v\mathbf{D}^T\mathbf{A} = \mathbf{G}, \tag{2.8}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(\xi_0) & 0 & \cdots & 0 \\ 0 & P_k(\xi_1) & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & P_k(\xi_N) \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}(\xi_0) \\ \mathbf{T}(\xi_1) \\ \vdots \\ \mathbf{T}(\xi_N) \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} g(\xi_0) \\ g(\xi_1) \\ \vdots \\ g(\xi_N) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix},$$

$$\overline{\mathbf{T}} = \begin{bmatrix} \mathbf{T}(\xi_0) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(\xi_1) & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{T}(\xi_N) \end{bmatrix}, \quad \overline{\mathbf{K}_v} = \begin{bmatrix} \mathbf{K} & 0 & \cdots & 0 \\ 0 & \mathbf{K} & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{K} \end{bmatrix}, \quad \overline{\mathbf{Q}}_v = \begin{bmatrix} \mathbf{Q}_v(\xi_0) \\ \mathbf{Q}_v(\xi_1) \\ \vdots \\ \mathbf{Q}_v(\xi_N) \end{bmatrix}.$$

Besides, the fundamental matrix equation Eq. (2.8) can be expressed in the form

$$\mathbf{WA} = \mathbf{G} \quad \Leftrightarrow \quad [\mathbf{W} : \mathbf{G}], \tag{2.9}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k \mathbf{T} \mathbf{B}^k \mathbf{D}^T - \overline{\mathbf{T}} \overline{\mathbf{K}}_v \overline{\mathbf{Q}}_v \mathbf{D}^T = [w_{mn}]; \quad m, n = 0, 1, \ldots N.$$

Now we can obtain the corresponding matrix form for the initial conditions (1.2), by means of the relation Eq. (2.3),

$$\mathbf{U}_s \mathbf{A} = \lambda_s \quad \Leftrightarrow \quad [\mathbf{U}_s : \lambda_s]; \quad s = 0, 1, \ldots m-1 \tag{2.10}$$

such that

$$\mathbf{U}_s = \sum_{k=0}^{m-1} (a_{sk} \mathbf{T}(a) + b_{sk} \mathbf{T}(b) + c_{sk} \mathbf{T}(c)) \mathbf{B}^k \mathbf{D}^T \mathbf{A} = \left[\begin{array}{cccc} u_{s0} & u_{s1} & \cdots & u_{sN} \end{array} \right]. \tag{2.11}$$

After substituting last m rows of the augmented matrix (2.9) with the m row matrices (2.11), we finally get the new matrix as the answer to the problem (1.1)-(1.2).

$$\widetilde{\mathbf{W}}\,\mathbf{A} = \widetilde{\mathbf{G}} \Rightarrow \left[\begin{array}{ccc} \widetilde{\mathbf{W}} & : & \widetilde{\mathbf{G}} \end{array} \right]. \tag{2.12}$$

In Eq. (2.12), if $rank\widetilde{\mathbf{W}} = rank \left[\begin{array}{ccc} \widetilde{\mathbf{W}} & : & \widetilde{\mathbf{G}} \end{array} \right] = N + 1$, then the coefficient matrix $\mathbf{A}$ is uniquely determined and the solution of the problem (1.1)-(1.2) is obtained as:

$$y_N(\xi) = \mathbf{L}(\xi)\mathbf{A} = \mathbf{T}(\xi)\mathbf{D}^T\mathbf{A}.$$

3. Accuracy of solutions

It's simple to verify the method's accuracy. Since the truncated Lucas series Eq. (1.3) is an approximate solution of Eq. (1.1), the equation that results when the function $y_N(\xi)$ and its derivatives are substituted must approximately satisfy the following conditions:

$$E(\xi_q) = \left\| \sum_{k=0}^{m} P_k(\xi_q) y^{(k)}(\xi_q) - \int_a^{\xi_q} K_v\left(\xi_q, \varsigma_q\right) y\left(\varsigma_q\right) d\varsigma - g\left(\xi_q\right) \right\| \cong 0 \tag{3.1}$$

for $\xi = \xi_q \in [a, b]$, $q = 0, 1, 2, \ldots$ and $E(\xi_q) \leq 10^{-k_q}$, k_q positive integer. If max $10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, when the difference $E(\xi_q)$ at each point is higher than the recommended 10^{-k}, the truncation limit N is increased [5], [13]-[15].

Moreover, the error can be estimated by the function

$$E_N(\xi) = \sum_{k=0}^{m} P_k y_N^{(k)} - \int_a^{\xi} K_v(\xi, \varsigma) y_N(\varsigma) d\varsigma - g(\xi).$$

If $E_N(\xi) \to 0$ the error decreases when N is sufficiently large enough. The $E_N(\xi)$ error function approaches zero, because the convergence of the proposed method is known from study of H. Brunner [4].

4. Numerical illustrations

In this section, problems frequently encountered in the literature will be given to explain the accuracy and efficiency of the Lucas matrix method based on collocation points presented for solving Eq. (1.1).

Problem 4.1 (*cf.* [22]). Let us first consider the second-order linear VIDE

$$y''(\xi)+\xi y'(\xi)-\xi y(\xi)=e^{\xi}+\frac{1}{2}\xi\cos(\xi)-\frac{1}{2}\int_0^{\xi}\cos(\xi)e^{-\varsigma}y(\varsigma)\,d\varsigma,$$

$0\le\xi,\varsigma\le 1$ with the initial conditions $y(0)=1$, $y'(0)=1$.

We approximate the solution $y(\xi)$ by the polinomial

$$y(\xi)=y_N(\xi)=\sum_{n=0}^{3}a_nL_n(\xi),\quad 0\le\xi\le 1,$$

$P_2(\xi)=1$, $P_1(\xi)=\xi$, $P_0\xi=-\xi$, $g(\xi)=e^{\xi}+\frac{1}{2}\xi\cos(\xi)$, $K_v(\xi,\varsigma)=-\frac{1}{2}\cos(\xi)e^{-\varsigma}$, and the collocation points for $a=0$, $b=1$ and $N=3$ are computed as

$$\left\{\xi_0=0,\quad \xi_1=\frac{1}{3},\quad \xi_2=\frac{2}{3}\quad \xi_3=1\right\}.$$

Finally we obtain following the procedure in Section 2,

$$\sum_{k=0}^{3}\mathbf{P}_k\mathbf{T}\mathbf{B}^k\mathbf{D}^T\mathbf{A}-\overline{\mathbf{T}\mathbf{K}}_v\overline{\mathbf{Q}}_v\mathbf{D}^T\mathbf{A}=\mathbf{G},$$

where

$$\mathbf{P}_0=\begin{bmatrix}0&0&0&0\\0&-\frac{1}{3}&0&0\\0&0&-\frac{2}{3}&0\\0&0&0&-1\end{bmatrix},\quad \mathbf{P}_1=\begin{bmatrix}0&0&0&0\\0&\frac{1}{3}&0&0\\0&0&\frac{2}{3}&0\\0&0&0&-1\end{bmatrix},\quad \mathbf{P}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \mathbf{T}=\begin{bmatrix}1&0&0&0\\1&\frac{1}{3}&\frac{1}{9}&\frac{1}{27}\\1&\frac{2}{3}&\frac{4}{9}&\frac{8}{27}\\1&1&1&1\end{bmatrix},$$

$$\mathbf{B}^0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix},\quad \mathbf{B}^2=\begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathbf{D}^T=\begin{bmatrix}2&0&2&0\\0&1&0&3\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$\mathbf{K}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}&-\frac{1}{4}&\frac{1}{12}\\0&0&0&0\\\frac{1}{4}&-\frac{1}{4}&\frac{1}{8}&-\frac{1}{24}\\0&0&0&0\end{bmatrix},\quad \overline{\mathbf{K}}=\begin{bmatrix}\mathbf{K}&0&0&0\\0&\mathbf{K}&0&0\\0&0&\mathbf{K}&0\\0&0&0&\mathbf{K}\end{bmatrix},\quad \mathbf{Q}(0)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\mathbf{Q}\left(\frac{1}{3}\right)=\begin{bmatrix}\frac{1}{3}&\frac{1}{18}&\frac{1}{81}&\frac{1}{324}\\\frac{1}{18}&\frac{1}{81}&\frac{1}{324}&\frac{1}{1215}\\\frac{1}{81}&\frac{1}{324}&\frac{1}{1215}&\frac{1}{4374}\\\frac{1}{324}&\frac{1}{1215}&\frac{1}{4374}&\frac{1}{15309}\end{bmatrix},\quad \mathbf{Q}\left(\frac{2}{3}\right)=\begin{bmatrix}\frac{2}{3}&\frac{2}{9}&\frac{8}{81}&\frac{4}{81}\\\frac{2}{9}&\frac{8}{81}&\frac{4}{81}&\frac{32}{1215}\\\frac{8}{81}&\frac{4}{81}&\frac{32}{1215}&\frac{32}{2187}\\\frac{4}{81}&\frac{32}{1215}&\frac{32}{2187}&\frac{128}{15309}\end{bmatrix},\quad \mathbf{Q}(1)=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\end{bmatrix},$$

$$\mathbf{W}=\begin{bmatrix}0&0&2&0\\-\frac{156}{391}&\frac{553}{2273}&\frac{1395}{779}&\frac{1845}{652}\\-\frac{2090}{2187}&\frac{529}{1902}&\frac{1134}{683}&\frac{3516}{635}\\-\frac{27}{16}&\frac{31}{480}&\frac{973}{720}&\frac{3346}{407}\end{bmatrix},\qquad \mathbf{G}=\begin{bmatrix}1\\\frac{775}{499}\\\frac{1823}{825}\\\frac{2842}{951}\end{bmatrix}.$$

This fundamental matrix equation's augmented matrix is

$$\left[\widetilde{\mathbf{W}}\quad;\quad\widetilde{\mathbf{G}}\right]=\begin{bmatrix}0&0&2&0&;&1\\-\frac{156}{391}&\frac{553}{2273}&\frac{1395}{779}&\frac{1845}{652}&;&\frac{775}{499}\\2&0&2&0&;&1\\0&1&0&3&;&1\end{bmatrix}.$$

Solving this system, **A** is obtained as $\mathbf{A}=\begin{bmatrix}0&\frac{505}{1238}&\frac{1}{2}&\frac{374}{1895}\end{bmatrix}$. Thus, the approximate solution of the problem becomes

$$y_3(\xi)=0.19736\xi^3+0.5\xi^2+\xi+1.$$

In a similar manner, we derive the approximate solutions of the problem for $N=12$,

$$\begin{aligned}y_{12}(\xi)\;=\;&3.19*10^{-9}\xi^{12}+2.22*10^{-8}\xi^{11}+2.80*10^{-7}\xi^{10}+2.75*10^{-6}\xi^{9}+0.00002\xi^{8}+0.00020\xi^{7}\\&+0.00139\xi^{6}+0.00833\xi^{5}+0.04167\xi^{4}+0.16667\xi^{3}+0.5\xi^{2}+\xi+1.\end{aligned}$$

ξ	**Exact solution**	**Bessel collocation (N=10) [22]**	**Present method (N=10)**	**Present method (N=12)**
0	1	1	1	1
0.2	1.2214027581602	1.2214027581600	1.2214027581600	1.2214027581602
0.4	1.4918246976413	1.4918246976409	1.4918246976409	1.4918246976413
0.6	1.8221188003905	1.8221188003899	1.8221188003900	1.8221188003905
0.8	2.2255409284925	2.2255409284915	2.2255409284916	2.2255409284925
1.0	2.7182818284590	2.7182818284279	2.7182818284282	2.7182818284590
CPU time			0.894 s	0.906 s

Table 1. Numerical results of solution in Problem 4.1

The solutions obtained by the present method and the Bessel collocation method are compared with the exact solutions in Table 1. Also, the comparison of absolute error functions corresponding to $N = 7, 10, 12$ is given in Table 2. In Figure 2, exact and the approximate solutions are depicted.

	Bessel collocation [22]		**Lucas collocation**		
ξ	e_7	e_{10}	e_7	e_{10}	e_{12}
0.2	3.2620e-09	1.6920e-13	3.2620e-09	1.6764e-13	2.2204e-16
0.4	6.7705e-09	3.6748e-13	6.7704e-09	3.5350e-13	4.4409e-16
0.6	1.0493e-08	5.7976e-13	1.0493e-08	5.3357e-13	4.4409e-16
0.8	2.3531e-08	9.4724e-13	2.3531e-08	8.3711e-13	2.2204e-15
1.0	2.9771e-07	3.1105e-11	2.9771e-07	3.0884e-11	7.0610e-14

Table 2. Comparison of the absolute error functions $e_N(\xi)$ in Problem 4.1

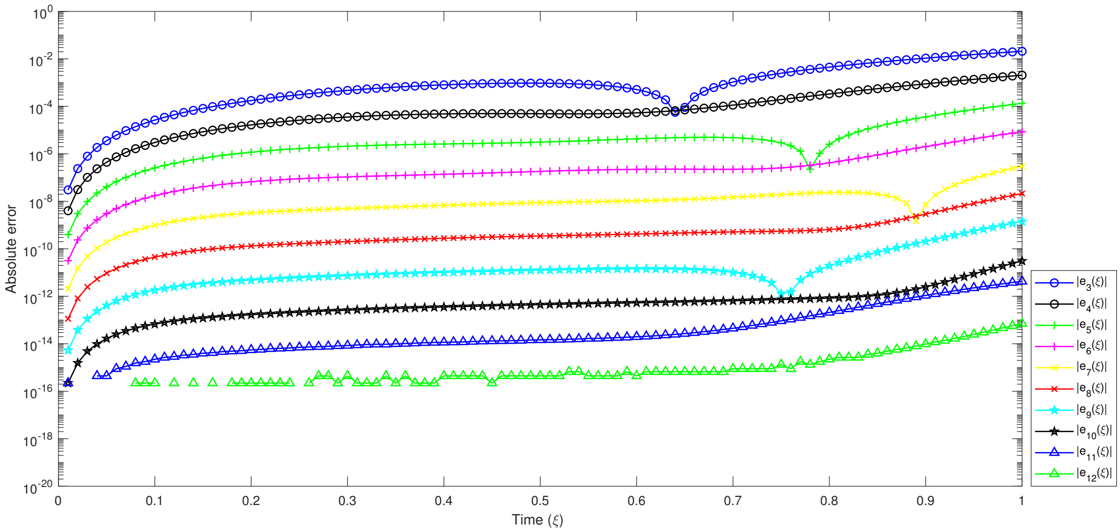

Figure 2. The absolute errors of Problem 4.1 for $3 \leq N \leq 12$

Problem 4.2 (*cf.* [16]). Let us consider another linear VIDE with initial condition

$$y'(\xi) = \cos(1) - \int_{-1}^{\xi} y(\varsigma)\, ds,$$

$-1 \leq \xi, \varsigma \leq 1$ with the initial condition $y(0) = 0$.

Following the procedure, for different values of N the polynomial solution is obtained as follows:

$$y_5(\xi) = 0.00795\xi^5 + 1.70 * 10^{-16}\xi^4 - 0.16648\xi^3 - 5.98 * 10^{-17}\xi^2 + 0.99998\xi - 3.54 * 10^{-17},$$

$$\begin{aligned} y_9(\xi) \quad = \quad & 2.69 * 10^{-6}\xi^9 + 4.71 * 10^{-17}\xi^8 - 0.00020\xi^7 - 1.61 * 10^{-16}\xi^6 + 0.00833\xi^5 + 1.26 * 10^{-16}\xi^4 - 0.16667\xi^3 \\ & +1.39 * 10^{-17}\xi^2 + \xi - 9.76 * 10^{-17}, \end{aligned}$$

$$\begin{aligned} y_{14}(\xi) \quad = \quad & 7.31 * 10^{-13}\xi^{14} + 1.59 * 10^{-10}\xi^{13} - 1.60 * 10^{-12}\xi^{12} - 2.51 * 10^{-8}\xi^{11} + 1.32 * 10^{-12}\xi^{10} + 2.76 * 10^{-6}\xi^9 \\ & +5.13 * 10^{-13}\xi^8 - 0.00020\xi^7 + 9.91 * 10^{-14}\xi^6 + 0.00833\xi^5 - 9.08 * 10^{-15}\xi^4 - 0.16667\xi^3 \\ & +3.85 * 10^{-16}\xi^2 + \xi - 1.09 * 10^{-16}. \end{aligned}$$

It will be seen that the Taylor expansion of the exact solution $y(\xi) = \sin(\xi)$ of the equation system is obtained when the solutions are examined.

Some absolute errors from the solutions of the problem are tabulated for $N = 5, 9, 14$ in Table 3. Figure 3 depicts the numerical solution of the absolute errors in Problem 4.2. As the integer N is increased, the error goes down.

ξ	e_5	e_9	e_{14}
-1.0	2.04679e-05	4.13929e-10	6.66134e-16
-0.75	2.07396e-06	2.10015e-11	3.33067e-16
-0.5	1.75065e-06	3.18773e-11	1.66533e-16
-0.25	3.06996e-06	7.61471e-12	1.38778e-16
0	3.53815e-17	9.75591e-17	1.08871e-16
0.25	3.06996e-06	7.61494e-12	8.32667e-17
0.5	1.75065e-06	3.18776e-11	0
0.75	2.07396e-06	2.10014e-11	2.22045e-16
1.0	2.04679e-05	4.13929e-10	4.70735e-14

Table 3. Comparisons of absolute errors for N= 5, 9, 14 in Problem 4.2

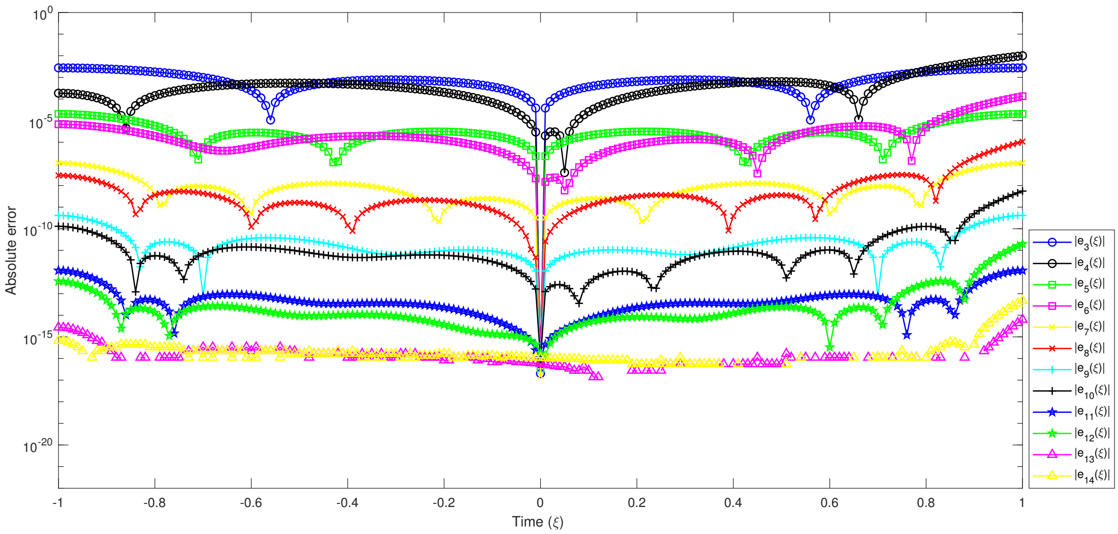

Figure 3. The absolute errors of Problem 4.2 for $3 \leq N \leq 14$

Problem 4.3 (*cf.* [1, 20, 22]). Consider the problem

$$y'(\xi) = 1 - \int_0^{\xi} y(\varsigma)d\varsigma, \quad 0 \leq \xi \leq 5$$

with the initial condition y(0) = 0.

The numerical results from this problem for several Lucas polynomials solutions for $N = 8, 12, 15$, exact solution and Haar wavelet solution [1] are tabulated in Table 4 using the suggested method. It is clearly seen that, the proposed method gives better results than Haar wavelet method.

ξ	**Exact solution**	**Haar wavelet (m=32) [1]**	**Lucas (N=8)**	**Lucas (N=12)**	**Lucas (N=15)**
0.078	0.07792093206	0.07765	0.07791721910	0.07792092659	0.07792093204
0.547	0.52012730711	0.51750	0.52008376937	0.52012728160	0.52012730705
1.016	0.85000774687	0.84614	0.84998276809	0.85000773092	0.85000774683
1.484	0.99623556303	0.99298	0.99622751498	0.99623555773	0.99623556302
1.953	0.92784499386	0.92645	0.92785133128	0.92784500056	0.92784499388
2.422	0.65907837191	0.66084	0.65910576840	0.65907838937	0.65907837195
2.891	0.24797814520	0.25324	0.24801190961	0.24797816924	0.24797814526
3.359	-0.21569872844	-0.20877	-0.21566023438	-0.21569870226	-0.21569872838
3.828	-0.63376226048	-0.62593	-0.63371679851	-0.63376223933	-0.63376226043
4.297	-0.91495941829	-0.90859	-0.91500720754	-0.91495940493	-0.91495941827
4.922	-0.97811192754	-0.97719	-0.97624529121	-0.97811089229	-0.97811192877

Table 4. Comparison of the results of the present method with Haar wavelet method [1] in Problem 4.3

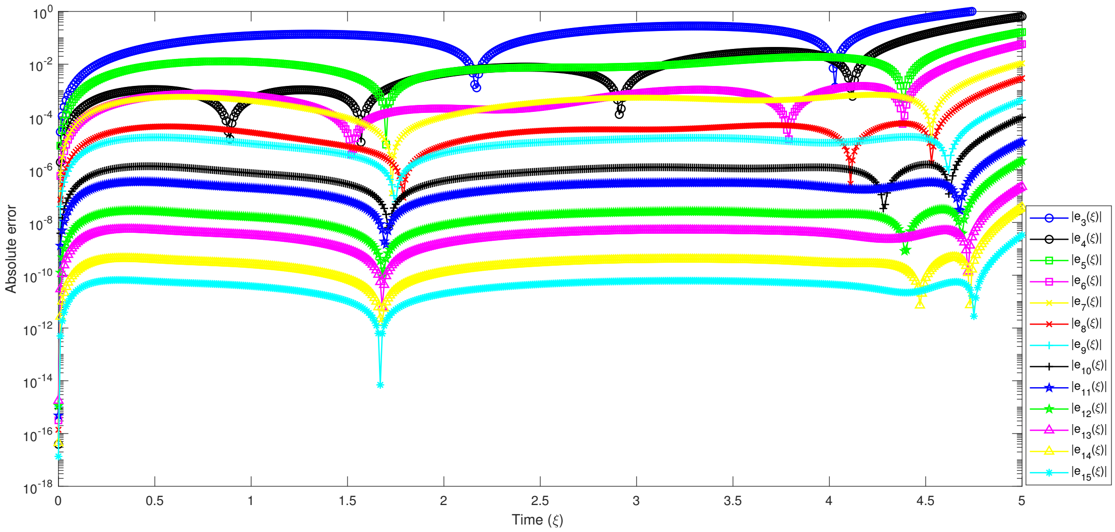

Figure 4. The absolute errors of Problem 4.3 for $3 \leq N \leq 15$

The absolute errors of the numerical solutions are depicted in Figure 4. As the integer N is increased, the error goes down.

Problem 4.4 (*cf.* [1]). Lastly, we consider the following first-order linear VIDE

$$y'(\xi) = \frac{1}{4} + \frac{3}{4}\xi + \sin(\xi) + \int_0^{\xi} y(\varsigma)d\varsigma, \qquad 0 < \xi \leq 1.1$$

with the initial condition y(0) = -1.

The solution of the problem for $N = 11$ becomes as follows:

$$\begin{aligned} y_{11}(\xi) &= 4.40 * 10^{-9}\xi^{11} + 2.10 * 10^{-7}\xi^{10} + 6.85 * 10^{-7}\xi^{9} - 6.20 * 10^{-6}\xi^{8} + 0.00005\xi^{7} \\ &\quad + 0.00104\xi^{6} + 0.00208\xi^{5} - 0.01042\xi^{4} + 0.04167\xi^{3} + 0.375\xi^{2} + 0.25\xi - 1. \end{aligned}$$

When the solution is examined, it is clear that the Taylor expansion of the exact solution $y(\xi) = \frac{1}{4}e^{\xi} - \frac{3}{4} - \frac{1}{2}\cos(\xi)$ of the equation system is obtained.

Figure 5 depicts the absolute errors to solution of Problem 4.4. As the number N is increased, the error decreases.

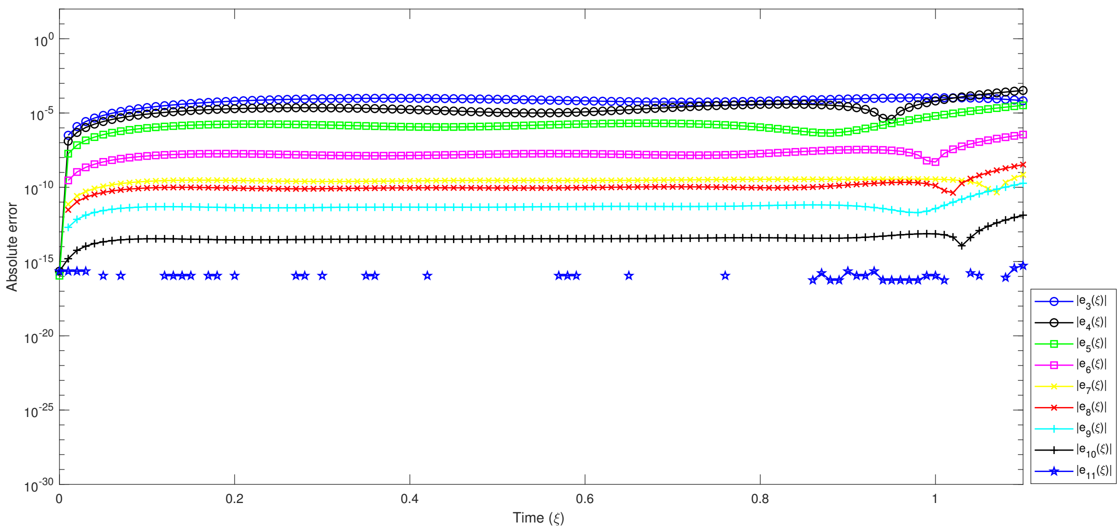

Figure 5. The absolute errors of Problem 4.4 for $3 \leq N \leq 13$

In Table 5, we compare the absolute errors of our results with Haar wavelet method [1]. From these comparison, it is seen that the proposed method gives better results than Haar wavelet method.

ξ	**Haar wavelet**[1]	**Present method** e_5	**Present method** e_9	**Present method** e_{11}
0.0344	0.000396	1.83931e-07	1.63047e-12	1.11022e-16
0.1719	0.000381	1.65277e-06	4.55014e-12	0
0.3094	0.000300	1.52695e-06	4.39249e-12	0
0.4469	0.000155	1.14887e-06	4.56934e-12	0
0.5844	0.000052	1.74981e-06	4.99856e-12	0
0.7219	0.000320	1.87612e-06	5.07505e-12	0
0.8594	0.000648	5.02056e-07	6.47737e-12	0
0.9969	0.001035	5.99281e-06	3.18934e-12	0
1.0656	0.001250	2.00396e-05	6.39383e-11	1.11022e-16

Table 5. Comparison of the absolute errors of the present method with Haar wavelet method [20] in Problem 4.4

5. Conclusion

Analytical solutions to high-order linear integro-differential equations are typically challenging. It is frequently necessary to approximate the solutions. If the solution function of the problem is polynomial, the approximate solution obtained by the Lucas collocation method will also be equal to the full solution by choosing the cut-off limit N less than or equal to the degree of the polynomial. As the values increase, the approximate solution will converge to the exact solution. Using the Lucas polynomials of the first kind, a collocation method is proposed in this study to numerically solve the high-order linear Volterra integro-differential equations. The method is exceedingly efficient and practical, as shown by a comparison of the results it produced with the precise solution and other methods. One of the method's many benefits is how simple it is to find the Lucas coefficients of the solution using the MATLAB code (R2022a). As a result, it is a useful strategy for resolving the mentioned problems, leading to both approximate and exact solutions.

Acknowledgments

The authors would like to thank the reviewers for their constructive feedbacks towards improving our manuscript.

Author Contributions: To the planning and carrying out of the study, to the findings analysis, and to the writing of the report, all authors contributed equally.

Conflict of Interest: The authors declare no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] I. S. Ali, *Haar wavelet collocation technique for solving linear volterra integro differential equations*, NeuroQuantology **18 (7)**, 39–44, 2020.
[2] N. Baykuş Savaşaneril and M. Sezer, *Hybrid Taylor-Lucas collocation method for numerical solution of high-order pantograph type Delay differential equations with variables Delays*, Appl. Math. Inf. Sci. **11 (6)**, 1795–1801, 2017; doi:10.18576/amis/110627.
[3] M. Bicknell, *A primer for the Fibonacci numbers VII*, Fibonacci Quart. **8**, 407–420, 1970.
[4] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations*, Cambridge University Press, 2004.
[5] B. Bülbül, M. Gülsu and M. Sezer, *A new Taylor collocation method for nonlinear Fredholm-Volterra integro-differential equations*, Numer. Methods Partial Differ. Equations **26 (5)**, 1006–1020, 2010.
[6] L. M. Delves and J. L. Mohamed, *Computational methods for integral equations*, Cambridge University Press, Cambridge, 1985.
[7] D. Elmacı and N. Baykuş Savaşaneril, *Euler polynomials method for solving linear integro differential equations*, New Trends in Mathematical Sciences **9 (3)**, 21–34, 2021.
[8] D. Elmacı, N. Baykuş Savaşaneril, F. Dal and M. Sezer, *On the application of Euler's method to linear integro differential equations and comparison with existing methods*, Turkish J. Math. **46 (1)**, 99–122, 2022.
[9] K. Erdem Biçer and H. G. Dağ, *Boole approximation method with residual error function to solve linear Volterra integro-differential equations*, Celal Bayar University Journal of Science **17 (1)**, 59–66, 2021.
[10] K. Erdem Biçer and M. Sezer, *Bernoulli matrix-collocation method for solving general functional integro-differential equations with hybrid delays*, J. Inequal. Spec. Funct. **8 (3)**, 85–99, 2017.
[11] S. Gümgüm, N. Baykuş Savaşaneril, Ö. K. Kürkçü and M. Sezer, *A numerical technique based on Lucas polynomials together with standard and Chebyshev-Lobatto collocation points for solving functional integro-differential equations involving variable delays*, Sakarya University Journal of Science **22 (6)**, 1659–1668, 2018; doi:10.16984/saufenbilder.384592.
[12] S. Gümgüm, N. Baykuş Savaşaneril, Ö. K. Kürkçü and M. Sezer, *Lucas polynomial solution of nonlinear differential equations with variable delay*, Hacet. J. Math. Stat. **49 (2)**, 553–564, 2020.
[13] N. Kurt and M. Sezer, *Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients*, J. Franklin Inst. **345 (8)**, 839–850, 2008.
[14] Ö. K. Kürkçü, E. Aslan and M. Sezer, *A novel collocation method based on residual error analysis for solving integro-differential equations using hybrid Dickson and Taylor polynomials*, Sains Malays **46**, 335–347, 2017.
[15] K. Maleknejad and Y. Mahmoudi, *Taylor polynomial solution of high-order nonlinear Volterra–Fredholm integro-differential equations*, Appl. Math. Comput. **145 (2-3)**, 641–653, 2003.
[16] T. Mollaoğlu, *Volterra tipi gecikmeli fonksiyonel integro-diferansiyel denklemler için Gegenbauer polinom yaklaşımı*, Master thesis, Manisa Celal Bayar University, Manisa, 2017.
[17] M. T. Rashed, *Numerical solution of functional differential, integral and integro-differential equations*, Appl. Numer. Math. **156**, 485–492, 2004.
[18] A. M. Wazwaz, *The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations*, Int. J. Comput. Math. **87 (5)**, 1131–1141, 2010.
[19] S. Yalçinbaş and K. Erdem, *Approximate solutions of nonlinear Volterra integral equation systems*, Internat. J. Modern Phys. B **24 (32)**, 6235–6258, 2010.
[20] S. Yalçinbaş and M. Sezer, *The approximate solution of high-order linear Volterra–Fredholm integro-differential equations in terms of Taylor polynomials*, Appl. Math. Comput. **112 (2-3)**, 291–308, 2000.
[21] Ş. Yüzbaşı and I. Nurbol, *An operational matrix method for solving linear Fredholm–Volterra integro-differential equations*, Turkish J. Math. **42 (1)**, 243–256, 2018.
[22] Ş. Yüzbaşı, N. Şahin and M. Sezer, *Bessel polynomial solutions of high-order linear Volterra integro-differential equations*, Comput. Math. Appl. **62 (4)**, 1940–1956, 2011.
[23] Ş. Yüzbaşı, N. Şahin and A. Yildirim, *A collocation approach for solving high-order linear Fredholm–Volterra integro-differential equations*, Math. Comput. Modelling **55 (3-4)**, 547–563, 2012.
[24] M. Zarebnia, *Sinc numerical solution for the Volterra integro-differential equation*, Commun. Nonlinear Sci. Numer. Simul. **15**, 700–706, 2010.
[25] H. Zuoshang, *Boundness of solutions to functional integro-differential equations*, Proc. Amer. Math. Soc. **114 (2)**, 617–625, 1992.