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Existence and nonexistence results for n^{th} order non-homogeneous three point boundary value problems

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Abstract

In this paper, we present criteria for the existence and nonexistence of positive solutions to n^{th} order differential equations

$$z^{(n)}(\tau) + q(\tau)f(z(\tau)) = 0, \quad 0 < \tau < 1,$$

fulfilling non-homogeneous three point conditions

$$z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \ \gamma z^{(n-2)}(1) - \beta z^{(n-2)}(\eta) = \nu,$$

where $n > 2$, $\eta \in (0, 1)$, $\gamma > 0$, $\beta \in (0, \frac{\gamma}{\eta})$ are constants and $\nu \in (0, \infty)$ is a parameter by an application of fixed point index theory.

Keywords: Green's function, differential equation, non-homogeneous three point conditions, positive solution, fixed point index theory

2010 MSC: 34B15, 34B25, 34B18

1. Introduction

The goal of studying differential equations is to analyze the real world problems which are formulated in mathematical models. These models are associated with either initial or boundary conditions. The study of the existence as well as nonexistence of positive solutions to problems has attracted a lot of significant attention in the domain of research, because these problems occur in a wide range of fascinating applications such as deflection of curved beams, catalytic theory, gas diffusion through porous media, electrostatics and so on.

In 1987, Il'in and Moiseev [4] demonstrated the existence of solutions to the problems associated with second order linear differential equations. Following that, many authors employed different methods to concentrate on various multi-point boundary value problems. Indeed, the fourth order differential equations are models for bending or deformation of elastic beams, which has been studied by Gupta [3] in 1988 and given by

$$\frac{d^4u}{dx^4} - \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,$$

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$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

In recent years, the researchers have explored more general nonlinear problems with homogeneous conditions [6, 11, 13], and non-homogeneous conditions [7]-[10], [12] to establish the positivity results. Due to the importance in both theory and applications, we wish to generalize the results for n^{th} order differential equation with non-homogeneous conditions.

Consider n^{th} order non-homogeneous three point boundary value problem

$$z^{(n)}(\tau) + q(\tau)f(z(\tau)) = 0, \quad 0 < \tau < 1, \tag{1.1}$$

$$z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \;\; \gamma z^{(n-2)}(1) - \beta z^{(n-2)}(\eta) = \nu, \tag{1.2}$$

where $n > 2$, $\eta \in (0, 1)$, $\gamma > 0, \beta \in \left(0, \frac{\gamma}{\eta}\right)$ are constants, $\nu \in (0, \infty)$ is a parameter, and used to prove the existence and nonexistence of positive solutions. The following conditions are assumed in order to establish the results.

(K1) $f \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$,

(K2) $q \in C\left([0, 1], \mathbb{R}^+\right)$ and q is not identically zero on any closed subinterval of $[0, 1]$.

Definition 1.1 (*cf.* [1])**.** Let $f_0 = \lim_{z\to 0^+} \frac{f(z)}{z}$ and $f_\infty = \lim_{z\to\infty} \frac{f(z)}{z}$. If $f_0 = 0$ and $f_\infty = \infty$ then f is said to be superlinear. If $f_0 = \infty$ and $f_\infty = 0$ then f is said to be sublinear.

The rest of the paper is structured as follows. In Section 2, the solution of the problem (1.1)-(1.2) is expressed as an equivalent integral equation in terms of kernels, and bounds for kernels are determined. In Section 3, we use the fixed point index theory to prove the existence and nonexistence results to the problem (1.1)-(1.2). As an application, we present our results via examples.

2. Kernels and their estimates

In this section, the solution of the problem (1.1)-(1.2) is expressed into an equivalent integral equation involving kernels and several inequalities for kernels are established.

Lemma 2.1. *If the assumptions* (K1) *and* (K2) *are fulfilled then the unique solution of the problem* (1.1)-(1.2) *is*

$$z(\tau) = \frac{\nu\tau^{n-1}}{(n-1)!(\gamma - \eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma - \eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta, \tag{2.1}$$

where

$$\mathcal{H}(\tau,\vartheta) = \begin{cases} \dfrac{1}{(n-1)!}[\tau^{n-1}(1-\vartheta) - (\tau-\vartheta)^{n-1}], & 0 \le \vartheta \le \tau \le 1, \\ \dfrac{1}{(n-1)!}\tau^{n-1}(1-\vartheta), & 0 \le \tau \le \vartheta \le 1, \end{cases} \tag{2.2}$$

and

$$\mathcal{H}_1(\eta,\vartheta) = \begin{cases} \vartheta(1-\eta), & 0 \le \vartheta \le \eta \le 1, \\ \eta(1-\vartheta), & 0 \le \eta \le \vartheta \le 1. \end{cases} \tag{2.3}$$

Proof. Let $z(\tau)$ be the solution of the problem (1.1)-(1.2). Then the corresponding integral equation of (1.1) is

$$z(\tau) = A_0 + A_1\tau + A_2\tau^2 + \cdots + A_{n-1}\tau^{n-1} - \frac{1}{(n-1)!}\int_0^\tau (\tau-\vartheta)^{n-1}q(\vartheta)f(z(\vartheta))d\vartheta.$$

Using (1.2), one can obtain

$$A_0 = A_1 = A_2 = \cdots = A_{n-2} = 0$$

and

$$A_{n-1} = \frac{\nu}{(n-1)!(\gamma-\eta\beta)} + \frac{\gamma}{(n-1)!(\gamma-\eta\beta)}\int_0^1 (1-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta - \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^\eta (\eta-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta.$$

Hence, the unique solution of (1.1)-(1.2) is

$$\begin{aligned}
z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} - \frac{1}{(n-1)!}\int_0^\tau (\tau-\vartheta)^{n-1}q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\gamma\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} \\
&\quad \int_0^1 (1-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta - \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^\eta (\eta-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} - \frac{1}{(n-1)!}\int_0^\tau (\tau-\vartheta)^{n-1}q(\vartheta)f(z(\vartheta))d\vartheta \\
&\quad + \frac{\tau^{n-1}}{(n-1)!}\int_0^1 (1-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\eta\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 (1-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\quad - \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^\eta (\eta-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \frac{1}{(n-1)!}\left[\int_0^\tau [\tau^{n-1}(1-\vartheta)-(\tau-\vartheta)^{n-1}]q(\vartheta)f(z(\vartheta))d\vartheta\right. \\
&\quad \left. + \int_\tau^1 \tau^{n-1}(1-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right] + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} \\
&\quad \left[\int_0^\eta \vartheta(1-\eta)q(\vartheta)f(z(\vartheta))d\vartheta + \int_\eta^1 \eta(1-\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right] \\
&= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta.
\end{aligned}$$

This concludes the proof. □

Lemma 2.2. *The kernels $\mathcal{H}(\tau,\vartheta)$ and $\mathcal{H}_1(\tau,\vartheta)$ given in* (2.2) *and* (2.3) *respectively, satisfy the inequalities mentioned below:*

(i) $\mathcal{H}(\tau,\vartheta) \geq 0$ *and* $\mathcal{H}_1(\tau,\vartheta) \geq 0$ *for all* $\tau,\vartheta \in [0,1]$,

(ii) $\mathcal{H}(\tau,\vartheta) \leq \mathcal{H}(1,\vartheta)$ *for all* $\tau,\vartheta \in [0,1]$,

(iii) $\mathcal{H}(\tau,\vartheta) \geq \frac{1}{4^{n-1}}\mathcal{H}(1,\vartheta)$ *for all* $\tau \in I$ *and* $\vartheta \in [0,1]$, *where* $I = \left[\frac{1}{4},\frac{3}{4}\right]$.

Proof. One can establish the inequalities by simple algebraic calculations. □

The fixed point index theorem presented below is the key technique used to establish the main results.

Theorem 2.3 (*cf.* [2, 5])**.** *Let $\mathcal{B}$ be a Banach space and $\mathcal{P}$ be a cone in $\mathcal{B}$. Suppose further that the operator $\mathcal{F}:\mathcal{P}_r \to \mathcal{P}$ is a completely continuous with $\mathcal{F}z \neq z$, for $z \in \partial\mathcal{P}_r$.*

(i) *If* $\|z\| \leq \|\mathcal{F}z\|$ *for* $z \in \partial\mathcal{P}_r$, *then* $i(\mathcal{F},\mathcal{P}_r,\mathcal{P}) = 0$.

(ii) *If* $\|z\| \geq \|\mathcal{F}z\|$ *for* $z \in \partial\mathcal{P}_r$, *then* $i(\mathcal{F},\mathcal{P}_r,\mathcal{P}) = 1$.

Here $i(\mathcal{F},\mathcal{P}_r,\mathcal{P})$ represents fixed point index of the operator $\mathcal{F}$ on $\mathcal{P}_r$ with respect to the cone $\mathcal{P}$,

$$\mathcal{P}_r = \{z \in \mathcal{P} \mid \|z\| < r\} \text{ and } \partial\mathcal{P}_r = \{z \in \mathcal{P} \mid \|z\| = r\}, \text{ for } r > 0.$$

3. Existence and nonexistence results

In this section, we prove the existence and nonexistence of positive solutions to the problem (1.1)-(1.2). For our construction, let $\mathcal{B} = \{z \mid z \in C[0,1]\}$ be the Banach space with norm $\|z\| = \max_{\tau\in[0,1]} |z(\tau)|$. Let us define a cone $\mathcal{P}$ in $\mathcal{B}$ as

$$\mathcal{P} = \Big\{z \in \mathcal{B} \mid z(\tau) \geq 0 \text{ on } \tau \in [0,1] \text{ and } \min_{\tau\in I} z(\tau) \geq \frac{1}{4^{n-1}}\|z\|\Big\}.$$

Using the integral equation (2.1), we define an operator $\mathcal{T} : \mathcal{P} \to \mathcal{B}$ as

$$\mathcal{T}z(\tau) = \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta. \tag{3.1}$$

Lemma 3.1. *The operator $\mathcal{T} : \mathcal{P} \to \mathcal{B}$ given in* (3.1) *is a self map on the cone $\mathcal{P}$.*

Proof. By positivity of kernels $\mathcal{H}(\tau,\vartheta)$ and $\mathcal{H}_1(\tau,\vartheta)$ and for $z \in \mathcal{P}$, $\mathcal{T}z(\tau) \geq 0$ on $\tau \in [0,1]$. Then from Lemma 2.2 and for $z \in \mathcal{P}$, we have

$$\begin{aligned}\mathcal{T}z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\ &\leq \frac{\nu}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta.\end{aligned}$$

So,

$$\|\mathcal{T}z(\tau)\| \leq \frac{\nu}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta. \tag{3.2}$$

Next, if $z \in \mathcal{P}$, by Lemma 2.2 and inequality (3.2), we obtain

$$\begin{aligned}\min_{\tau\in I}\mathcal{T}z(\tau) &= \min_{\tau\in I}\left\{\frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right\} \\ &\geq \frac{1}{4^{n-1}}\left[\frac{\nu}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right] \\ &\geq \frac{1}{4^{n-1}}\|\mathcal{T}z(\tau)\|.\end{aligned}$$

Hence, $\mathcal{T} : \mathcal{P} \to \mathcal{P}$. □

Furthermore, $\mathcal{T}$ is a completely continuous operator based on Arzela–Ascoli theorem. Let

$$L_1 = \frac{1}{2}\left\{\int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\right\}^{-1} \tag{3.3}$$

and

$$L_2 = \left\{\frac{1}{4^{2n-2}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\right]\right\}^{-1}. \tag{3.4}$$

Theorem 3.2. *Suppose that the assumptions* (K1) *and* (K2) *are fulfilled. If $f_0 = 0$ and $f_\infty = \infty$, then the problem* (1.1)-(1.2) *has at least one positive solution for ν small enough and no positive solution for ν large enough.*

Proof. The proof is divided into two steps.

Step 1: We now prove that for sufficiently small $\nu > 0$, the problem (1.1)-(1.2) has at least one positive solution.

Since $f_0 = 0$, there exist $L_1 > 0$ and $H_1 > 0$ such that

$$f(z) \leq L_1 z, \quad 0 < z \leq H_1. \tag{3.5}$$

Let ν satisfy

$$0 < \nu \leq \frac{(n-1)!(\gamma - \eta\beta)H_1}{2}. \tag{3.6}$$

Choose $z \in \mathcal{P}$ and $\|z\| = H_1$. Then from Lemma 2.2 and (3.5), (3.6) that

$$\begin{aligned}
\mathcal{T}z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\leq \frac{\nu}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\leq \frac{H_1}{2} + L_1\left[\int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)z(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)z(\vartheta)d\vartheta\right] \\
&\leq \frac{H_1}{2} + L_1\left[\int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\right]\|z\| \\
&= \frac{H_1}{2} + \frac{H_1}{2} = H_1,
\end{aligned}$$

so that

$$\mathcal{T}z(\tau) \leq H_1.$$

Therefore, $\|\mathcal{T}z\| \leq H_1 = \|z\|$. If we take

$$\Omega_1 = \{z \in \mathcal{B} \mid \|z\| < H_1\},$$

then

$$\|\mathcal{T}z\| \leq \|z\|, \quad \text{for } z \in \mathcal{P} \cap \partial\Omega_1. \tag{3.7}$$

Since $f_\infty = \infty$, there exist $L_2 > 0$ and $\bar{H}_2 > 0$ such that

$$f(z) \geq L_2 z, \; z \geq \bar{H}_2. \tag{3.8}$$

Let $H_2 = \max\left\{2H_1, 4^{n-1}\bar{H}_2\right\}$. Choose $z \in \mathcal{P}$ with $\|z\| = H_2$. Then

$$\min_{\tau\in I} z(\tau) \geq \frac{1}{4^{n-1}}\|z\| \geq \bar{H}_2. \tag{3.9}$$

Then from Lemma 2.2, (3.8) and (3.9), one can obtain

$$\begin{aligned}
\mathcal{T}z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\geq \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\geq \min_{\tau\in I}\left\{\int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right\} \\
&\geq \frac{1}{4^{n-1}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right] \\
&\geq \frac{L_2}{4^{n-1}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)z(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)z(\vartheta)d\vartheta\right] \\
&\geq \frac{L_2}{4^{2n-2}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\right]\|z\| \\
&= \|z\|,
\end{aligned}$$

so that

$$\mathcal{T}z(\tau) \geq \|z\|.$$

Hence, $\|\mathcal{T}z\| \geq \|z\|$. Set

$$\Omega_2 = \{z \in \mathcal{B} \mid \|z\| < H_2\},$$

then

$$\|\mathcal{T}z\| \geq \|z\|, \quad \text{for } z \in \mathcal{P} \cap \partial\Omega_2. \tag{3.10}$$

Hence by using Theorem 2.3, (3.7), (3.10) and the property of fixed point index, we can see that $i(\mathcal{T}, \mathcal{P} \cap (\Omega_2 \backslash \overline{\Omega_1}), \mathcal{P}) = -1$. It follows that $\mathcal{T}$ has at least one fixed point $z \in \mathcal{P} \cap (\Omega_2 \backslash \overline{\Omega_1})$ and that z is the positive solution of (1.1)-(1.2).

Step 2: We verify that for ν large enough, the problem (1.1)-(1.2) has no positive solution.

We assume that, there exist $0 < \nu_1 < \nu_2 < \cdots < \nu_m \cdots$ with $\lim_{m\to\infty} \nu_m = \infty$ such that for any $m \in \mathbb{Z}^+$, the problem

$$z^{(n)}(\tau) + q(\tau)f(z(\tau)) = 0, \quad 0 < \tau < 1,$$
$$z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \quad \gamma z^{(n-2)}(1) - \beta z^{(n-2)}(\eta) = \nu_m,$$

has a positive solution $z_m(\tau)$. By the equation (2.1), we can see that

$$\begin{aligned} z_m(1) &= \frac{\nu_m}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\ &\geq \frac{\nu_m}{(n-1)!(\gamma-\eta\beta)} \to \infty \quad \text{as} \quad m \to \infty. \end{aligned}$$

Thus, $\|z_m\| \to \infty$ as $m \to \infty$.

Since $f_\infty = \infty$, there exist $L_2 > 0$ and $\overline{H} > 0$ such that

$$f(z) \geq 2L_2 z, \ z \geq \overline{H}.$$

Let m be the large enough that $\|z_m\| \geq \overline{H}$. Then

$$\begin{aligned} \|z_m\| &\geq z_m(1) \\ &= \frac{\nu_m}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(z_m(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z_m(\vartheta))d\vartheta \\ &\geq \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(z_m(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z_m(\vartheta))d\vartheta \\ &\geq 2L_2\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)z_m(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)z_m(\vartheta)d\vartheta\right] \\ &\geq \frac{2L_2}{4^{2n-2}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\right]\|z_m\| \\ &= 2\|z_m\|, \end{aligned}$$

which is a contradiction, and hence the theorem. □

Theorem 3.3. *Suppose that the assumptions* (K1) *and* (K2) *are fulfilled. If $f_0 = \infty$ and $f_\infty = 0$, then the problem* (1.1)-(1.2) *has at least one positive solution for any $\nu \in (0, \infty)$.*

Proof. Since $f_0 = \infty$, there exist $L_3 > 0$ and $J_1 > 0$ such that

$$f(z) \geq L_3 z, \ \ 0 < z \leq J_1,$$

where $L_3 \geq L_2$ and L_2 is given in (3.4). Then for any $z \in \mathcal{P}$ with $\|z\| = J_1$, we obtain

$$\begin{aligned}
\mathcal{T}z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\geq \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\geq \min_{\tau\in I}\left\{\int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right\} \\
&\geq \frac{1}{4^{n-1}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta\right] \\
&\geq \frac{L_2}{4^{n-1}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)z(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)z(\vartheta)d\vartheta\right] \\
&\geq \frac{L_2}{4^{2n-2}}\left[\int_{\vartheta\in I} \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_{\vartheta\in I} \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\right]\|z\| \\
&= \|z\|.
\end{aligned}$$

Therefore, $\|\mathcal{T}z\| \geq \|z\|$. Now, if we set

$$\Omega_3 = \{z \in \mathcal{B} \mid \|z\| < J_1\},$$

then

$$\|\mathcal{T}z\| \geq \|z\|, \text{ for } z \in \mathcal{P} \cap \partial\Omega_3. \tag{3.11}$$

Further, since $f^\infty = 0$, there exist $L_4 > 0$ and $\bar{J}_2 > 0$ such that

$$f(z) \leq L_4 z, \quad z \geq \bar{J}_2,$$

where $L_4 \leq L_1$ and L_1 is given in (3.3).

We establish the result by considering the following.

Case (i): Assume that the function f is bounded. Then there exists a constant $N > 0$ such that

$$f(z) \leq N, \text{ for } 0 < z < \infty.$$

We now take

$$J_2 \geq \max\left\{2J_1,\ \frac{N}{L_1},\ \frac{2\nu}{(n-1)!(\gamma-\eta\beta)}\right\},$$

and then for any $z \in \mathcal{P}$ with $\|z\| = J_2$, we obtain

$$\begin{aligned}
\mathcal{T}z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\leq \frac{\nu}{(n-1)!(\gamma-\eta\beta)} + N\int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta N}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta \\
&\leq \frac{J_2}{2} + \frac{J_2}{2} = J_2.
\end{aligned}$$

So, $\|\mathcal{T}z\| \leq \|z\|$.

Case (ii): Assume that the function f is unbounded. Then, we take

$$J_2 \geq \max\left\{2J_1,\ \bar{J}_2,\ \frac{2\nu}{(n-1)!(\gamma-\eta\beta)}\right\},$$

with

$$f(z) \leq f(J_2), \text{ for } 0 < z \leq J_2.$$

Then for any $z \in \mathcal{P}$ with $\|z\| = J_2$, we can obtain

$$\begin{aligned}
\mathcal{T}z(\tau) &= \frac{\nu\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(\tau,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta + \frac{\beta\tau^{n-1}}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(z(\vartheta))d\vartheta \\
&\le \frac{\nu}{(n-1)!(\gamma-\eta\beta)} + \int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)f(J_2)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)f(J_2)d\vartheta \\
&\le \frac{J_2}{2} + L_1J_2\bigg(\int_0^1 \mathcal{H}(1,\vartheta)q(\vartheta)d\vartheta + \frac{\beta}{(n-1)!(\gamma-\eta\beta)}\int_0^1 \mathcal{H}_1(\eta,\vartheta)q(\vartheta)d\vartheta\bigg) \\
&= \frac{J_2}{2} + \frac{J_2}{2} = J_2.
\end{aligned}$$

So, $\|\mathcal{T}z\| \le \|z\|$. Therefore, in either case by setting

$$\Omega_4 = \{z \in \mathcal{B} \mid \|z\| < J_2\},$$

then

$$\|\mathcal{T}z\| \le \|z\|, \text{ for } z \in \mathcal{P} \cap \partial\Omega_4. \tag{3.12}$$

Hence by Theorem 2.3, (3.11), (3.12) and the property of fixed point index, we obtain $i(\mathcal{T}, \mathcal{P} \cap (\Omega_4 \setminus \bar{\Omega}_3), \mathcal{P}) = 1$. It follows that the operator $\mathcal{T}$ has a fixed point $z \in \mathcal{P} \cap (\Omega_4 \setminus \bar{\Omega}_3)$ and that z is the positive solution of (1.1)-(1.2). □

4. Examples

Consider the following examples to demonstrate our results.

Example 4.1. Consider the problem

$$\left.\begin{aligned}
z^{(4)}(\tau) + \frac{1}{\sqrt{\tau(1-\tau^2)}}z^2(\tau) = 0, \quad 0 < \tau < 1, \\
z(0) = z'(0) = z''(0) = 0, \ 2z''(1) - 3z''\Big(\frac{1}{2}\Big) = \nu.
\end{aligned}\right\} \tag{4.1}$$

Then $f_0 = \lim_{z\to 0^+} \frac{z^2}{z} = 0$ and $f_\infty = \lim_{z\to\infty} \frac{z^2}{z} = \infty$. So, all the assumptions of Theorem 3.2 are fulfilled. By choosing ν such that $0 < \nu \le \frac{3}{2}H_1$, where $H_1 > 0$, the problem (4.1) has at least one positive solution for sufficiently small $\nu > 0$ and no positive solution for ν large enough.

Example 4.2. Consider the problem

$$\left.\begin{aligned}
z^{(4)}(\tau) + \frac{1}{\sqrt{\tau(1-\tau^2)}}sin^2 z(\tau) = 0, \quad 0 < \tau < 1, \\
z(0) = z'(0) = z''(0) = 0, \ 2z''(1) - 3z''\Big(\frac{1}{2}\Big) = \nu.
\end{aligned}\right\} \tag{4.2}$$

Then $f_0 = \lim_{z\to 0^+} \frac{sin^2 z}{z} = \infty$ and $f_\infty = \lim_{z\to\infty} \frac{sin^2 z}{z} = 0$. So, all the assumptions of Theorem 3.3 are fulfilled and hence, the problem (4.2) has at least one positive solution for any $\nu \in (0, \infty)$.

5. Conclusion

We proved the existence and nonexistence of positive solutions to n^{th} order nonlinear differential equations with non-homogeneous three point boundary conditions by employing fixed point index theory.

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