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On Some Pexider Type Sum Form Functional Equations

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Abstract

Since its genesis, an equation of Pexider type has captivated the attention of the mathematical fraternity around the world. Over the decades, several Pexiderized forms of various functional equations have been studied meticulously. In comparison to the functional equations, such forms are less analysed for sum form functional equations and require substantial study. Taking lead from it, this paper is devoted to obtain the general solution of some Pexiderized forms of a sum form functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} T\left(p_i q_j\right) = \sum_{i=1}^{n} T\left(p_i\right) \sum_{j=1}^{m} T\left(q_j\right) + (m-n)T(0) \sum_{j=1}^{m} T\left(q_j\right) + m(n-1)T(0),$$

where T is a real-valued mapping with the domain $I = [0, 1]$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $n \geq 3$, $m \geq 3$ are fixed integers.

Keywords: Pexider's equation, Sum form functional equation, Additive mapping, Multiplicative mapping, Entropy

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1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers; $\mathbb{R}$ denotes the set of real numbers; I denotes the closed unit interval $[0, 1]$ and I^0 denotes the interior of I, i.e. the open interval $]0, 1[$. For $n \in \mathbb{N}$, let

$$\Gamma_n = \left\{(p_1, \ldots, p_n);\ p_i \geq 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} p_i = 1\right\}$$

denote the set of all finite n-component complete discrete probability distributions with nonnegative elements.

In current mathematical literature, *Pexider's equation* and equation of *Pexider type* appear frequently. It is worth a mention that it was introduced by a Czech mathematician J. V. Pexider around early twentieth century [2]. Today, more than a hundred years later, Pexider's equation appears quite often in pure and applied mathematics. One such area of mathematics where this phenomenon has played a significant role is the *sum form functional equations*.

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The study of sum form functional equations, amalgamated from mathematical branches of functional equations and statistical thermodynamics. It first came into existence in the year 1960 with the seminal paper of Chaundy and Mcleod [3]. It was observed that continuous solution obtained by them [3] for an equation under consideration coincided with Shannon's entropy [16]. Thereafter, the analysis of sum form functional equations and entropies emerging from information theory gained a lot of momentum. In fact, these two branches (sum form functional equations and information theory) are well interweaved and have resulted in many novel results. Recently, with the aim to further delve in this direction: Madan, Grover and Singh [6]; Nath and Singh [11, 13, 15]; Singh and Grover [18]-[22] addressed few sum form functional equations. These papers have not only focussed on sum form functional equations related to information theory but also discussed Pexiderized forms and stability for some of them leaving behind many more yet to be explored. Taking cue from these studies, this paper is devoted to study the functional equations

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_iq_j\right)=\sum_{i=1}^{n} h\left(p_i\right)\sum_{j=1}^{m} f\left(q_j\right)+(m-n)f(0)\sum_{j=1}^{m} f\left(q_j\right)+m(n-1)f(0), \tag{1.1}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_iq_j\right)=\sum_{i=1}^{n} f\left(p_i\right)\sum_{j=1}^{m} h\left(q_j\right)+(m-n)f(0)\sum_{j=1}^{m} h\left(q_j\right)+m(n-1)f(0), \tag{1.2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_iq_j\right)=\sum_{i=1}^{n} h\left(p_i\right)\sum_{j=1}^{m} f\left(q_j\right)+(m-n)h(0)\sum_{j=1}^{m} f\left(q_j\right)+m(n-1)f(0) \tag{1.3}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_iq_j\right)=\sum_{i=1}^{n} h\left(p_i\right)\sum_{j=1}^{m} h\left(q_j\right)+(m-n)h(0)\sum_{j=1}^{m} h\left(q_j\right)+m(n-1)f(0), \tag{1.4}$$

where f, h are real-valued mappings each having the domain I and $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$.

It can be observed that equations (1.1)-(1.4) are Pexider type equations of the sum form functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} T\left(p_iq_j\right)=\sum_{i=1}^{n} T\left(p_i\right)\sum_{j=1}^{m} T\left(q_j\right)+(m-n)T(0)\sum_{j=1}^{m} T\left(q_j\right)+m(n-1)T(0). \tag{1.5}$$

Nath and Singh [7] were the first ones to come across this equation. They obtained its general solutions for $n \geq 3$, $m \geq 3$ being fixed integers. Further, Nath and Singh [10] obtained the general solution of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \bar{T}\left(p_iq_j\right)=\sum_{i=1}^{n} \bar{T}\left(p_i\right)\sum_{j=1}^{m} \bar{T}\left(q_j\right)+(n-m)\bar{T}(0)\sum_{i=1}^{n} \bar{T}\left(p_i\right)+n(m-1)\bar{T}(0) \tag{1.6}$$

by defining a mapping $\bar{T}(x) : I \to \mathbb{R}$ as $\bar{T}(x) = T(x) + (n-m)T(0)x$ for all $x \in I$. The equation (1.6) plays a vital role in solving several sum form functional equations. Thus, it needs to be remarked that equation (1.5) has been occurring *naturally* and played a key role in obtaining general solutions of several other sum form functional equations in [7, 10, 12, 14, 17].

Apart from this, another significant feature of the equations (1.1)-(1.4) that motivated us to address them is: If the last two terms of all these equations are omitted, then we obtain Pexiderized forms of a renowned multiplicative type sum form functional equation studied by Losonczi and Maksa [5], i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_iq_j\right)=\sum_{i=1}^{n} f\left(p_i\right)\sum_{j=1}^{m} f\left(q_j\right). \tag{1.7}$$

It is in this sense, equations (1.1)-(1.4) may be considered as an enlargement of sum form functional equation (1.7). Consequently, the importance of equations (1.1)-(1.4) is undeniable and their study becomes necessary.

This paper is structured as follows:
In Section 1, we have briefly touched upon a naturally occurring equation which initiated examination of the functional equations (1.1)-(1.4). Section 2 gives a concise overview of all definitions followed by some preliminary results required to develop the subsequent sections. Section 3 presents the general solutions of the functional equation (1.1) for $n \geq 3$, $m \geq 3$ being fixed integers. In Section 4, we reflect upon the general solutions of the functional equations (1.2)-(1.4) for $n \geq 3$, $m \geq 3$ being fixed integers.

2. Some preliminary results

This section presents some relevant definitions and known results.

Definition 2.1. A real-valued mapping $a : I \to \mathbb{R}$ is said to be additive on the closed unit interval I or on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if the equation $a(x + y) = a(x) + a(y)$ holds for all $(x, y) \in \Delta$. Analogously, a real-valued mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on the set of real numbers if the equation $A(x + y) = A(x) + A(y)$ holds for all $x \in \mathbb{R}$, $y \in \mathbb{R}$ by Acžel [1]. An interesting relation between these real-valued additive mappings has been established by Daróczy and Losonczi [4]. They proved that a real-valued additive mapping $a : I \to \mathbb{R}$ can be uniquely extended to the set of real numbers.

Definition 2.2. A real-valued mapping $m : I \to \mathbb{R}$ is said to be multiplicative on the closed unit interval I if $m(0) = 0$, $m(1) = 1$ and the equation $m(xy) = m(x)m(y)$ holds for all $x \in I^0$, $y \in I^0$.

Lemma 2.3 (*cf.* [5])**.** *Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geq 3$ is a fixed integer and c is a real constant. Then there exists an additive mapping $\overline{a} : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = \overline{a}(p) - \frac{1}{n}\overline{a}(1) + \frac{c}{n}$ for all $p \in I$.*

Lemma 2.4 (*cf.* [7])**.** *Let $n \geq 3$, $m \geq 3$ be fixed integers and $T : I \to \mathbb{R}$ be a mapping which satisfies the functional equation* (1.5)*. Then, for all $p \in I$, real-valued mapping T is of the form*

$$T(p) = A(p) + T(0),$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$\left.\begin{array}{l} A(1) = -mT(0) \neq -1 + T(1) - T(0) \quad \text{or} \\ A(1) = 1 - mT(0) = T(1) - T(0) \end{array}\right\}$$

or

$$T(p) = M(p) - B(p) + T(0),\ B(1) = mT(0),$$

where $B : \mathbb{R} \to \mathbb{R}$ is an additive mapping; $M : I \to \mathbb{R}$ is a nonconstant nonadditive multiplicative mapping with $M(0) = 0$ and $M(1) = 1$.

Lemma 2.5 (*cf.* [8])**.** *Let $n \geq 3$, $m \geq 3$ be fixed integers and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional equation*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} f\left(p_i\right) \sum_{j=1}^{m} h\left(q_j\right) \tag{2.1}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. Then, for all $p \in I$, any general solution (f, h) of (2.1) *is of the form*

$$\left.\begin{array}{ll} \text{(i)} & f(p) = e(p),\ e(1) = 0, \\ \text{(ii)} & h \text{ an arbitrary real-valued mapping} \end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=e_1(p)-\frac{1}{n}e_1(1)+(m-1)f(0)[h(1)+(m-1)h(0)-1]^{-1}, f(0)\neq 0,\\ & e_1(1)=nf(0)[m-h(1)-(m-1)h(0)][h(1)+(m-1)h(0)-1]^{-1},\\ \text{(ii)} & h(p)=e_2(p)-\frac{1}{m}e_2(1)+\frac{1}{m}[h(1)+(m-1)h(0)]\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=e_3(p),\ e_3(1)=f(1),\ f(1)\neq 0,\\ \text{(ii)} & h(p)=e_4(p)+h(0),\ e_4(1)=1-mh(0)\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=f(1)[M(p)-e_5(p)]+E(p),\ f(1)\neq 0,\\ & e_5(1)=mh(0), E(1)=mf(1)h(0),\\ \text{(ii)} & h(p)=M(p)-e_5(p)+h(0),\ e_5(1)=mh(0)\end{array}\right\},$$

where $e:\mathbb{R}\to\mathbb{R}$, $E:\mathbb{R}\to\mathbb{R}$ and $e_i:\mathbb{R}\to\mathbb{R}$ ($i=1$ to 5) are additive mappings; $M: I\to\mathbb{R}$ is a nonconstant nonadditive multiplicative mapping with $M(0)=0$ and $M(1)=1$.

Lemma 2.6 (*cf.* [9])**.** *Let $n\geq 3$, $m\geq 3$ be fixed integers and $f: I\to\mathbb{R}$, $h: I\to\mathbb{R}$ be mappings which satisfy the functional equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right)=\sum_{i=1}^{n} h\left(p_i\right)\sum_{j=1}^{m} h\left(q_j\right) \tag{2.2}$$

for all $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$. Then, for all $p\in I$, any general solution (f,h) of (2.2) is of the form

$$\left.\begin{array}{ll}\text{(i)} & f(p)=d_1(p)+f(0),\\ \text{(ii)} & h(p)=d_2(p)+h(0)\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=d_1(p)+f(0),\\ \text{(ii)} & h(p)=d_3(p)+h(0)\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(m-1)h(0)]^2 d_4(p)+D(p)+f(0),\\ \text{(ii)} & h(p)=[h(1)+(m-1)h(0)]d_4(p)+h(0), [h(1)+(m-1)h(0)]\neq 0\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(n-1)h(0)]^2 d_5(p)+D^*(p)+f(0),\\ \text{(ii)} & h(p)=[h(1)+(n-1)h(0)]d_5(p)+h(0), [h(1)+(n-1)h(0)]\neq 0\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(m-1)h(0)]^2[M(p)-d_6(p)]+D(p)+f(0),\\ \text{(ii)} & h(p)=[h(1)+(m-1)h(0)][M(p)-d_6(p)]+h(0), [h(1)+(m-1)h(0)]\neq 0\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(n-1)h(0)]^2[M(p)-d_7(p)]+D^*(p)+f(0),\\ \text{(ii)} & h(p)=[h(1)+(n-1)h(0)][M(p)-d_7(p)]+h(0), [h(1)+(n-1)h(0)]\neq 0\end{array}\right\}$$

with

$$(n-m)h(0)=0,$$

where $f(0), h(0)$ *are arbitrary constants;* $D:\mathbb{R}\to\mathbb{R}$, $D^*:\mathbb{R}\to\mathbb{R}$ *and* $d_j:\mathbb{R}\to\mathbb{R}$ $(j=1 \text{ to } 7)$ *are additive mappings such that*

$$\left.\begin{array}{l}
d_1(1)=-nmf(0)\\
d_2(1)=-nh(0)\\
d_3(1)=-mh(0)\\
d_4(1)=1-m[h(1)+(m-1)h(0)]^{-1}h(0)\\
d_5(1)=1-n[h(1)+(n-1)h(0)]^{-1}h(0)\\
d_6(1)=n[h(1)+(m-1)h(0)]^{-1}h(0)\\
d_7(1)=m[h(1)+(n-1)h(0)]^{-1}h(0)\\
D(1)=n\{[h(1)+(m-1)h(0)]h(0)-mf(0)\}\\
D^*(1)=m\{[h(1)+(n-1)h(0)]h(0)-nf(0)\}
\end{array}\right\}$$

and $M:I\to\mathbb{R}$ *is a nonconstant nonadditive multiplicative mapping with* $M(0)=0$ *and* $M(1)=1$.

3. The general solutions of the functional equation (1.1)

The main result of this section is the following theorem:

Theorem 3.1. *Let* $n\geq 3$, $m\geq 3$ *be fixed integers and* $f:I\to\mathbb{R}$, $h:I\to\mathbb{R}$ *be mappings which satisfy the functional equation* (1.1) *for all* $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$. *Then for all* $p\in I$, *any general solution* (f,h) *of* (1.1) *is of the form*

$$\left.\begin{array}{ll}
\text{(i)} & f(p)=a_1(p)+f(0),\ a_1(1)=-mf(0),\\
\text{(ii)} & h(p)=a_2(p)+h(0),\ a_2(1)=(n-m)f(0)-nh(0)
\end{array}\right\}\tag{3.1}$$

or

$$\left.\begin{array}{ll}
\text{(i)} & f(p)=a(p)+f(0),\ a(1)=-mf(0),\\
\text{(ii)} & h \text{ an arbitrary real-valued mapping}
\end{array}\right\}\tag{3.2}$$

or

$$\left.\begin{array}{ll}
\text{(i)} & f(p)=[f(1)+(m-1)f(0)]\,a_3(p)+f(0),\ f(1)+(m-1)f(0)\neq 0,\\
 & \quad a_3(1)=1-mf(0)\,[f(1)+(m-1)f(0)]^{-1},\\
\text{(ii)} & h(p)=a_4(p)+h(0),\ a_4(1)=1+(n-m)f(0)-nh(0)
\end{array}\right\}\tag{3.3}$$

or

$$\left.\begin{array}{ll}
\text{(i)} & f(p)=[f(1)+(m-1)f(0)]\,[M(p)-a_5(p)]+f(0), f(1)+(m-1)f(0)\neq 0,\\
 & a_5(1)=mf(0)\,[f(1)+(m-1)f(0)]^{-1},\\
\text{(ii)} & h(p)=M(p)+a_6(p)+h(0),\ a_6(1)=(n-m)f(0)-nh(0)
\end{array}\right\},\tag{3.4}$$

where $a_i:\mathbb{R}\to\mathbb{R}$ $(i=1 \text{ to } 6)$ *and* $a:\mathbb{R}\to\mathbb{R}$ *are additive mappings;* $M:I\to\mathbb{R}$ *is a nonconstant nonadditive multiplicative mapping with* $M(0)=0$ *and* $M(1)=1$.

Proof. Let us write equation (1.1) as

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right)=\left[\sum_{i=1}^{n} h\left(p_i\right) + (m-n)f(0)\right]\sum_{j=1}^{m} f\left(q_j\right) + m(n-1)f(0). \tag{3.5}$$

Case 1. $\sum\limits_{i=1}^{n} h(p_i) + (m-n)f(0)$ vanishes identically on Γ_n.

This implies

$$\sum_{i=1}^{n} h(p_i) = (n-m)f(0). \tag{3.6}$$

By Lemma 2.3, there exists an additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ such that

$$h(p) = a_2(p) - \frac{1}{n}a_2(1) + \frac{1}{n}(n-m)f(0) \tag{3.7}$$

for all $p \in I$. The substitution $p = 0$ in (3.7) gives

$$a_2(1) = (n-m)f(0) - nh(0). \tag{3.8}$$

Thus, from equations (3.7) and (3.8), solution (3.1)(ii) follows. Also, with the aid of (3.6), equation (3.5) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right)=m(n-1)f(0). \tag{3.9}$$

Now, on putting $q_1 = 1, q_2 = \ldots q_m = 0$ in equation (3.9), we obtain

$$\sum_{i=1}^{n} f(p_i) = (n-m)f(0).$$

By Lemma 2.3, there exists an additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = a_1(p) - \frac{1}{n}a_1(1) + \frac{1}{n}(n-m)f(0) \tag{3.10}$$

for all $p \in I$. The substitution $p = 0$ in (3.10) gives

$$a_1(1) = -mf(0). \tag{3.11}$$

From equations (3.10) and (3.11), the solution (3.1)(i) follows.

Case 2. $\sum\limits_{i=1}^{n} h(p_i) + (m-n)f(0)$ does not vanish identically on Γ_n.

In this case, there exists a probability distribution $(p_1^*, \ldots, p_n^*) \in \Gamma_n$ such that

$$\sum_{i=1}^{n} h(p_i^*) + (m-n)f(0) \neq 0. \tag{3.12}$$

Putting $q_1 = 1, q_2 = \cdots = q_m = 0$ in (1.1), we obtain

$$\sum_{i=1}^{n} \{f(p_i) - [f(1)+(m-1)f(0)]\, h(p_i)\} = (m-n)\,[f(1)+(m-1)f(0)]\, f(0) + (n-m)f(0).$$

By Lemma 2.3, there exists an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = [f(1)+(m-1)f(0)]\, h(p) + A_1(p) - \frac{1}{n}A_1(1) + \frac{1}{n}\Big\{(m-n)\,[f(1)+(m-1)f(0)]\, f(0) + (n-m)f(0)\Big\} \tag{3.13}$$

for all $p \in I$. The substitution $p = 0$ in (3.13) gives

$$A_1(1) = [f(1)+(m-1)f(0)]\,[nh(0) + (m-n)f(0)] - mf(0). \tag{3.14}$$

From equations (3.13) and (3.14), it follows that

$$f(p) = [f(1)+(m-1)f(0)]\,[h(p) - h(0)] + A_1(p) + f(0). \tag{3.15}$$

With the aid of equations (3.15) and (3.14), we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) =& [f(1)+(m-1)f(0)] \sum_{i=1}^{n}\sum_{j=1}^{m} h\left(p_i q_j\right) - n(m-1)\,[f(1)+(m-1)f(0)]\, h(0)\\ &+ (m-n)\,[f(1)+(m-1)f(0)]\, f(0) + m(n-1)f(0).\end{aligned} \tag{3.16}$$

Substituting (3.16) in (3.5), we get

$$\begin{aligned}&[f(1)+(m-1)f(0)] \sum_{i=1}^{n}\sum_{j=1}^{m} h\left(p_i q_j\right) = \left[\sum_{i=1}^{n} h(p_i) + (m-n)f(0)\right] \sum_{j=1}^{m} f\left(q_j\right)\\ &+ [f(1)+(m-1)f(0)]\,[n(m-1)h(0) - (m-n)f(0)]\,.\end{aligned} \tag{3.17}$$

Case 2.1. $f(1) + (m-1)f(0) = 0$.

In this case, equation (3.17) reduces to

$$\left[\sum_{i=1}^{n} h(p_i) + (m-n)f(0)\right] \sum_{j=1}^{m} f\left(q_j\right) = 0. \tag{3.18}$$

From equations (3.12) and (3.18), it can easily be concluded that

$$\sum_{j=1}^{m} f\left(q_j\right) = 0.$$

By Lemma 2.3, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = a(p) - \frac{1}{m}a(1) \tag{3.19}$$

for all $p \in I$. The substitution $p = 0$ in (3.19) gives

$$a(1) = -mf(0). \tag{3.20}$$

From equations (3.19) and (3.20), the solution (3.2)(i) follows. Now on substituting (3.2)(i) in (1.1) and using equation (3.20), we observe that "h is an arbitrary real-valued mapping". Hence solution (3.2) is obtained.

Case 2.2. $f(1) + (m-1)f(0) \neq 0$.

In this case, equation (3.15) gives

$$h(p) = [f(1)+(m-1)f(0)]^{-1}\,[f(p) - f(0)] + A_2(p) + h(0), \tag{3.21}$$

where $A_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping defined as

$$A_2(x) = \frac{-A_1(x)}{[f(1)+(m-1)f(0)]} \tag{3.22}$$

for all $x \in \mathbb{R}$. With the help of equations (3.21), (3.22) and (3.14), functional equation (1.1) can be rewritten as

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \frac{1}{[f(1)+(m-1)f(0)]}\sum_{i=1}^{n} f\left(p_i\right)\sum_{j=1}^{m} f\left(q_j\right) + \frac{(m-n)f(0)}{[f(1)+(m-1)f(0)]}\sum_{j=1}^{m} f\left(q_j\right) + m(n-1)f(0). \tag{3.23}$$

Define $\overline{f} : I \to \mathbb{R}$ as

$$\overline{f}(x) = \frac{f(x)}{[f(1)+(m-1)f(0)]} \tag{3.24}$$

for all $x \in I$. From equation (3.24), it can easily be observed that

$$\overline{f}(0) = \frac{f(0)}{[f(1)+(m-1)f(0)]} \tag{3.25}$$

and

$$\overline{f}(1)+(m-1)\overline{f}(0) = 1. \tag{3.26}$$

With the help of equations (3.23) and (3.24), the functional equation (1.5) follows. Now, by Lemma 2.4 with (3.26), we obtain

$$\overline{f}(p) = a_3(p) + \overline{f}(0), \tag{3.27}$$

where $a_3 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_3(1) = 1 - m\,\overline{f}(0)$ or

$$\overline{f}(p) = M(p) - a_5(p) + \overline{f}(0), \tag{3.28}$$

where $a_5 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_5(1) = m\,\overline{f}(0)$ and $M : I \to \mathbb{R}$ is a nonconstant nonadditive multiplicative mapping with $M(0) = 0$ and $M(1) = 1$. The solution (3.3)(i) follows from equations (3.24) and (3.27). Further, on substituting (3.3)(i) in equation (3.21), the solution (3.3)(ii) is obtained by defining the additive mapping $a_4 : \mathbb{R} \to \mathbb{R}$ as $a_4(x) = a_3(x) + A_2(x)$. Similarly, the solution (3.4) can be attained from equations (3.24), (3.28) and (3.21) by defining the additive mapping $a_6 : \mathbb{R} \to \mathbb{R}$ as $a_6(x) = -a_5(x) + A_2(x)$. This completes the proof. □

4. The general solutions of the functional equations (1.2)-(1.4)

Theorem 4.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional equation (1.2) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. Then, for all $p \in I$, any general solution (f, h) of (1.2) is of the form*

$$\left.\begin{array}{ll} \text{(i)} & f(p) = e^*(p) + f(0),\ e^*(1) = -mf(0), \\ \text{(ii)} & h \text{ an arbitrary real-valued mapping} \end{array}\right\} \tag{4.1}$$

or

$$\left.\begin{array}{ll} \text{(i)} & f(p) = e_1^*(p) + f(0),\ e_1^*(1) = -mf(0), \\ \text{(ii)} & h(p) = e_2^*(p) + h(0),\ e_2^*(1) = -mh(0) \end{array}\right\} \tag{4.2}$$

or

$$\left.\begin{array}{ll} \text{(i)} & f(p)=e_3^*(p)+f(0), \quad e_3^*(1)=f(1)-f(0), \quad f(1)+(m-1)f(0) \neq 0, \\ \text{(ii)} & h(p)=e_4^*(p)+h(0), \quad e_4^*(1)=1-mh(0) \end{array}\right\} \tag{4.3}$$

or

$$\left.\begin{array}{ll} \text{(i)} & f(p)=[f(1)+(m-1)f(0)]\,[M(p)-e_5^*(p)]+E^*(p)+f(0), \quad f(1)+(m-1)f(0)\neq 0, \quad e_5^*(1)=mh(0), \\ & E^*(1)=m\,[f(1)+(m-1)f(0)]\,h(0)-mf(0), \\ \text{(ii)} & h(p)=M(p)-e_5^*(p)+h(0),\ e_5^*(1)=mh(0) \end{array}\right\}, \tag{4.4}$$

where $e^* : \mathbb{R} \to \mathbb{R}$, $E^* : \mathbb{R} \to \mathbb{R}$ *and* $e_i^* : \mathbb{R} \to \mathbb{R}$ ($i = 1$ to 5) *are additive mappings;* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive multiplicative mapping with* $M(0) = 0$ *and* $M(1) = 1$.

Proof. Let us write equation (1.2) as

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) - m(n-1)f(0) = \left[\sum_{i=1}^{n} f(p_i) + (m-n)f(0)\right]\sum_{j=1}^{m} h\left(q_j\right). \tag{4.5}$$

Define $F : I \to \mathbb{R}$ as

$$F(x) = f(x) + mf(0)x - f(0) \tag{4.6}$$

and $H : I \to \mathbb{R}$ as

$$H(x) = h(x) + mh(0)x - h(0) \tag{4.7}$$

for all $x \in I$. From equations (4.6) and (4.7) it can easily be deduced that

$$F(0) = 0, \qquad F(1) = f(1) + (m-1)f(0) \tag{4.8}$$

and

$$H(0) = 0, \qquad H(1) = h(1) + (m-1)h(0). \tag{4.9}$$

Using (4.6) and (4.7), equation (4.5) reduces to functional equation (2.1), i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F\left(p_i q_j\right) = \sum_{i=1}^{n} F(p_i)\sum_{j=1}^{m} H\left(q_j\right), \tag{4.10}$$

where F, H are real-valued mappings each having the domain I and $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_n) \in \Gamma_m$.

Case 1. $\sum_{i=1}^{n} F(p_i)$ vanishes identically on Γ_n.

In this case, we get

$$\sum_{i=1}^{n} F(p_i) = 0$$

for all $(p_1, \ldots p_n) \in \Gamma_n$. Consequently by Lemma 2.3, there exists an additive mapping $\hat{e} : \mathbb{R} \to \mathbb{R}$ such that

$$F(p) = \hat{e}(p) - \frac{1}{n}\hat{e}(1) \tag{4.11}$$

for all $p \in I$. The substitution $p = 0$ in (4.11), using (4.8), we have

$$\hat{e}(1) = 0. \tag{4.12}$$

From equations (4.11), (4.12) and (4.6), the solution (4.1)(i) follows by defining the additive mapping $e^* : \mathbb{R} \to \mathbb{R}$ as $e^*(x) = \hat{e}(x) - mf(0)x$. Moreover, the solution (4.1)(ii) is attained from equations (4.1)(i), (4.10) and (4.7).

Case 2. $\sum_{j=1}^{m} H(q_j)$ vanishes identically on Γ_m.

In this case, we get that for all $(q_1, \ldots q_m) \in \Gamma_m$

$$\sum_{j=1}^{m} H(q_j) = 0.$$

By Lemma 2.3, there exists an additive mapping $E_2 : \mathbb{R} \to \mathbb{R}$ such that

$$H(q) = E_2(q) - \frac{1}{m} E_2(1) \tag{4.13}$$

for all $q \in I$. Using the substitution $q = 0$ in (4.13) along with equation (4.9), we obtain

$$E_2(1) = 0. \tag{4.14}$$

From equations (4.13), (4.14) and (4.7), the solution (4.2)(ii) is obtained by defining the additive mapping $e_2^* : \mathbb{R} \to \mathbb{R}$ as $e_2^*(x) = E_2(x) - mh(0)x$. From (4.2)(ii) and (4.10), we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F\left(p_i q_j\right) = 0. \tag{4.15}$$

Now on putting $p_1 = 1$, $p_2 = \ldots p_n = 0$ in (4.15), we get

$$\sum_{j=1}^{m} F(q_j) = 0$$

for all $(q_1, \ldots q_m) \in \Gamma_m$. Consequently by Lemma 2.3, there exists an additive mapping $E_1 : \mathbb{R} \to \mathbb{R}$ such that

$$F(p) = E_1(p) - \frac{1}{n} E_1(1) \tag{4.16}$$

for all $p \in I$. The substitution $p = 0$ in (4.16) along with the equation (4.8), it follows that

$$E_1(1) = 0. \tag{4.17}$$

From equations (4.16), (4.17) and (4.6), the solution (4.2)(i) follows by defining the additive mapping $e_1^* : \mathbb{R} \to \mathbb{R}$ as $e_1^*(x) = E_1(x) - mf(0)x$. This completes the solution (4.2).

Case 3. Neither $\sum_{i=1}^{n} F(p_i)$ vanishes on Γ_n nor $\sum_{j=1}^{m} H(q_j)$ vanishes on Γ_m.

In this case, by applying Lemma 2.5 on (4.10) and considering only those solutions which satisfy $F(0) = 0$ and $H(0) = 0$, we obtain solutions (4.3) and (4.4) from equations (4.6), (4.7), (4.8) and (4.9). The details are omitted for the sake of brevity. □

Theorem 4.2. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional equation* (1.3) *for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. Then for all $p \in I$, any general solution (f, h) of* (1.3) *is of the form*

$$\left.\begin{array}{ll} \text{(i)} & f(p) = b^*(p) + f(0),\ b^*(1) = -mf(0), \\ \text{(ii)} & h \text{ an arbitrary real-valued mapping} \end{array}\right\}$$

or

$$\left.\begin{array}{ll} \text{(i)} & f(p) = b_1^*(p) + f(0), \quad b_1^*(1) = -mf(0), \\ \text{(ii)} & h(p) = b_2^*(p) + h(0), \quad b_2^*(1) = -mh(0) \end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=b_3^*(p)+f(0), \quad b_3^*(1)=f(1)-f(0), \quad f(1)+(m-1)f(0)\neq 0,\\ \text{(ii)} & h(p)=b_4^*(p)+h(0), \quad b_4^*(1)=1-mh(0)\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[f(1)+(m-1)f(0)]\,[M(p)-b_5^*(p)]+B^*(p)+f(0), \quad b_5^*(1)=mh(0), \quad f(1)+(m-1)f(0)\neq 0,\\ & B^*(1)=m\,[f(1)+(m-1)f(0)]\,h(0)-mf(0),\\ \text{(ii)} & h(p)=M(p)-b_5^*(p)+h(0), \ b_5^*(1)=mh(0)\end{array}\right\},$$

where $b^*:\mathbb{R}\to\mathbb{R}$, $B^*:\mathbb{R}\to\mathbb{R}$ *and* $b_i^*:\mathbb{R}\to\mathbb{R}$ ($i=1$ to 5) *are additive mappings;* $M:I\to\mathbb{R}$ *is a nonconstant nonadditive multiplicative mapping with* $M(0)=0$ *and* $M(1)=1$.

The proof of this theorem is omitted as it is similar to that of Theorem 4.1.

Theorem 4.3. *Let* $n\geq 3$, $m\geq 3$ *be fixed integers and* $f:I\to\mathbb{R}$, $h:I\to\mathbb{R}$ *be mappings which satisfy the functional equation* (1.4) *for all* $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$. *Then for all* $p\in I$, *any general solution* (f,h) *of* (1.4) *is of the form*

$$\left.\begin{array}{ll}\text{(i)} & f(p)=d_1^*(p)+f(0),\\ \text{(ii)} & h(p)=d_2^*(p)+h(0)\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(m-1)h(0)]^2 d_3^*(p)+\overline{D}(p)+f(0),\\ \text{(ii)} & h(p)=[h(1)+(m-1)h(0)]d_3^*(p)+h(0), [h(1)+(m-1)h(0)]\neq 0\end{array}\right\}$$

or

$$\left.\begin{array}{ll}\text{(i)} & f(p)=[h(1)+(m-1)h(0)]^2[M(p)-d_4^*(p)]+\overline{D}(p)+f(0),\\ \text{(ii)} & h(p)=[h(1)+(m-1)h(0)][M(p)-d_4^*(p)]+h(0), [h(1)+(m-1)h(0)]\neq 0\end{array}\right\}$$

where $\overline{D}:\mathbb{R}\to\mathbb{R}$ *and* $d_j^*:\mathbb{R}\to\mathbb{R}$ ($j=1,2,3,4$) *are additive mappings such that*

$$\left.\begin{array}{l}d_1^*(1)=-mf(0)\\ d_2^*(1)=-mh(0)\\ d_3^*(1)=1-m[h(1)+(m-1)h(0)]^{-1}h(0)\\ d_4^*(1)=m[h(1)+(m-1)h(0)]^{-1}h(0)\\ \overline{D}(1)=m[h(1)+(m-1)h(0)]h(0)-mf(0)\end{array}\right\}$$

and $M:I\to\mathbb{R}$ *is a nonconstant nonadditive multiplicative mapping with* $M(0)=0$ *and* $M(1)=1$.

The proof of this theorem can easily be obtained by primarily defining the real-valued mappings F and H as in Theorem 4.1 (i.e. (4.6) and (4.7)), followed by applying Lemma 2.6. The details are omitted to avoid repetition.

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