Montes Taurus J. Pure Appl. Math. * (*), 1–7, 2022

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Mahgoub Transform Method for the Hyers-Ulam Stability of Differential Equations

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Abstract

In this paper we prove the Hyers-Ulam stability of general second-order linear differential equations by using Mahgoub integral transform method. Furthermore we provide some examples to illustrate main results.

Keywords: Hyers-Ulam stability, Second-order linear differential equations, Mahgoub integral transformation

2010 MSC: 34K20, 34K30, 34K05, 34B30, 39B82

1. Introduction

Mahgoub Transform is derived from the Fourier integral and was introduced by Mahgoub (*cf.* [6]) to facilitate the process of solving ordinary and partial differential equations. Based on the mathematical simplicity of the Mahgoub transform and its fundamental properties, we will study the Hyers-Ulam stability of general second-order differential equations. Aruldass *et al.* (*cf.* [3]) investigated the Hyers-Ulam stability of second-order differential equations of the form, $y^{''}(t) + \mu\vartheta(t) = r(t)$, by using Mahgoub transformation for a continuous function $r(t)$. Also, by using Mahgoub integral transform Jung *et al.* (*cf.* [5]) established the various forms of the stability of the first-order linear differential equations.

In this paper, by using the Mahgoub transform method we want to study various forms of the stability of the following equations,

$$\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) = 0 \tag{1.1}$$

and

$$\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) = r(t), \tag{1.2}$$

where λ and μ are scalars and $r(t)$ is a continuous function of exponential order.

Historically, Hyers-Ulam stability started with Ulam's following question (*cf.* [15]): *"If we change the assumptions of a mathematical statement a little, what can we say about that statement?"* and for the stability of the additive

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Received:19 September 2022, Accepted:16 December 2022
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functional equation, Hyers (*cf.* [4]) answered to Ulam. And for the concept of an approximate solution (*cf.* [11, 13]), it has been widely generalized.

After that, for the differential equations, Obłoza (*cf.* [9, 10]) generalized Ulam's question. Since then, many mathematicians investigated various forms of Hyers-Ulam stability of the differential equations by different methods. In 2008, Wang *et al.* (*cf.* [16]) used the integral factor method for the stability of differential equations. In 2013, the Laplace transform method was used by Rezaei *et al.* (*cf.* [14]) (see also *cf.* [2]) for the Hyers-Ulam stability of the form

$$\vartheta^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \vartheta^{(k)}(t) = f(t).$$

In 2019, Murali *et al.* (*cf.* [7]) used the Fourier transform method to prove the Hyers-Ulam stability of the linear differential equation (see also *cf.* [12]).

Then, recently, Murali *et al.* (*cf.* [8]) used the Abooth transform method to prove various forms of Hyers-Ulam stability of second-order linear differential equation of the form $\vartheta''(t) + \mu^2\vartheta(t) = r(t)$. In addition, many mathematicians have tried various methods. In this paper, we will try Mahgoub transformation to prove the Hyers-Ulam stability of the differential equations.

2. Preliminaries

In this section, we will recall notations, definitions and useful proposition.

In this paper, the field $\mathbb{K}$ denotes the field $\mathbb{R}$ or the field $\mathbb{C}$. And for a function $f : [0,\infty) \to \mathbb{K}$, it is said to be of exponential order when we can obtain $A, B \in \mathbb{R}$ with $|f(t)| \le Ae^{Bt}$ for all $t \ge 0$. Similarly, for a function $g : (-\infty, 0] \to \mathbb{K}$, it is said to be of exponential order when we obtain $A, B \in \mathbb{R}$ with $|g(t)| \le Ae^{Bt}$ for all $t \le 0$.

Now, we introduce the class $\mathcal{G}$,

$$\mathcal{G} = \{f \mid f \in C^2[0,\infty), \mathbb{K}] \text{ and it is of exponential order}\}.$$

For $f \in \mathcal{G}$, the integral transformation

$$\mathcal{M}\{f(t)\} = F(u) = u\int_0^\infty f(s)e^{-us}ds,$$

is called as the Mahgoub integral transform of the function f. And we denote $f(t) = \mathcal{M}^{-1}\{F(u)\}$ and we call $f(t)$ as the inverse Mahgoub transform of $F(u)$.

Also, one obtains the following properties for the Mahgoub integral transform:

Proposition 2.1 (*cf.* [1, 6]). *Assume that $f, g : [0,\infty) \to \mathbb{K}$ belong to $\mathcal{G}$. If $\mathcal{M}\{f(t)\}$ and $\mathcal{M}\{g(t)\}$ exist, then*

$$\mathcal{M}\{f * g(t)\} = \frac{1}{u}\mathcal{M}\{f(t)\}\mathcal{M}\{g(t)\},$$

where $f * g(t)$ is the convolution of two functions $f(t)$ and $g(t)$. Moreover, one has

$$\mathcal{M}\{f^{(n)}(t)\} = u^n F(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0).$$

Now, we will introduce the Mittag-Leffler function which depends on two complex parameters. The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

where $\Gamma(x)$ is the gamma function and $\alpha, \beta \in \mathbb{C}$. In this paper, we consider the Mittag-Leffler function $E_\nu(t)$ which depends on one positive real parameter ν:

$$E_\nu(t) := \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\nu n + 1)}.$$

One can note that for $\nu = 1$, we obtain

$$E_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

Definition 2.2. Let $\phi : [0,\infty) \to (0,\infty)$ be a positive real function. If there exists $K > 0$ for which the following is true for any $\varepsilon > 0$, then we say that the equation (1.2) has the Hyers-Ulam ϕ-stability (for the class $\mathcal{G}$): If a function $\vartheta \in \mathcal{G}$ satisfies

$$\left|\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)\right| \leq \rho(t)\varepsilon \tag{2.1}$$

for all $t \geq 0$, then there exists a solution $z \in \mathcal{G}$ to the equation (1.2) such that

$$|\vartheta(t) - z(t)| \leq K\rho(t)\varepsilon$$

for any $t \geq 0$. In this case, the constant K is called a Hyers-Ulam ϕ-stability constant.

Remark 2.3. In Definition 2.2, if $\phi(t) = 1$ then the equation (1.2) is said to have the Hyers-Ulam stability and we call K a Hyers-Ulam stability constant. Also, if the function $\phi(t) = E_\nu(t)$, then the equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam stability and we call K a Mittag-Leffler-Hyers-Ulam stability constant. Finally, if $\phi(t) = \rho(t)E_\nu(t)$, then we say that the equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam ϕ-stability and we call K a Mittag-Leffler-Hyers-Ulam ϕ-stability constant.

3. Main results

In this section, we will prove several forms of Hyers-Ulam stability of the differential equation (1.2) by using Mahgoub transform method. Before we prove main theorem we note several notations. For any $l, m \in \mathbb{K}$, we use the notations $\Re(l)$ and $\Re(m)$ for the real part of l and m, respectively.

Theorem 3.1. *Let $\phi : [0,\infty) \to (0,\infty)$ be an increasing function and $r : [0,\infty) \to \mathbb{K}$ be a continuous function of exponential order. Also we assume that the characteristic equation, $u^2 + \mu u + \lambda = 0$, of the equation* (1.1) *has two distinct roots l and m with $\Re(l) < 0$ and $\Re(m) < 0$. Then there exist a constant K such that, for any $\varepsilon > 0$, if a function $\vartheta \in \mathcal{G}$ satisfies the inequality*

$$\left|\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)\right| \leq \rho(t)\varepsilon \tag{3.1}$$

for all $t \geq 0$, then there exists a solution $z \in \mathcal{G}$ to the differential equation (1.2) *such that*

$$|\vartheta(t) - z(t)| \leq K\rho(t)\varepsilon$$

for all $t \geq 0$. For the case, the equation (1.2) *is said to have the Hyers-Ulam ϕ-stability (for the class $\mathcal{G}$) with a Hyers-Ulam ϕ-stability constant $K = \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|}\right)$.*

Proof. We assume that $\vartheta \in \mathcal{G}$ satisfies the inequality (3.1) for any $t \geq 0$ and a function $p : [0,\infty) \to \mathbb{K}$ by $p(t) := \vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)$ for all $t \geq 0$. Then, by (3.1) we have $|p(t)| \leq \rho(t)\varepsilon$ for each $t \geq 0$.

By Proposition 2.1, we have

$$\begin{aligned}
\mathcal{M}\{p(t)\} = P(u) &= \mathcal{M}\{\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)\} \\
&= \mathcal{M}\{\vartheta''(t)\} + \mu\mathcal{M}\{\vartheta'(t)\} + \lambda\mathcal{M}\{\vartheta(t)\} - \mathcal{M}\{r(t)\} \\
&= u^2\vartheta(u) - u\vartheta'(0) - u^2\vartheta(0) + \mu(u\vartheta(u) - u\vartheta(0)) + \lambda\vartheta(u) - R(u) \\
&= \vartheta(u)(u^2 + \mu u + \lambda) - \vartheta(0)(u^2 + \mu u) - u\vartheta'(0) - R(u),
\end{aligned}$$

which means that

$$\begin{aligned}\mathcal{M}\{\vartheta(t)\} = \vartheta(u) &= \frac{\vartheta(0)(u^2+\mu u) + u\vartheta'(0) + P(u) + R(u)}{u^2+\mu u+\lambda} \\ &= \frac{\vartheta(0)(u^2+\mu u) + u\vartheta'(0) + P(u) + R(u)}{(u-l)(u-m)},\end{aligned} \tag{3.2}$$

where l and m are the roots of the characteristic equation of (1.1). We note that $u^2+\mu u+\lambda = (u-l)(u-m)$, $l+m=-\mu$, and $lm=\lambda$.

If we define $G_{l,m}(t) := \frac{e^{lt}-e^{mt}}{l-m}$ and

$$z(t) := \vartheta(0)\frac{le^{lt}-me^{mt}}{l-m} + (\mu\vartheta(0)+\vartheta'(0))G_{l,m}(t) + (G_{l,m} * r)(t),$$

then $z(0)=\vartheta(0)$, $z'(0)=\vartheta'(0)$, and $z \in \mathcal{G}$. Moreover, we have $\mathcal{M}\{G_{l,m}(t)\} = \frac{u}{(u-l)(u-m)}$ and

$$\mathcal{M}\{z(t)\} = Z(u) = \frac{\vartheta(0)u^2}{(u-l)(u-m)} + \frac{(\mu\vartheta(0)+\vartheta'(0))u}{(u-l)(u-m)} + \frac{R(u)}{(u-l)(u-m)}. \tag{3.3}$$

On the other hand, we get

$$\mathcal{M}\{z''(t)+\mu z'(t)+\lambda z(t)\} = (u^2+\mu u+\lambda)Z(u) - (u^2+\mu u)z(0) - uz'(0).$$

The relation, together with (3.3), implies that

$$\mathcal{M}\{z''(t)+\mu z'(t)+\lambda z(t)\} = R(u) = \mathcal{M}\{r(t)\}.$$

So, we have $z''(t)+\mu z'(t)+\lambda z(t) = r(t)$ which means that $z(t)$ is a solution to the equation (1.2).

Now, by (3.2) and (3.3), one has

$$\mathcal{M}\{\vartheta(t)-z(t)\} = \mathcal{M}\{\vartheta(t)\} - \mathcal{M}\{z(t)\} = \frac{1}{u}P(u)\frac{u}{(u-l)(u-m)} = \mathcal{M}\{(p * G_{l,m})(t)\},$$

where $\mathcal{M}\{G_{l,m}(t)\} = \frac{u}{(u-l)(u-m)}$. Therefore, we see that $\vartheta(t)-z(t) = (p * G_{l,m})(t)$ for any $t \geq 0$.

Taking modulus on both sides, we obtain

$$\begin{aligned}|\vartheta(t)-z(t)| &= \left|\int_0^t p(s)G_{l,m}(t-s)ds\right| \leq \int_0^t |p(s)||G_{l,m}(t-s)|ds \\ &\leq \int_0^t \rho(s)\varepsilon|G_{l,m}(t-s)|ds \leq \rho(t)\varepsilon\int_0^t \left|\frac{e^{l(t-s)}-e^{m(t-s)}}{l-m}\right|ds \\ &\leq \frac{\phi(t)\varepsilon}{|l-m|}\left(e^{\Re(l)t}\int_0^t e^{-\Re(l)s}ds + e^{\Re(m)t}\int_0^t e^{-\Re(m)s}ds\right) \\ &\leq K\rho(t)\varepsilon\end{aligned}$$

for any $t \geq 0$ and $K = \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|}+\frac{1}{|\Re(m)|}\right)$. This completes the proof. □

Since $\phi(t) \equiv 1$ also satisfies all the assumptions of Theorem 3.1, we obtain the Hyers-Ulam stability of the linear differential equation (1.2).

Corollary 3.2. *Let $r : [0,\infty) \to \mathbb{K}$ be a continuous function of exponential order. Also we assume that the characteristic equation, $u^2+\mu u+\lambda = 0$, of the equation (1.1) has two distinct roots l and m with $\Re(l) < 0$ and $\Re(m) < 0$. Then the equation (1.2) has the Hyers-Ulam stability (for the class $\mathcal{G}$) with a Hyers-Ulam stability constant $K = \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|}+\frac{1}{|\Re(m)|}\right)$.*

Next, one note that the Mittag-Leffler function $E_\nu(t)$ is an increasing function when ν is a positive real number. So, if we replace $\phi(t)$ of Theorem 3.1 with $E_\nu(t)$ then we obtain the Mittag-Leffler-Hyers-Ulam stability of the linear differential equation (1.2).

Corollary 3.3. *Let $r : [0,\infty) \to \mathbb{K}$ be a continuous function of exponential order. Also we assume that the characteristic equation, $u^2 + \mu u + \lambda = 0$, of the equation* (1.1) *has two distinct roots l and m with $\Re(l) < 0$ and $\Re(m) < 0$. Then the equation* (1.2) *has the Mittag-Leffler-Hyers-Ulam stability (for the class $\mathcal{G}$) with a Mittag-Leffler-Hyers-Ulam stability constant $K = \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|}\right)$, where ν is a positive real number.*

Finally, one note that $\phi(t)E_\nu(t)$ is an increasing function when ν is a positive real number and $\phi(t)$ is an increasing function. So, if we replace $\phi(t)$ of Theorem 3.1 with $\phi(t)E_\nu(t)$ then one has the Mittag-Leffler-Hyers-Ulam ϕ-stability of the linear differential equation (1.2).

Corollary 3.4. *Let $\phi : [0,\infty) \to (0,\infty)$ be an increasing function and $r : [0,\infty) \to \mathbb{K}$ is a continuous function of exponential order. We also assume that the characteristic equation, $u^2 + \mu u + \lambda = 0$, of the equation* (1.1) *has two distinct roots l and m with $\Re(l) < 0$ and $\Re(m) < 0$. Then the equation* (1.2) *has the Mittag-Leffler-Hyers-Ulam ϕ-stability (for the class $\mathcal{G}$) with a Mittag-Leffler-Hyers-Ulam ϕ-stability constant $K = \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|}\right)$, where ν is a positive real number.*

4. Examples

Here, we will introduce examples to help the reader understand main theorems.

Example 4.1. Let us consider the following equation:

$$\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) = 0. \tag{4.1}$$

Assume that a function $\vartheta : [0,\infty) \to (0,\infty)$ of exponential order satisfies the inequality

$$\left|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t)\right| \le \varepsilon$$

for any $t \ge 0$ and for some $\varepsilon > 0$. Then $l = -2$ and $m = -1$ are distinct roots of the corresponding characteristic equation $u^2 + 3u + 2 = 0$. Hence, by Theorem 3.1, we obtain a solution $z \in \mathcal{G}$ of the equation (4.1) such that

$$|\vartheta(t) - z(t)| \le \frac{3}{2}\varepsilon$$

for any $t \ge 0$. In particular, $z(t) = c_1 e^{-2t} + c_2 e^{-t}$ for some constants $c_1, c_2 \in \mathbb{K}$.

Example 4.2. Let us consider the equation,

$$\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) = 4e^{3t}. \tag{4.2}$$

Assume that a function $\vartheta : [0,\infty) \to (0,\infty)$ of exponential order satisfies the inequality

$$\left|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) - 4e^{3t}\right| \le \varepsilon$$

for any $t \ge 0$ and for some $\varepsilon > 0$. Then $l = -2$ and $m = -1$ are distinct roots of the corresponding characteristic equation $u^2 + 3u + 2 = 0$. Hence, Theorem 3.1 implies that there exists a solution $z \in \mathcal{G}$ of differential equation (4.2) such that

$$|\vartheta(t) - z(t)| \le \frac{3}{2}\varepsilon$$

for all $t \ge 0$. In particular, $z(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{5}e^{3t}$ for some constants $c_1, c_2 \in \mathbb{K}$.

Example 4.3. We now assume that a function $\vartheta : [0,\infty) \to (0,\infty)$ of exponential order satisfies the inequality $\left|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t)\right| \le \varepsilon t$ for any $t \ge 0$ and for some $\varepsilon > 0$. Then $l = -2$ and $m = -1$ are distinct roots of the corresponding characteristic equation $u^2 + 3u + 2 = 0$. Since $\phi(t) = t$ is an increasing function, by Theorem 3.1, we obtain a solution $z \in \mathcal{G}$ of the equation (4.1) such that

$$|\vartheta(t) - z(t)| \le \frac{3}{2}\varepsilon t$$

for any $t \ge 0$. In particular, $z(t) = c_1 e^{-2t} + c_2 e^{-t}$ for some constants $c_1, c_2 \in \mathbb{K}$.

Example 4.4. We now assume that a function $\vartheta : [0,\infty) \to (0,\infty)$ of exponential order satisfies the differential inequality

$$\left|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) - 4e^{3t}\right| \le \varepsilon E_\nu(t)$$

for any $t \ge 0$ and for some $\varepsilon > 0$, where ν is a fixed positive real number. Then, $l = -2$ and $m = -1$ are distinct roots of the corresponding characteristic equation $u^2 + 3u + 2 = 0$.
Since $E_\nu(t)$ is an increasing function, by Corollary 3.3, we have a solution $z \in \mathcal{G}$ of the equation (4.2) such that $|\vartheta(t) - z(t)| \le \dfrac{3}{2}\varepsilon E_\nu(t)$ for all $t \ge 0$. In particular, $z(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{5}e^{3t}$ for some constants $c_1, c_2 \in \mathbb{K}$.

5. Discussion

In this paper, we introduced another new method to study various forms of the Hyers-Ulam stability of general second-order linear differential equations. In this section, we want to see what we can say about our main results when the domain is non-positive real numbers $t \ge 0$. For arbitrary $\varepsilon > 0$, let us consider

$$\left|\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)\right| \le \varepsilon \quad (\text{for } t \le 0), \tag{5.1}$$

where $\vartheta : (-\infty, 0] \to \mathbb{K}$ and $r : (-\infty, 0] \to \mathbb{K}$ be a continuous functions of exponential order.

Now, we want to define a new functions by $y_1(t) = y(-t)$ and $r_1(t) = r(-t)$ for any $t \ge 0$, then due to (5.1) we have

$$\left|y_1''(t) - \mu y_1'(t) + \lambda y_1(t) - r_1(t)\right| \le \varepsilon \tag{5.2}$$

for any $t \ge 0$.

Therefore, if the quadratic equation, $u^2 - \mu u + \lambda = 0$, has two distinct roots l and m with $\Re(l) < 0$ and $\Re(m) < 0$, then by Theorem 3.1 and (5.2), we have the solution function $z_1 : [0,\infty) \to \mathbb{K}$ which satisfies

$$z_1''(t) - \mu z_1'(t) + \lambda z_1(t) = r_1(t) \tag{5.3}$$

and

$$|y_1(t) - z_1(t)| \le \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|}\right)\varepsilon \tag{5.4}$$

for any $t \ge 0$. Hence, by defining as $z(t) = z_1(-t)$ for each $t \le 0$, by (5.4) we have

$$|\vartheta(t) - z(t)| \le \frac{1}{|l-m|}\left(\frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|}\right)\varepsilon$$

for each $t \le 0$.

In other words, we can obtain theorems similar to our main theorems in the domain $(-\infty, 0]$.

6. Conclusion

In this paper, we proved various forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the linear differential equations of general second-order with constant coefficients using the Mahgoub transform method. In other words, we established sufficient criteria for the Hyers-Ulam stability of general second-order linear differential equations with constant coefficients using the Mahgoub transform method.

Moreover, we provided a new method to investigate the Hyers-Ulam stability of differential equations. This is the first attempt to use the Mahgoub transform to prove the Hyers-Ulam stability for linear differential equations of the general second-order. Furthermore, we showed that the Mahgoub transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients.

Acknowledgment

This paper is dedicated to Professor Themistocles M. Rassias on the occasion of his 70th birthday.

The authors are very grateful to the editor for the worthwhile comments. Also, we would like to thank the referees for their valuable suggestions to the improvement of this paper.

Author Contributions: The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Conflict of Interest: The authors declare that they have no conflicts of interest.

Funding (Financial Disclosure): Not applicable.

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