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r-Bell polynomials and derangement polynomials identities using exponential partial Bell polynomials

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Abstract

With exponential partial Bell polynomials we manage to study r-Bell polynomials and derangement polynomials to revisit corresponding explicit formulae. This study leads to some interesting identities which give new link between these polynomials, and the explicit formula of the Bell-based Bernoulli polynomials of higher order.

Keywords: Bell polynomials, derangement polynomials, generating functions

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1. Introduction

In the literature numerous works are done on r-Bell and derangement polynomials. Serafin [23] studied these polynomials and derived new congruences that generalize these already known. He also emphasized the links between these polynomials. Mező [15] studied r-Bell polynomials and obtained some combinatorial, algebraic and analytical properties which satisfy. In [18], Mihoubi et al. provided some associated linear recurrences. In [17], the authors studied arithmetic properties and constructed new congruences satisfied by r-Bell numbers. These polynomials are a generalization of the Bell polynomials, they are not the only ones, there is also reciprocal degenerate Bell polynomials recently investigated in [12]. In another work Kim et al. [13] investigated derangement polynomials using umbral calculus techniques, and introduced the higher-order r-derangement polynomials. The link of these numbers to set partitions is explained in [16]. Given their importance in combinatorial analysis Jang et al. [10] obtained some identities which involve derangement polynomials. In this work we give new explicit formulae of r-Bell and derangement polynomials, we derive some identities which relate r-Bell polynomials to derangement polynomials via the exponential partial Bell polynomials (*cf.* [2, 6]). We revisit some results given by Mező and Serafin [15, 23] and we compute the explicit formula of the Bell-based Bernoulli polynomials of higher order introduced by Duran et al. [7]. The Bell numbers $B_n^\star$ (*cf.* [22, Chapter 5 p. 459]) are the number of partitions of a set of n objects and the r-Bell numbers $B_{n,r}^\star$ count partitions of a set of $n+r$ objects such that r chosen objects are separated (*cf.* [15]). The r-Bell polynomials $B_{n,r}^\star(x)$ and derangement polynomials are defined respectively by the exponential generating functions

$$\mathcal{B}(x,z) = e^{x(e^z-1)+rz} = \sum_{n\geq 0} B_{n,r}^\star(x)\frac{z^n}{n!} \tag{1.1}$$

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and

$$\mathcal{D}(x,z) = \frac{e^{(x-1)z}}{1-z} = \sum_{n\geq 0} \mathcal{D}_n(x)\frac{z^n}{n!}. \tag{1.2}$$

For some properties of these polynomials we refer to [3, 18]. Clearly $B^\star_{n,r}(1) = B^\star_{n,r}$, $B^\star_{n,r}(0) = r^n$ and $B^\star_n = B^\star_{n,0}(1)$ are so called Touchard numbers. The derangement numbers are

$$\mathcal{D}_n = \mathcal{D}_n(0) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

For more properties of these numbers one can consult [21] and references therein. The polynomials $B^\star_{n,r}(x)$ are linked to r-Stirling number of second kind (*cf.* [4]) by the following relation (*cf.* [18, 23]):

$$B^\star_{n,r}(x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k. \tag{1.3}$$

The r-Stirling number of second kind $\left\{ {n \atop k} \right\}_r$ admits the following evaluation

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{i=0}^{k} \frac{(-1)^{k-i}(i+r)^n}{i!(k-i)!}, \tag{1.4}$$

which takes the form

$$\left\{ {n+r \atop k+r} \right\}_r = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}(i+r)^n. \tag{1.5}$$

The polynomials $B^\star_{n,r}(x)$ are topical, Duran et al. [7] defined them as follows

$$e^{y(e^z-1)}e^{xz} = \sum_{n\geq 0} Bel_n(x;y)\frac{z^n}{n!}.$$

In the case $x = 0$ they get the polynomials $Bel_n(y)$, which can be calculated as follows (*cf.* [11]):

$$Bel_n(y) = \sum_{k=0}^{n} S(n,k) y^k, \tag{1.6}$$

where $S(n,k)$ are the Stirling numbers of second kind, defined by the generating function

$$\frac{1}{k!}(e^z - 1)^k = \sum_{n\geq 0} S(n,k)\frac{z^n}{n}.$$

We have $B^\star_{n,r}(x) = Bel_n(r;x)$ and $B^\star_n(x) = Bel_n(x)$. The Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ (*cf.* [1, Chapter 12.11, p. 264]) are given respectively by the generating functions

$$\frac{z}{e^z - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!} \tag{1.7}$$

and

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n\geq 0} B_n(x) \frac{t^n}{n!}. \tag{1.8}$$

One generalization of Bernoulli numbers is Bernoulli numbers of higher order $\alpha \in \mathbb{C}$, namely $B^{(\alpha)}_n$ (*cf.* [19, p. 129]), which are defined by the generating function

$$\left(\frac{z}{e^z - 1}\right)^\alpha = \sum_{n\geq 0} B^{(\alpha)}_n \frac{z^n}{n!}. \tag{1.9}$$

Duran et al. [7] even introduced the following generalization

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{y(e^z-1)}e^{xz} = \sum_{n\geq 0} {}_{Bel}B_n^{(\alpha)}(x;y)\frac{z^n}{n!}, \tag{1.10}$$

namely the Bell-based Bernoulli polynomials of order α.

2. Connection of *r*-Bell and derangement polynomials to Bell polynomials

The exponential partial Bell polynomial $B_{n,k} = B_{n,k}(x_1,\ldots,x_n)$ (*cf.* [2, 6]) is defined by the exponential generating function

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}\frac{z^n}{n!}, \tag{2.1}$$

which admits for explicit formula the expression

$$B_{n,k} = \sum_{\substack{k_1+\cdots+k_n=k \\ \sum_{r=1}^{n} rk_r=n}} \frac{n!}{k_1!\ldots k_n!}\prod_{r=1}^{n}\left(\frac{x_r}{r!}\right)^{k_r}. \tag{2.2}$$

The (exponential) complete Bell polynomials (*cf.* [6]) $Y_n = Y_n(x_1,x_2,\ldots,x_n)$ are defined by

$$Y_n = \sum_{k=1}^{n} B_{n,k},\ Y_0 = 1 \tag{2.3}$$

and generated by

$$\exp\left(\sum_{n\geq 1} x_n\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} Y_n\frac{z^n}{n!}. \tag{2.4}$$

The polynomials $B_{n,k}$ satisfy the following recursive relation

$$B_{n,k}(x_1,x_2,\ldots,x_n) = \sum_{j=0}^{k}\binom{n}{j}x_1^j B_{n-j,k-j}(0,x_2,x_3,\ldots,x_n). \tag{2.5}$$

By convention we consider $B_{0,0} = 1$ and $B_{n,0} = 0$ for $n \geq 1$. The Stirling number of second kind $S(n,k)$ is equal to $B_{n,k}(1,1,\ldots,1)$, and their explicit formula is

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

These numbers coincide with r-stirling numbers of second kind for $r = 0$, and we write $S(n,k) = \left\{{n \atop k}\right\}$. Zhang-Yang [25] computed particular value $B_{n,k}(0,x_2,x_3,\ldots,x_n)$ by the relation

$$B_{n,k}(0,1,1,\ldots,1) = \sum_{j=0}^{k}\binom{n}{j}(-1)^j S(n-j,k-j). \tag{2.6}$$

F. Qi [20] stated that

$$B_{n,k}\left(\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n+1}\right) = \frac{n!}{(n+k)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{n+k}{k-j}S(n+j,j).$$

The exponential Bell polynomials take an important place in number theory and discrete mathematics. Especially for the series expansion of analytic functions. The series play an important role in the different branches of mathematics, some calculation techniques on series are well explained in [24]. The r-Bell polynomials and derangement polynomials are connected to complete Bell polynomial by the relations

$$B^{\star}_{n,r}(x) = Y_n\left(x + r, x^2, x^3, \dots, x^n\right) \tag{2.7}$$

and

$$\mathcal{D}_n(x) = Y_n\left(x, 1!, \dots, (n-1)!\right). \tag{2.8}$$

The proof consist to write

$$\mathcal{B}(x,z) = e^{(x+r)z+\sum_{n=2}^{\infty}\frac{x^n z^n}{n!}} = \sum_{n=0}^{\infty} Y_n\left(x + r, x^2, x^3, \dots, x^n\right)\frac{z^n}{n!}$$

and

$$\mathcal{D}(x,z) = e^{xz+\sum_{n=2}^{\infty}\frac{z^n}{n}} = \sum_{n=0}^{\infty} Y_n\left(x, 1!, \dots, (n-1)!\right)\frac{z^n}{n!}.$$

2.1. *Explicit formulae of r-Bell polynomials*

By means of the identity (2.7), a new explicit formula of r-Bell polynomials is given by the following theorem.

Theorem 2.1. *We have*

$$B^{\star}_{n,r}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j} B_{n-j,k-j}\left(0, 1, 1, \dots, 1\right)(x + r)^j x^{n-j},\ n \geq 0. \tag{2.9}$$

Proof. We have

$$Y_n\left(x + r, x^2, x^3, \dots, x^n\right) = \sum_{k=0}^{n} B_{n,k}\left(x + r, x^2, x^3, \dots, x^n\right),$$

but

$$B_{n,k}\left(x + r, x^2, x^3, \dots, x^n\right) = \sum_{j=0}^{k}\binom{n}{j}(x + r)^j B_{n-j,k-j}\left(0, x^2, x^3, \dots, x^n\right)$$

and

$$B_{n-j,k-j}\left(0, x^2, x^3, \dots, x^n\right) = x^{n-j} B_{n-j,k-j}\left(0, 1, 1, \dots, 1\right).$$

Then

$$Y_n\left(x + r, x^2, x^3, \dots, x^n\right) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j}(x + r)^j B_{n-j,k-j}\left(0, 1, 1, \dots, 1\right) x^{n-j}.$$

Combining (2.7) with the above equation, the proof of Theorem 2.1 is complete. □

As a special case, for $x = 1$ in (2.9), we have

$$B^{\star}_{n,r} = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j} B_{n-j,k-j}\left(0, 1, 1, \dots, 1\right)(r + 1)^j.$$

If $r = -1$, then

$$B^{\star}_{n,-1} = \sum_{k=0}^{n} B_{n,k}\left(0, 1, 1, \dots, 1\right) = Y_n\left(0, 1, 1, \dots, 1\right).$$

In general for two generating functions $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$, the composition $f \circ g$ is given by the generating function (*cf.* [8]):

$$f \circ g(z) = \sum_{n\geq 0} \sum_{k=0}^{n} B_{n,k}\left(1!b_1, 2!b_2, \ldots, n!b_n\right) f^{(k)}(b_0) \frac{z^n}{n!}.$$

We use this identity to give another proof of the following theorem.

Theorem 2.2 (*cf.* [15, Eq. (4), p. 2])**.** *The connection between $B^{\star}_{n,r}(x)$ and $S(n,k)$ is given by the relation*

$$B^{\star}_{n,r}(x) = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j} x^k. \tag{2.10}$$

Proof. Let the functions $f(x,z) = e^{xz}$ $g(z) = e^z - 1$ and $h(z) = e^{rz}$. Then we have $\mathcal{B}(x,z) = h(z)(f \circ g)(z)$. Since the series expansions of the functions h, g and f are

$$h(z) = \sum_{n=0}^{\infty} r^n \frac{z^n}{n!},\ g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \text{ and } f(x,z) = \sum_{n=0}^{\infty} x^n \frac{z^n}{n!},$$

then

$$f \circ g(x,z) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} B_{n,j}(1,1,\ldots) \left(\frac{\partial f(x,z)}{\partial z}|_{z=0} \right)^j \frac{z^n}{n!}.$$

Thus

$$f \circ g(z) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} S(n,j) x^j \frac{z^n}{n!}.$$

By Cauchy product of generating functions we get

$$h(z)(f \circ g)(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} S(j,k) x^k r^{n-j} \right) \frac{z^n}{n!}.$$

Thereafter

$$B^{\star}_{n,r}(x) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} S(j,k) x^k r^{n-j}$$

and

$$B^{\star}_{n,r}(x) = \sum_{k=0}^{n} \binom{n}{j} \sum_{j=k}^{n} S(j,k) x^k r^{n-j}.$$

□

Replace x by y and letting $r = 0$ to deduce the identity (1.6). The comparison with the identity (1.3) permit to give another proof of the following result.

Corollary 2.3 (*cf.* [15, Eq. (3), p. 2])**.** *For $k \geq 1$ we have*

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r = \sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j}. \tag{2.11}$$

Proof. Since $S(n,k) = 0$ for $0 \le n < k$ then

$$\sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j} = \sum_{j=0}^{n} \binom{n}{j} S(j,k) r^{n-j}$$

and

$$\sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \sum_{\ell=0}^{n} \binom{n}{j} i^{\ell} r^{n-j}$$

but we have $\sum_{j=0}^{n} \binom{n}{j} i^j r^{n-j} = (i+r)^n$. Then

$$\sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j} = \sum_{i=0}^{k} \frac{(i+r)^n (-1)^{k-i}}{i!(k-i)!}.$$

□

Corollary 2.4.

$$Y_n\left(x+r, x^2, x^3, \ldots, x^n\right) = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j} x^k. \tag{2.12}$$

Proof. The connection between the identities (2.7) and (2.10) permits to conclude the desired result. □

If $r = 0$ we have

$$Y_n\left(x, x^2, x^3, \ldots, x^n\right) = \sum_{k=0}^{n} S(n,k) x^k. \tag{2.13}$$

Some properties of these polynomials are given by Carlitz in [5]. Using exponential partial Bell polynomials we compute the explicit formula of $B_n^{(\alpha)}$ as follows. First we remember for $\sum_{n\ge 0} b_n z^n$ and $b_0 \ne 0$ that

$$\left(\sum_{n\ge 0} b_n z^n\right)^{\alpha} = b_0^{\alpha} + \sum_{n\ge 1} \sum_{k=0}^{n} (\alpha)_k b_0^{\alpha-k} B_{n,k}\left(1!b_1, 2!b_2, \ldots, n!b_n\right) \frac{z^n}{n!},$$

where $(\alpha)_k = \alpha(\alpha-1)\ldots(\alpha-k+1)$. For more information about this identity we refer to [9] and reference therein. The application to $\left(\frac{t}{e^z-1}\right)^{\alpha}$ states that

$$\left(\frac{t}{e^z-1}\right)^{\alpha} = \left(\sum_{n\ge 0} \frac{z^n}{(n+1)!}\right)^{-\alpha} = 1 + \sum_{n\ge 1} \sum_{k=1}^{n} (-\alpha)_k B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}\right) \frac{z^n}{n!}.$$

Then

$$B_n^{(\alpha)} = \sum_{k=1}^{n} (-\alpha)_k B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}\right).$$

Finally

$$B_n^{(\alpha)} = \sum_{k=1}^{n} \sum_{j=0}^{k} \frac{n!(-1)^{k-j}(-\alpha)_k}{(n+k)!} \binom{n+k}{k-j} S(n+j, j).$$

In addition to the results already established in [7]. The following theorem gives the explicit formula of ${}_{Bel}B_m^{(\alpha)}(x;y)$

Theorem 2.5.

$${}_{Bel}B_m^{(\alpha)}(x;y) = B_{m,x}^{\star}(y) + \sum_{n=1}^{m} \sum_{k=1}^{n} \sum_{j=0}^{k} \frac{m!(-1)^{k-j}(-\alpha)_k}{(n+k)!(m-n)!} \binom{n+k}{k-j} S(n+j, j) B_{m-n,x}^{\star}(y).$$

Proof. We have

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{y(e^z-1)}e^{xz} = \left(\sum_{n\geq 0} B_n^{(\alpha)}\frac{z^n}{n!}\right)\left(\sum_{n\geq 0} B_{n,x}^{\star}(y)\frac{z^n}{n!}\right).$$

Then we obtain

$$_{Bel}B_m^{(\alpha)}(x;y) = \sum_{n=0}^{m}\binom{m}{n}B_n^{(\alpha)}B_{m-n,x}^{\star}(y)$$

and the desired result follows. □

2.2. Explicit formulae of derangement polynomials

Regarding the generating function of $\mathcal{D}_n(x)$ it is easily checked that

$$\mathcal{D}_n(x) = \sum_{k=0}^{n}\binom{n}{k}k!(x-1)^{n-k}. \tag{2.14}$$

Clearly $\mathcal{D}_n(1) = n!$. Since $Y_n(x, 1!, \dots (n-1)!, \dots) = \mathcal{D}_n(x)$, then we have

$$Y_n(x, 1!, \dots (n-1)!, \dots) = \sum_{k=0}^{n}\binom{n}{k}k!(x-1)^{n-k}. \tag{2.15}$$

Some properties of $\mathcal{D}_n(x)$ and $\mathcal{D}_n$ are developed in [14]. We write $\mathcal{D}_n(x)$ in the following form

$$\mathcal{D}_n(x) = n!\sum_{k=0}^{n}\frac{1}{k!}(x-1)^k. \tag{2.16}$$

The identity

$$\mathcal{D}_{n+m}(x) \equiv (x-1)^m\mathcal{D}_n(x) \pmod{m}.$$

follows from [23, Proposition 1, p. 39]. In what follows we give another proof in a different way. We have

$$\mathcal{D}_{n+m}(x) = (n+m)!\sum_{k=0}^{m-1}\frac{1}{k!}(x-1)^k + (n+m)!\sum_{k=m}^{n+m}\frac{1}{k!}(x-1)^k.$$

Since $\frac{(n+m)!}{k!} \equiv 0 \pmod{m}$ for $0 \leq k \leq m-1$, then

$$\mathcal{D}_{n+m}(x) \equiv (n+m)!\sum_{k=m}^{n+m}\frac{1}{k!}(x-1)^k \pmod{m}.$$

But

$$\sum_{k=m}^{n+m}\frac{1}{k!}(x-1)^k = (x-1)^m\sum_{k=0}^{n+m}\frac{1}{(k+m)!}(x-1)^k.$$

The result follows from the fact that $\frac{(n+m)!}{(k+m)!}$ is an integer which satisfies $\frac{(n+m)!}{(k+m)!} \equiv \frac{n!}{k!} \pmod{m}$. Moreover with the identity

$$e^{(1-x)z}\mathcal{D}(x,z) = \frac{1}{1-z}$$

we obtain (*cf.* [23, Identity 5, p. 4]):

$$\sum_{k=0}^{n}\binom{n}{k}(1-x)^k\,\mathcal{D}_{n-k}(x) = n!. \tag{2.17}$$

Using the identity

$$(1-z)\,\mathcal{D}(x,z) = e^{(x-1)z},$$

to deduce the following identity of [23, p. 4]

$$\mathcal{D}_n(x) - n\mathcal{D}_{n-1}(x) = (x-1)^n\,,\ n \geq 1. \tag{2.18}$$

3. *r*-Bell-derangement polynomials identities

Serafin [23, Proposition 2, p. 5] showed that

$$n!B^{\star}_{n,r}(x) = \sum_{i=0}^{n}(i+r)^n x^i \binom{n}{i}\mathcal{D}_{n-i}(1-x). \tag{3.1}$$

Additional r-Bell-Derangement polynomials identity is given by the following theorem.

Theorem 3.1. *Let us denote*

$$\sum_{n,j,k}^{m} = \sum_{n=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{n-j}.$$

Then for $m \geq 1$ *the algebraic distance* $B^{\star}_{m,r}(x) - \mathcal{D}_m(x)$ *is a sequence of polynomials given by the expression*

$$\begin{aligned} B^{\star}_{m,r}(x) - \mathcal{D}_m(x) \quad = \quad & \sum_{n,j,k}^{m}\binom{m}{n}\binom{n}{j}B_{n-j,k}(0,1,1,\dots)\,(r+1-j)\,(r+1)^{j-1}\,\mathcal{D}_{m-n}(x)x^k \\ & + \sum_{n=1}^{m}\sum_{k=0}^{n}\binom{m}{n}B_{n,k}(0,1,1,\dots)\mathcal{D}_{m-n}(x)x^k. \end{aligned} \tag{3.2}$$

Proof. First we remark that

$$e^{x(e^z-1)+rz} = e^{x(e^z-1-z)+(r+1)z}e^{(x-1)z},$$

which means that

$$\mathcal{B}(x,z) = (1-z)e^{x(e^z-1-z)+(r+1)z}\mathcal{D}(x,z).$$

Let $\lambda(z)$ be the generating function

$$\lambda(z) = e^z - 1 - z = \sum_{n\geq 2}\frac{z^n}{n!}.$$

Then we obtain

$$e^{x\lambda(z)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}B_{n,k}(0,1,1,\dots,1)x^k\frac{z^n}{n!}.$$

But we have

$$(1-z)e^{(r+1)z} = \sum_{n=0}^{\infty}(r+1)^n\frac{z^n}{n!} - \sum_{n=1}^{\infty}n(r+1)^{n-1}\frac{z^n}{n!}.$$

Let a_n be the sequence defined by $a_0 = 1$ and $a_n = (r+1-n)\,(r+1)^{n-1}$, then

$$(1-z)e^{(r+1)z} = 1 + \sum_{n=1}^{\infty}(r+1-n)\,(r+1)^{n-1}\,\frac{z^n}{n!} = \sum_{n=0}^{\infty}a_n\frac{z^n}{n!}.$$

The Cauchy product of $(1-z)e^{(r+1)z}$ and $e^{x\lambda(z)}$ conducts to

$$(1-z)e^{(r+1)z+x\lambda(z)} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}B_{n-j,k}(0,1,1,\dots,1)a_j x^k\frac{z^n}{n!}.$$

Let the auxiliary sequence β_n given by the expression

$$\beta_n = \sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}B_{n-j,k}(0,1,1,\dots,1)a_j x^k.$$

It is obvious that $\beta_0 = 1$ and for $n \geq 1$;

$$\beta_n = \sum_{j=1}^{n} \binom{n}{j} \sum_{k=0}^{n-j} B_{n-j,k}(0,1,1,\ldots,1)\,(r+1-j)\,(r+1)^{j-1}\,x^k + \sum_{k=0}^{n} B_{n,k}(0,1,1,\ldots,1)x^k.$$

Finally for $m \geq 1$ we obtain

$$B^{\star}_{m,r}(x) = \mathcal{D}_m(x) + \sum_{n=1}^{m} \binom{m}{n} \beta_n \mathcal{D}_{m-n}(x).$$

□

For the special cases $x \in \{0, 1\}$, the following corollary is immediate.

Corollary 3.2. *We have*

$$\begin{aligned} B^{\star}_{m,r} &= m! + \sum_{n,j,k}^{m} \binom{m}{n}\binom{n}{j}(m-n)!B_{n-j,k}(0,1,1,\ldots,1)\,(r+1-j)\,(r+1)^{j-1} \\ &\quad + \sum_{n=1}^{m}\sum_{k=0}^{n}\binom{m}{n}(m-n)!B_{n,k}(0,1,1,\ldots) \end{aligned} \tag{3.3}$$

and

$$r^m = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} + \sum_{n=1}^{m}\sum_{k=0}^{m-n} \frac{m!}{k!n!}(-1)^k\,(r+1-n)\,(r+1)^{n-1}\,. \tag{3.4}$$

Proof. The identity (3.3) follows from the identity (3.2) by taking $x = 1$. The identity (3.4) follows by taking $x = 0$ in the identity (3.2). □

One can write

$$r^m = \mathcal{D}_m + \sum_{n=1}^{m} \binom{m}{n} (r+1-n)\,(r+1)^{n-1}\,\mathcal{D}_{m-n}$$

and

$$r^m = \mathcal{D}_m + \sum_{n=1}^{m} \binom{m}{n} (r+1)^n\,\mathcal{D}_{m-n} - \sum_{n=1}^{m} \binom{m}{n} n\,(r+1)^{n-1}\,\mathcal{D}_{m-n}.$$

Thus

$$r^m = \mathcal{D}_m + (r+1)^m - m\mathcal{D}_{m-1} + \sum_{n=1}^{m-1} \binom{m}{n} (r+1)^n\,\mathcal{D}_{m-n} - \sum_{n=1}^{m-1} \binom{m}{n+1} (n+1)\,(r+1)^n\,\mathcal{D}_{m-n-1}.$$

But

$$\frac{m!}{n!(m-n)!}\,(r+1)^n\,\mathcal{D}_{m-n} - \frac{m!}{n!(m-n-1)!}\,(r+1)^n\,\mathcal{D}_{m-n-1} = \frac{m!(r+1)^n}{n!(m-n-1)!}\left(\frac{\mathcal{D}_{m-n}}{m-n} - \mathcal{D}_{m-n-1}\right)$$

and

$$\frac{\mathcal{D}_{m-n}}{m-n} - \mathcal{D}_{m-n-1} = (m-n-1)!\frac{(-1)^{m-n}}{(m-n)!} = \frac{(-1)^{m-n}}{m-n}.$$

Then the identity (3.4) is another reformulation of the combinatorial identity:

$$r^m = \sum_{n=0}^{m} \binom{m}{n} (-1)^{m-n}\,(r+1)^n\,,$$

which is obtained from the fact that $r^m = (r+1-1)^m$. Using Cauchy product of generating functions, we prove the following theorem.

Theorem 3.3. *Let us denote*

$$\sum_{n,k,j,\ell}^{m} = \sum_{n=1}^{m}\sum_{k=1}^{n}\sum_{j=0}^{k}\sum_{\ell=0}^{m-n}.$$

Then for $m \geq 1$ we have

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{D}_n(x)B^{\star}_{m-n,r}(x) = m!\sum_{n,k,j,\ell}^{m}\frac{(r-1)^{\ell}}{n!\ell!}\binom{n}{j}2^{j}B_{n-j,k-j}\left(0,1,1,\ldots,1\right)x^{k} + m!\sum_{k=0}^{m}\frac{(r-1)^{k}}{k!}. \tag{3.5}$$

Proof. We have

$$\mathcal{B}(x,z)\mathcal{D}(x,z) = \frac{e^{x(e^{z}+z-1)+(r-1)z}}{1-z}.$$

But

$$\frac{e^{(r-1)z}}{1-z} = \left(\sum_{n=0}^{\infty}z^{n}\right)\left(\sum_{n=0}^{\infty}(r-1)^{n}\frac{z^{n}}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(r-1)^{k}}{k!}\right)z^{n}.$$

Otherwise

$$e^{x(e^{z}+z-1)} = e^{x\left(2z+\sum_{n\geq 2}\frac{z^{n}}{n!}\right)} = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}B_{n,k}\left(2,1,1,\ldots,1\right)x^{k}\frac{z^{n}}{n!}.$$

From the property (2.5) we conclude that

$$B_{n,k}\left(2,1,\ldots,1\right) = \sum_{j=0}^{k}\binom{n}{j}2^{j}B_{n-j,k-j}\left(0,1,1,\ldots,1\right)$$

and

$$e^{x\left(e^{t}+t-1\right)} = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{j}2^{j}B_{n-j,k-j}\left(0,1,1,\ldots,1\right)x^{k}\frac{t^{n}}{n!}.$$

Let b_n and c_n be to sequence defined respectively by $b_n = \sum_{k=0}^{n}\frac{(r-1)^{k}}{k!}$, $c_0 = 1$ and

$$c_n = \frac{1}{n!}\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{j}2^{j}B_{n-j,k-j}\left(0,1,1,\ldots,1\right)x^{k};\ n \geq 1.$$

The Cauchy product of the functions $e^{x(e^{z}+z-1)}$ and $(1-z)^{-1}\,e^{(r-1)z}$ conducts to

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{D}_n(x)B^{\star}_{m-n,r}(x) = m!\sum_{n=0}^{m}c_nb_{m-n} = m!b_m + m!\sum_{n=1}^{m}c_nb_{m-n}$$

and the desired result follows. □

Corollary 3.4.

$$\sum_{n=0}^{m}\binom{m}{n}n!B^{\star}_{m-n,r} = m!\sum_{n=1}^{m}\sum_{k=1}^{n}\sum_{j=0}^{k}\sum_{\ell=0}^{m-n}\frac{(r-1)^{\ell}}{n!\ell!}\binom{n}{j}2^{j}B_{n-j,k-j}\left(0,1,1,\ldots,1\right) + m!\sum_{k=0}^{m}\frac{(r-1)^{k}}{k!} \tag{3.6}$$

and

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{D}_n r^{m-n} = m!\sum_{k=0}^{m}\frac{(r-1)^{k}}{k!}. \tag{3.7}$$

Proof. Since $\mathcal{D}_n(1) = n!$, then identity (3.6) follows directly from the identity (3.5). But if we take $x = 0$, we get the identity (3.7). □

4. Conclusion

In this work we are interested by r-Bell and derangement polynomials. We have shown that they are special cases of complete Bell polynomials. This made it easier to give new expressions of these polynomials and link between them. We established new proofs of some results in the literature due to Mező [15] and Serafin [23]. What encouraged us to give the explicit formula of Bell-based Bernoulli polynomials of higher order recently introduced by Duran et al. [7], this is both a generalization of Bell and Bernoulli polynomials.

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