



# Mahgoub Transform Method for the Hyers-Ulam Stability of Differential Equations

Ponmana Selvan Arumugam<sup>a</sup>, Sandra Pinelas<sup>b</sup>

<sup>a</sup>Department of Mathematics, Kings Engineering College, Irungattukottai, Sri Perumbudur, Chennai – 602 117, Tamil Nadu, India

<sup>b</sup>Department of Exact Science and Engineering, Academia Militar, Lisboa, 2720-113, Amadora, Portugal

## Abstract

In this paper we prove the Hyers-Ulam stability of general second-order linear differential equations by using Mahgoub integral transform method. Furthermore we provide some examples to illustrate main results.

**Keywords:** Hyers-Ulam stability, Second-order linear differential equations, Mahgoub integral transformation

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## 1. Introduction

Mahgoub Transform is derived from the Fourier integral and was introduced by Mahgoub (*cf.* [6]) to facilitate the process of solving ordinary and partial differential equations. Based on the mathematical simplicity of the Mahgoub transform and its fundamental properties, we will study the Hyers-Ulam stability of general second-order differential equations. Aruldass *et al.* (*cf.* [3]) investigated the Hyers-Ulam stability of second-order differential equations of the form,  $y''(t) + \mu\vartheta(t) = r(t)$ , by using Mahgoub transformation for a continuous function  $r(t)$ . Also, by using Mahgoub integral transform Jung *et al.* (*cf.* [5]) established the various forms of the stability of the first-order linear differential equations.

In this paper, by using the Mahgoub transform method we want to study various forms of the stability of the following equations,

$$\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) = 0 \quad (1.1)$$

and

$$\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) = r(t), \quad (1.2)$$

where  $\lambda$  and  $\mu$  are scalars and  $r(t)$  is a continuous function of exponential order.

Historically, Hyers-Ulam stability started with Ulam's following question (*cf.* [15]): "If we change the assumptions of a mathematical statement a little, what can we say about that statement?" and for the stability of the additive

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Email addresses: [selvaharry@yahoo.com](mailto:selvaharry@yahoo.com) (Ponmana Selvan Arumugam), [sandra.pinelas@gmail.com](mailto:sandra.pinelas@gmail.com) (Sandra Pinelas)

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\*Corresponding Author: Sandra Pinelas



functional equation, Hyers (cf. [4]) answered to Ulam. And for the concept of an approximate solution (cf. [11, 13]), it has been widely generalized.

After that, for the differential equations, Obloza (cf. [9, 10]) generalized Ulam’s question. Since then, many mathematicians investigated various forms of Hyers-Ulam stability of the differential equations by different methods. In 2008, Wang *et al.* (cf. [16]) used the integral factor method for the stability of differential equations. In 2013, the Laplace transform method was used by Rezaei *et al.* (cf. [14]) (see also cf. [2]) for the Hyers-Ulam stability of the form

$$\vartheta^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \vartheta^{(k)}(t) = f(t).$$

In 2019, Murali *et al.* (cf. [7]) used the Fourier transform method to prove the Hyers-Ulam stability of the linear differential equation (see also cf. [12]).

Then, recently, Murali *et al.* (cf. [8]) used the Abooth transform method to prove various forms of Hyers-Ulam stability of second-order linear differential equation of the form  $\vartheta''(t) + \mu^2 \vartheta(t) = r(t)$ . In addition, many mathematicians have tried various methods. In this paper, we will try Mahgoub transformation to prove the Hyers-Ulam stability of the differential equations.

## 2. Preliminaries

In this section, we will recall notations, definitions and useful proposition.

In this paper, the field  $\mathbb{K}$  denotes the field  $\mathbb{R}$  or the field  $\mathbb{C}$ . And for a function  $f : [0, \infty) \rightarrow \mathbb{K}$ , it is said to be of exponential order when we can obtain  $A, B \in \mathbb{R}$  with  $|f(t)| \leq Ae^{Bt}$  for all  $t \geq 0$ . Similarly, for a function  $g : (-\infty, 0] \rightarrow \mathbb{K}$ , it is said to be of exponential order when we obtain  $A, B \in \mathbb{R}$  with  $|g(t)| \leq Ae^{Bt}$  for all  $t \leq 0$ .

Now, we introduce the class  $\mathcal{G}$ ,

$$\mathcal{G} = \{f \mid f \in C^2[0, \infty), \mathbb{K}\} \text{ and it is of exponential order}.$$

For  $f \in \mathcal{G}$ , the integral transformation

$$\mathcal{M}\{f(t)\} = F(u) = u \int_0^\infty f(s)e^{-us} ds,$$

is called as the Mahgoub integral transform of the function  $f$ . And we denote  $f(t) = \mathcal{M}^{-1}\{F(u)\}$  and we call  $f(t)$  as the inverse Mahgoub transform of  $F(u)$ .

Also, one obtains the following properties for the Mahgoub integral transform:

**Proposition 2.1** (cf. [1, 6]). *Assume that  $f, g : [0, \infty) \rightarrow \mathbb{K}$  belong to  $\mathcal{G}$ . If  $\mathcal{M}\{f(t)\}$  and  $\mathcal{M}\{g(t)\}$  exist, then*

$$\mathcal{M}\{f * g(t)\} = \frac{1}{u} \mathcal{M}\{f(t)\} \mathcal{M}\{g(t)\},$$

where  $f * g(t)$  is the convolution of two functions  $f(t)$  and  $g(t)$ . Moreover, one has

$$\mathcal{M}\{f^{(n)}(t)\} = u^n F(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0).$$

Now, we will introduce the Mittag-Leffler function which depends on two complex parameters. The Mittag-Leffler function  $E_{\alpha,\beta}(z)$  is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)},$$

where  $\Gamma(x)$  is the gamma function and  $\alpha, \beta \in \mathbb{C}$ . In this paper, we consider the Mittag-Leffler function  $E_\nu(t)$  which depends on one positive real parameter  $\nu$ :

$$E_\nu(t) := \sum_{n=0}^\infty \frac{t^n}{\Gamma(\nu n + 1)}.$$

One can note that for  $\nu = 1$ , we obtain

$$E_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

**Definition 2.2.** Let  $\phi : [0, \infty) \rightarrow (0, \infty)$  be a positive real function. If there exists  $K > 0$  for which the following is true for any  $\varepsilon > 0$ , then we say that the equation (1.2) has the Hyers-Ulam  $\phi$ -stability (for the class  $\mathcal{G}$ ): If a function  $\vartheta \in \mathcal{G}$  satisfies

$$|\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)| \leq \rho(t)\varepsilon \tag{2.1}$$

for all  $t \geq 0$ , then there exists a solution  $z \in \mathcal{G}$  to the equation (1.2) such that

$$|\vartheta(t) - z(t)| \leq K\rho(t)\varepsilon$$

for any  $t \geq 0$ . In this case, the constant  $K$  is called a Hyers-Ulam  $\phi$ -stability constant.

*Remark 2.3.* In Definition 2.2, if  $\phi(t) = 1$  then the equation (1.2) is said to have the Hyers-Ulam stability and we call  $K$  a Hyers-Ulam stability constant. Also, if the function  $\phi(t) = E_\nu(t)$ , then the equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam stability and we call  $K$  a Mittag-Leffler-Hyers-Ulam stability constant. Finally, if  $\phi(t) = \rho(t)E_\nu(t)$ , then we say that the equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam  $\phi$ -stability and we call  $K$  a Mittag-Leffler-Hyers-Ulam  $\phi$ -stability constant.

### 3. Main results

In this section, we will prove several forms of Hyers-Ulam stability of the differential equation (1.2) by using Mahgoub transform method. Before we prove main theorem we note several notations. For any  $l, m \in \mathbb{K}$ , we use the notations  $\Re(l)$  and  $\Re(m)$  for the real part of  $l$  and  $m$ , respectively.

**Theorem 3.1.** Let  $\phi : [0, \infty) \rightarrow (0, \infty)$  be an increasing function and  $r : [0, \infty) \rightarrow \mathbb{K}$  be a continuous function of exponential order. Also we assume that the characteristic equation,  $u^2 + \mu u + \lambda = 0$ , of the equation (1.1) has two distinct roots  $l$  and  $m$  with  $\Re(l) < 0$  and  $\Re(m) < 0$ . Then there exist a constant  $K$  such that, for any  $\varepsilon > 0$ , if a function  $\vartheta \in \mathcal{G}$  satisfies the inequality

$$|\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)| \leq \rho(t)\varepsilon \tag{3.1}$$

for all  $t \geq 0$ , then there exists a solution  $z \in \mathcal{G}$  to the differential equation (1.2) such that

$$|\vartheta(t) - z(t)| \leq K\rho(t)\varepsilon$$

for all  $t \geq 0$ . For the case, the equation (1.2) is said to have the Hyers-Ulam  $\phi$ -stability (for the class  $\mathcal{G}$ ) with a Hyers-Ulam  $\phi$ -stability constant  $K = \frac{1}{|l-m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right)$ .

*Proof.* We assume that  $\vartheta \in \mathcal{G}$  satisfies the inequality (3.1) for any  $t \geq 0$  and a function  $p : [0, \infty) \rightarrow \mathbb{K}$  by  $p(t) := \vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)$  for all  $t \geq 0$ . Then, by (3.1) we have  $|p(t)| \leq \rho(t)\varepsilon$  for each  $t \geq 0$ .

By Proposition 2.1, we have

$$\begin{aligned} \mathcal{M}\{p(t)\} &= P(u) = \mathcal{M}\{\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)\} \\ &= \mathcal{M}\{\vartheta''(t)\} + \mu\mathcal{M}\{\vartheta'(t)\} + \lambda\mathcal{M}\{\vartheta(t)\} - \mathcal{M}\{r(t)\} \\ &= u^2\vartheta(u) - u\vartheta'(0) - u^2\vartheta(0) + \mu(u\vartheta(u) - u\vartheta(0)) + \lambda\vartheta(u) - R(u) \\ &= \vartheta(u)(u^2 + \mu u + \lambda) - \vartheta(0)(u^2 + \mu u) - u\vartheta'(0) - R(u), \end{aligned}$$

which means that

$$\begin{aligned} \mathcal{M}\{\vartheta(t)\} = \vartheta(u) &= \frac{\vartheta(0)(u^2 + \mu u) + u\vartheta'(0) + P(u) + R(u)}{u^2 + \mu u + \lambda} \\ &= \frac{\vartheta(0)(u^2 + \mu u) + u\vartheta'(0) + P(u) + R(u)}{(u-l)(u-m)}, \end{aligned} \tag{3.2}$$

where  $l$  and  $m$  are the roots of the characteristic equation of (1.1). We note that  $u^2 + \mu u + \lambda = (u-l)(u-m)$ ,  $l+m = -\mu$ , and  $lm = \lambda$ .

If we define  $G_{l,m}(t) := \frac{e^{lt} - e^{mt}}{l-m}$  and

$$z(t) := \vartheta(0) \frac{le^{lt} - me^{mt}}{l-m} + (\mu\vartheta(0) + \vartheta'(0))G_{l,m}(t) + (G_{l,m} * r)(t),$$

then  $z(0) = \vartheta(0)$ ,  $z'(0) = \vartheta'(0)$ , and  $z \in \mathcal{G}$ . Moreover, we have  $\mathcal{M}\{G_{l,m}(t)\} = \frac{u}{(u-l)(u-m)}$  and

$$\mathcal{M}\{z(t)\} = Z(u) = \frac{\vartheta(0)u^2}{(u-l)(u-m)} + \frac{(\mu\vartheta(0) + \vartheta'(0))u}{(u-l)(u-m)} + \frac{R(u)}{(u-l)(u-m)}. \tag{3.3}$$

On the other hand, we get

$$\mathcal{M}\{z''(t) + \mu z'(t) + \lambda z(t)\} = (u^2 + \mu u + \lambda)Z(u) - (u^2 + \mu u)z(0) - uz'(0).$$

The relation, together with (3.3), implies that

$$\mathcal{M}\{z''(t) + \mu z'(t) + \lambda z(t)\} = R(u) = \mathcal{M}\{r(t)\}.$$

So, we have  $z''(t) + \mu z'(t) + \lambda z(t) = r(t)$  which means that  $z(t)$  is a solution to the equation (1.2).

Now, by (3.2) and (3.3), one has

$$\mathcal{M}\{\vartheta(t) - z(t)\} = \mathcal{M}\{\vartheta(t)\} - \mathcal{M}\{z(t)\} = \frac{1}{u}P(u)\frac{u}{(u-l)(u-m)} = \mathcal{M}\{(p * G_{l,m})(t)\},$$

where  $\mathcal{M}\{G_{l,m}(t)\} = \frac{u}{(u-l)(u-m)}$ . Therefore, we see that  $\vartheta(t) - z(t) = (p * G_{l,m})(t)$  for any  $t \geq 0$ .

Taking modulus on both sides, we obtain

$$\begin{aligned} |\vartheta(t) - z(t)| &= \left| \int_0^t p(s)G_{l,m}(t-s)ds \right| \leq \int_0^t |p(s)||G_{l,m}(t-s)|ds \\ &\leq \int_0^t \rho(s)\varepsilon|G_{l,m}(t-s)|ds \leq \rho(t)\varepsilon \int_0^t \left| \frac{e^{l(t-s)} - e^{m(t-s)}}{l-m} \right| ds \\ &\leq \frac{\phi(t)\varepsilon}{|l-m|} \left( e^{\Re(l)t} \int_0^t e^{-\Re(l)s} ds + e^{\Re(m)t} \int_0^t e^{-\Re(m)s} ds \right) \\ &\leq K\rho(t)\varepsilon \end{aligned}$$

for any  $t \geq 0$  and  $K = \frac{1}{|l-m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right)$ . This completes the proof. □

Since  $\phi(t) \equiv 1$  also satisfies all the assumptions of Theorem 3.1, we obtain the Hyers-Ulam stability of the linear differential equation (1.2).

**Corollary 3.2.** *Let  $r : [0, \infty) \rightarrow \mathbb{K}$  be a continuous function of exponential order. Also we assume that the characteristic equation,  $u^2 + \mu u + \lambda = 0$ , of the equation (1.1) has two distinct roots  $l$  and  $m$  with  $\Re(l) < 0$  and  $\Re(m) < 0$ . Then the equation (1.2) has the Hyers-Ulam stability (for the class  $\mathcal{G}$ ) with a Hyers-Ulam stability constant  $K = \frac{1}{|l-m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right)$ .*

Next, one note that the Mittag-Leffler function  $E_\nu(t)$  is an increasing function when  $\nu$  is a positive real number. So, if we replace  $\phi(t)$  of Theorem 3.1 with  $E_\nu(t)$  then we obtain the Mittag-Leffler-Hyers-Ulam stability of the linear differential equation (1.2).

**Corollary 3.3.** *Let  $r : [0, \infty) \rightarrow \mathbb{K}$  be a continuous function of exponential order. Also we assume that the characteristic equation,  $u^2 + \mu u + \lambda = 0$ , of the equation (1.1) has two distinct roots  $l$  and  $m$  with  $\Re(l) < 0$  and  $\Re(m) < 0$ . Then the equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability (for the class  $\mathcal{G}$ ) with a Mittag-Leffler-Hyers-Ulam stability constant  $K = \frac{1}{|l-m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right)$ , where  $\nu$  is a positive real number.*

Finally, one note that  $\phi(t)E_\nu(t)$  is an increasing function when  $\nu$  is a positive real number and  $\phi(t)$  is an increasing function. So, if we replace  $\phi(t)$  of Theorem 3.1 with  $\phi(t)E_\nu(t)$  then one has the Mittag-Leffler-Hyers-Ulam  $\phi$ -stability of the linear differential equation (1.2).

**Corollary 3.4.** *Let  $\phi : [0, \infty) \rightarrow (0, \infty)$  be an increasing function and  $r : [0, \infty) \rightarrow \mathbb{K}$  is a continuous function of exponential order. We also assume that the characteristic equation,  $u^2 + \mu u + \lambda = 0$ , of the equation (1.1) has two distinct roots  $l$  and  $m$  with  $\Re(l) < 0$  and  $\Re(m) < 0$ . Then the equation (1.2) has the Mittag-Leffler-Hyers-Ulam  $\phi$ -stability (for the class  $\mathcal{G}$ ) with a Mittag-Leffler-Hyers-Ulam  $\phi$ -stability constant  $K = \frac{1}{|l-m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right)$ , where  $\nu$  is a positive real number.*

#### 4. Examples

Here, we will introduce examples to help the reader understand main theorems.

**Example 4.1.** Let us consider the following equation:

$$\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) = 0. \tag{4.1}$$

Assume that a function  $\vartheta : [0, \infty) \rightarrow (0, \infty)$  of exponential order satisfies the inequality

$$|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t)| \leq \varepsilon$$

for any  $t \geq 0$  and for some  $\varepsilon > 0$ . Then  $l = -2$  and  $m = -1$  are distinct roots of the corresponding characteristic equation  $u^2 + 3u + 2 = 0$ . Hence, by Theorem 3.1, we obtain a solution  $z \in \mathcal{G}$  of the equation (4.1) such that

$$|\vartheta(t) - z(t)| \leq \frac{3}{2}\varepsilon$$

for any  $t \geq 0$ . In particular,  $z(t) = c_1e^{-2t} + c_2e^{-t}$  for some constants  $c_1, c_2 \in \mathbb{K}$ .

**Example 4.2.** Let us consider the equation,

$$\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) = 4e^{3t}. \tag{4.2}$$

Assume that a function  $\vartheta : [0, \infty) \rightarrow (0, \infty)$  of exponential order satisfies the inequality

$$|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) - 4e^{3t}| \leq \varepsilon$$

for any  $t \geq 0$  and for some  $\varepsilon > 0$ . Then  $l = -2$  and  $m = -1$  are distinct roots of the corresponding characteristic equation  $u^2 + 3u + 2 = 0$ . Hence, Theorem 3.1 implies that there exists a solution  $z \in \mathcal{G}$  of differential equation (4.2) such that

$$|\vartheta(t) - z(t)| \leq \frac{3}{2}\varepsilon$$

for all  $t \geq 0$ . In particular,  $z(t) = c_1e^{-2t} + c_2e^{-t} + \frac{1}{3}e^{3t}$  for some constants  $c_1, c_2 \in \mathbb{K}$ .

**Example 4.3.** We now assume that a function  $\vartheta : [0, \infty) \rightarrow (0, \infty)$  of exponential order satisfies the inequality  $|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t)| \leq \varepsilon t$  for any  $t \geq 0$  and for some  $\varepsilon > 0$ . Then  $l = -2$  and  $m = -1$  are distinct roots of the corresponding characteristic equation  $u^2 + 3u + 2 = 0$ . Since  $\phi(t) = t$  is an increasing function, by Theorem 3.1, we obtain a solution  $z \in \mathcal{G}$  of the equation (4.1) such that

$$|\vartheta(t) - z(t)| \leq \frac{3}{2}\varepsilon t$$

for any  $t \geq 0$ . In particular,  $z(t) = c_1 e^{-2t} + c_2 e^{-t}$  for some constants  $c_1, c_2 \in \mathbb{K}$ .

**Example 4.4.** We now assume that a function  $\vartheta : [0, \infty) \rightarrow (0, \infty)$  of exponential order satisfies the differential inequality

$$|\vartheta''(t) + 3\vartheta'(t) + 2\vartheta(t) - 4e^{3t}| \leq \varepsilon E_\nu(t)$$

for any  $t \geq 0$  and for some  $\varepsilon > 0$ , where  $\nu$  is a fixed positive real number. Then,  $l = -2$  and  $m = -1$  are distinct roots of the corresponding characteristic equation  $u^2 + 3u + 2 = 0$ .

Since  $E_\nu(t)$  is an increasing function, by Corollary 3.3, we have a solution  $z \in \mathcal{G}$  of the equation (4.2) such that  $|\vartheta(t) - z(t)| \leq \frac{3}{2}\varepsilon E_\nu(t)$  for all  $t \geq 0$ . In particular,  $z(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{5}e^{3t}$  for some constants  $c_1, c_2 \in \mathbb{K}$ .

### 5. Discussion

In this paper, we introduced another new method to study various forms of the Hyers-Ulam stability of general second-order linear differential equations. In this section, we want to see what we can say about our main results when the domain is non-positive real numbers  $t \geq 0$ . For arbitrary  $\varepsilon > 0$ , let us consider

$$|\vartheta''(t) + \mu\vartheta'(t) + \lambda\vartheta(t) - r(t)| \leq \varepsilon \quad (\text{for } t \leq 0), \tag{5.1}$$

where  $\vartheta : (-\infty, 0] \rightarrow \mathbb{K}$  and  $r : (-\infty, 0] \rightarrow \mathbb{K}$  be a continuous functions of exponential order.

Now, we want to define a new functions by  $y_1(t) = y(-t)$  and  $r_1(t) = r(-t)$  for any  $t \geq 0$ , then due to (5.1) we have

$$|y_1''(t) - \mu y_1'(t) + \lambda y_1(t) - r_1(t)| \leq \varepsilon \tag{5.2}$$

for any  $t \geq 0$ .

Therefore, if the quadratic equation,  $u^2 - \mu u + \lambda = 0$ , has two distinct roots  $l$  and  $m$  with  $\Re(l) < 0$  and  $\Re(m) < 0$ , then by Theorem 3.1 and (5.2), we have the solution function  $z_1 : [0, \infty) \rightarrow \mathbb{K}$  which satisfies

$$z_1''(t) - \mu z_1'(t) + \lambda z_1(t) = r_1(t) \tag{5.3}$$

and

$$|y_1(t) - z_1(t)| \leq \frac{1}{|l - m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right) \varepsilon \tag{5.4}$$

for any  $t \geq 0$ . Hence, by defining as  $z(t) = z_1(-t)$  for each  $t \leq 0$ , by (5.4) we have

$$|\vartheta(t) - z(t)| \leq \frac{1}{|l - m|} \left( \frac{1}{|\Re(l)|} + \frac{1}{|\Re(m)|} \right) \varepsilon$$

for each  $t \leq 0$ .

In other words, we can obtain theorems similar to our main theorems in the domain  $(-\infty, 0]$ .

## 6. Conclusion

In this paper, we proved various forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the linear differential equations of general second-order with constant coefficients using the Mahgoub transform method. In other words, we established sufficient criteria for the Hyers-Ulam stability of general second-order linear differential equations with constant coefficients using the Mahgoub transform method.

Moreover, we provided a new method to investigate the Hyers-Ulam stability of differential equations. This is the first attempt to use the Mahgoub transform to prove the Hyers-Ulam stability for linear differential equations of the general second-order. Furthermore, we showed that the Mahgoub transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients.

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