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Bilinear multipliers theory on some function spaces

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Abstract

In this work, we define the bilinear multipliers of the spaces with fractional wavelet transform and consider the basic properties of these bilinear multipliers. Moreover, we give construction examples of bilinear multipliers.

Keywords: Bilinear multipliers, fractional wavelet transform, weighted Lebesgue spaces

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1. Introduction

In this paper, one denotes by $C_c^{\infty}(\mathbb{R})$ and $C_c(\mathbb{R})$ the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}$ and the space of all continuous, complex-valued functions with compact support on $\mathbb{R}$, respectively (*cf.* [4]). Let $1 \leq p \leq \infty$. $L^p(\mathbb{R})$ denotes the usual Lebesgue space. A continuous function ω satisfying $1 \leq \omega(x)$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for $x, y \in \mathbb{R}$ will be called a weight function on $\mathbb{R}$. If $\omega_1(x) \leq \omega_2(x)$ for all $x \in \mathbb{R}$, it is said that $\omega_1 \leq \omega_2$. One says that a weight function υ_s polynomial type, if $\upsilon_s(x) = (1+|x|)^s$ for $s \geq 0$. The weighted Lebesgue space is defined by $L^p_\omega(\mathbb{R}) = \{f : f\omega \in L^p(\mathbb{R})\}$ and it is a Banach space under the norm $\|f\|_{p,\omega} = \|f\omega\|_p$. For $f \in L^1(\mathbb{R})$, the Fourier transform of f is denoted by $\hat{f}$. The Fourier transform is a continuous function on $\mathbb{R}$, which vanishes at infinity and it has the inequality $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$ (*cf.* [4]). Assume that f is a complex valued measurable function on $\mathbb{R}$. The translation and character operators T_x and M_x are defined by $T_x f(y) = f(y-x)$ and $M_x f(y) = e^{2\pi i x y} f(y)$, respectively for $x, y \in \mathbb{R}$. One has the equalities $(T_x f)\hat{}\,(\xi) = M_{-x}\hat{f}(\xi)$ and $(M_x f)\hat{}\,(\xi) = T_x \hat{f}(\xi)$ (*cf.* [2]). $M(\mathbb{R})$ denotes the space of bounded regular Borel measures and also $M(\omega)$ denotes the space of μ in $M(\mathbb{R})$ such that

$$\|\mu\|_\omega = \int_{\mathbb{R}} \omega \, d|\mu| < \infty.$$

Let $\mu \in M(\mathbb{R})$. Then the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$ (*cf.* [13]).

The fractional wavelet transform with an angle θ of $f \in L^2(\mathbb{R})$ is defined by

$$W_\psi^\theta f(b,a) = \int_{\mathbb{R}} f(t)\overline{\psi_{b,a,\theta}(t)}\,dt,$$

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where $\psi_{b,a,\theta}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)e^{-\frac{i}{2}(t^2-b^2)}\cot\theta$ for $a \in \mathbb{R}^+$, $b \in \mathbb{R}$. Since $\psi_{b,a,\frac{\pi}{2}} = T_b D_a \psi$, the fractional wavelet transform with $\theta = \frac{\pi}{2}$ corresponds to the classical wavelet transform. It is a time-frequency operator which can provide the time-frequency information of a signal in the successful way. Also it is a generalization of classical wavelet transform in time-frequency analysis. The fractional wavelet transform can be written as a convolution $W_{\psi}^{\theta}f(b,a) = e^{\frac{-i}{2}b^2\cot\theta}\left(e^{\frac{i}{2}(.)^2\cot\theta}f * \overline{\psi_a^*}\right)(b)$, (*cf.* [11, 12]).

Let ω_1, ω_2 be weight functions on $\mathbb{R}$ and let $0 \neq \psi \in C_c^{\infty}(\mathbb{R})$, $1 \leq p, q < \infty$. The space $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ consist of all $f \in L_{\omega_1}^p(\mathbb{R})$ such that their fractional wavelet transforms $W_{\psi}^{\theta}f$ are in $L_{\omega_2}^q(\mathbb{R})$, where the scale a is fixed. This space is equipped with the following norm:

$$\|f\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a} = \|f\|_{p,\omega_1} + \left\|W_{\psi}^{\theta}f\right\|_{q,\omega_2}.$$

The space $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ is a Banach space with this norm. Moreover, it is invariant under the translations and the following norm inequality is satisfied:

$$\|T_u f\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a} \leq \tilde{\omega}(u)\|f\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a},$$

where $\tilde{\omega}(u) = \max\{\omega_1(u), \omega_2(u)\}$, (*cf.* [3]).

In the literature, some studies have been done on the bilinear multipliers of function spaces such as Lebesgue, Lorentz, amalgam and their variable exponents, and also small Lebesgue (*cf.* [5], [7]-[10]). Also, many function spaces have been defined using fractional time-frequency transforms and some properties of these spaces have been investigated (*cf.* [3], [14]-[16]). Kulak considered the bilinear multipliers of the space with wavelet transform (*cf.* [6]). In our work, we study the bilinear multipliers of the space $\left(FW_{\omega,\upsilon}^{\theta,p,q}\right)_a(\mathbb{R})$, the space of functions whose fractional wavelet transforms in the weighted Lebesgue space, and give some examples.

2. Main results

Lemma 2.1. *Let ω,υ be weight functions of polynomial type. Then $C_c^{\infty}(\mathbb{R})$ is dense in $(FW_{\omega,\upsilon}^{\theta,p,q})_a(\mathbb{R})$.*

Proof. It's known that $\overline{C_c(\mathbb{R})} = (FW_{\omega,\upsilon}^{\theta,p,q})_a(\mathbb{R})$ (*cf.* [3]). Also by the inclusion $C_c^{\infty}(\mathbb{R}) \subset C_c(\mathbb{R})$, so we can write $\overline{C_c^{\infty}(\mathbb{R})} \subset (FW_{\omega,\upsilon}^{\theta,p,q})_a(\mathbb{R})$. On the other hand since $\overline{C_c^{\infty}(\mathbb{R})} = L_{\omega}^p(\mathbb{R})$ and $\overline{C_c^{\infty}(\mathbb{R})} = L_{\upsilon}^q(\mathbb{R})$, we can easily prove $\overline{C_c^{\infty}(\mathbb{R})} = (FW_{\omega,\upsilon}^{\theta,p,q})_a(\mathbb{R})$ using the proof technique of Theorem 2.4 in [3]. □

Definition 2.2. Let $1 \leq p_1, q_1, p_2, q_2, p_3, q_3 < \infty$, $a_1, a_2, a_3 \in \mathbb{R}^+$ and $\omega_1, \upsilon_1, \omega_2, \upsilon_2, \omega_3, \upsilon_3$ be weight functions on $\mathbb{R}$. Assume that $\omega_1, \upsilon_1, \omega_2, \upsilon_2$ are weight functions of polynomial type and $m(\xi,\eta)$ is a bounded, measurable function on $\mathbb{R}^2$. Define

$$B_m(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle\xi+\eta,x\rangle}d\xi d\eta$$

for all $f, g \in C_c^{\infty}(\mathbb{R})$. m said to be a bilinear multiplier on $\mathbb{R}$ of type $[FW]$, if there exists $C > 0$ such that

$$\|B_m(f,g)\|_{\left(FW_{\omega_3,\upsilon_3}^{\theta_3,p_3,q_3}\right)_{a_3}} \leq C\|f\|_{\left(FW_{\omega_1,\upsilon_1}^{\theta_1,p_1,q_1}\right)_{a_1}}\|g\|_{\left(FW_{\omega_2,\upsilon_2}^{\theta_2,p_2,q_2}\right)_{a_2}}$$

for all $f, g \in C_c^{\infty}(\mathbb{R})$. It is mean that B_m extends to a bounded bilinear operator from $\left(FW_{\omega_1,\upsilon_1}^{\theta_1,p_1,q_1}\right)_{a_1}(\mathbb{R}) \times \left(FW_{\omega_2,\upsilon_2}^{\theta_2,p_2,q_2}\right)_{a_2}(\mathbb{R})$ to $\left(FW_{\omega_3,\upsilon_3}^{\theta_3,p_3,q_3}\right)_{a_3}(\mathbb{R})$.

We denote by $BM\left[FW(p_1, q_1, \omega_1, \upsilon_1, a_1, \theta_1; p_2, q_2, \omega_2, \upsilon_2, a_2, \theta_2; p_3, q_3, \omega_3, \upsilon_3, a_3, \theta_3)\right]$ the space of all bilinear multipliers of type $[FW]$. Also we use this set notation shortly $BM\left[FW(p_i, q_i, \omega_i, \upsilon_i, a_i, \theta_i)\right]$ and

$$\|m\|_{FW} = \|B_m\|$$

$$= sup\left\{\frac{\|B_m(f,g)\|_{\left(FW_{\omega_3,\upsilon_3}^{\theta_3,p_3,q_3}\right)_{a_3}}}{\|f\|_{\left(FW_{\omega_1,\upsilon_1}^{\theta_1,p_1,q_1}\right)_{a_1}}\|g\|_{\left(FW_{\omega_2,\upsilon_2}^{\theta_2,p_2,q_2}\right)_{a_2}}} : \|f\|_{\left(FW_{\omega_1,\upsilon_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \leq 1, \|g\|_{\left(FW_{\omega_2,\upsilon_2}^{\theta_2,p_2,q_2}\right)_{a_2}} \leq 1\right\}.$$

Lemma 2.3 (Hölder type inequality for the space $\left(FW^{\theta,p,q}_{\omega,\upsilon}\right)_a(\mathbb{R})$)**.** *Let $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. Then there exists $C>0$ such that*

$$\|fg\|_{\left(FW^{\theta_3,p,p}_{\omega,\omega}\right)_{a_3}} \le C\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega,\upsilon_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega,\upsilon_2}\right)_{a_2}}$$

for all $f\in\left(FW^{\theta_1,p_1,q_1}_{\omega,\upsilon_1}\right)_{a_1}(\mathbb{R})$ and $g\in\left(FW^{\theta_2,p_2,q_2}_{\omega,\upsilon_2}\right)_{a_2}(\mathbb{R})$. In other words

$$\left(FW^{\theta_1,p_1,q_1}_{\omega,\upsilon_1}\right)_{a_1}(\mathbb{R})\left(FW^{\theta_2,p_2,q_2}_{\omega,\upsilon_2}\right)_{a_2}(\mathbb{R})\subset\left(FW^{\theta_3,p,p}_{\omega,\omega}\right)_{a_3}(\mathbb{R}).$$

Proof. Take $f\in\left(FW^{\theta_1,p_1,q_1}_{\omega,\upsilon_1}\right)_{a_1}(\mathbb{R})$ and $g\in\left(FW^{\theta_2,p_2,q_2}_{\omega,\upsilon_2}\right)_{a_2}(\mathbb{R})$. We can write the equality

$$W^{\theta_3}_{\Psi}(fg)(b,a)=e^{\frac{-i}{2}b^2\cot\theta_3}\left(e^{\frac{i}{2}(.)^2\cot\theta_3}(fg)*\overline{\Psi^*_a}\right)(b).$$

From last equality and Hölder inequality, we have

$$\begin{aligned}\|fg\|_{(FW^{\theta_3,p,p}_{\omega,\omega})_{a_3}} &= \|fg\|_{p,\omega}+\left\|W^{\theta_3}_{\Psi}(f,g)\right\|_{p,\omega}\\ &\le \|f\|_{p_{1,\omega}}\|g\|_{p_2,\omega}+\left\|e^{\frac{i}{2}(.)^2\cot\theta_3}(fg)*\overline{\Psi^*_a}\right\|_{p,\omega}. \qquad (2.1)\end{aligned}$$

Moreover since $L^p_\omega(\mathbb{R})$ is a Banach module over $L^1_\omega(\mathbb{R})$ and using the inequality (2.1), we achieve

$$\begin{aligned}\|fg\|_{(FW^{\theta_3,p,p}_{\omega,\omega})_{a_3}} &\le \|f\|_{p_{1,\omega}}\|g\|_{p_2,\omega}+\left\|e^{\frac{i}{2}(.)^2\cot\theta_3}(fg)\right\|_{p,\omega}\left\|\Psi^*_a\right\|_{1,\omega}\\ &= \|f\|_{p_{1,\omega}}\|g\|_{p_2,\omega}+\|fg\|_{p,\omega}\left\|\overline{\Psi^*_a}\right\|_{1,\omega}\\ &= \left(1+\left\|\Psi^*_a\right\|_{1,\omega}\right)\|f\|_{p_{1,\omega}}\|g\|_{p_2,\omega}\\ &\le \left(1+\left\|\Psi^*_a\right\|_{1,\omega}\right)\left(\|f\|_{p_{1,\omega}}+\left\|W^{\theta_1}_{\Psi_1}f\right\|_{q_1,\upsilon_1}\right)\left(\|g\|_{p_{2,\omega}}+\left\|W^{\theta_2}_{\Psi_2}g\right\|_{q_2,\upsilon_2}\right)\\ &= C\,\|f\|_{(FW^{\theta_1,p_1,q_1}_{\omega,\upsilon_1})_{a_1}}\|g\|_{(FW^{\theta_2,p_2,q_2}_{\omega,\upsilon_2})_{a_2}},\end{aligned}$$

where $C=1+\left\|\Psi^*_a\right\|_{1,\omega}$. □

Example 2.4. Let $m(\xi,\eta)=a$ and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. Take any $f,\,g\in C^\infty_c(\mathbb{R})$. Then we can write

$$\begin{aligned}B_m(f,g)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{f}(\xi)\widehat{g}(\eta)a e^{2\pi i\langle\xi+\eta,x\rangle}d\xi d\eta\\ &= a\left(\int_{\mathbb{R}}\widehat{f}(\xi)e^{2\pi i x\xi}d\xi\right)\left(\int_{\mathbb{R}}\widehat{g}(\eta)e^{2\pi i x\eta}d\eta\right)\\ &= af(x)g(x).\end{aligned}$$

So by Lemma 2.3, we get for $C>0$

$$\begin{aligned}\|B_m(f,g)\|_{\left(FW^{\theta,p,p}_{\omega,\omega}\right)_{a_3}} &= |a|\,\|fg\|_{\left(FW^{\theta,p,p}_{\omega,\omega}\right)_{a_3}}\\ &\le |a|\,C\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega,v_1}\right)_{a_1}}\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega,v_2}\right)_{a_2}}.\end{aligned}$$

Thus, we obtain

$$m\in BM\left[FW(p_1,q_1,\omega,v_1,a_1,\theta_1;p_2,q_2,\omega,v_2,a_2,\theta_2;p,p,\omega,\omega,a_3,\theta_3)\right].$$

Now let us give an example to bilinear multiplier in the following theorem.

Theorem 2.5. *Assume that ω_3 is polynomial type weight function. Let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$. If $K \in L^1_\omega(\mathbb{R})$ such that $\omega(y) = \widetilde{\omega_1}(y)\widetilde{\omega_2}(-y)$, $\widetilde{\omega_1} = \max\{\omega_3, \upsilon_1\}$, $\widetilde{\omega_2} = \max\{\omega_3, \upsilon_2\}$, then*

$$m(\xi,\eta) = \widehat{K(\xi-\eta)} \in BM\left[FW(p_1, q_1, \omega_3, \upsilon_1, a_1, \theta_1; p_2, q_2, \omega_3, \upsilon_2, a_2, \theta_2; p_3, p_3, \omega_3, \omega_3, a_3, \theta_3)\right].$$

Moreover, there exists $C > 0$ such that

$$\|m\|_{FW} \leq C\,\|K\|_{1,\omega}.$$

Proof. Given any $f, g \in C_c^\infty(\mathbb{R})$. The following equation is written from (*cf.* [7]),

$$B_m(f,g)(t) = \int_{\mathbb{R}} f(t-y)g(t+y)K(y)dy. \tag{2.2}$$

Furthermore, it's known that $T_y f \in \left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}(\mathbb{R})$, $T_{-y} g \in \left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}(\mathbb{R})$ and

$$\left\|T_y f\right\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}} \leq \widetilde{\omega_1}(y)\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}}, \tag{2.3}$$

$$\left\|T_{-y} g\right\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}} \leq \widetilde{\omega_2}(-y)\,\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}}. \tag{2.4}$$

Thus using (2.2), (2.3), (2.4) and Lemma 2.3, we get

$$\begin{aligned}
\|B_m(f,g)\|_{\left(FW^{\theta_3,p_3,p_3}_{\omega_3,\omega_3}\right)_{a_3}} &= \left\|\int_{\mathbb{R}} f(t-y)g(t+y)K(y)dy\right\|_{\left(FW^{\theta_3,p_3,p_3}_{\omega_3,\omega_3}\right)_{a_3}} \\
&\leq \int_{\mathbb{R}} \|f(t-y)g(t+y)K(y)\|_{\left(FW^{\theta_3,p_3,p_3}_{\omega_3,\omega_3}\right)_{a_3}} dy \\
&= \int_{\mathbb{R}} |K(y)|\,\|f(t-y)g(t+y)\|_{\left(FW^{\theta_3,p_3,p_3}_{\omega_3,\omega_3}\right)_{a_3}} dy \\
&\leq C\int_{\mathbb{R}} |K(y)|\left\|T_y f\right\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}}\left\|T_{-y} g\right\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}} dy \\
&\leq C\int_{\mathbb{R}} |K(y)|\,\widetilde{\omega_1}(y)\widetilde{\omega_2}(-y)\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}}\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}} dy \\
&= C\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}}\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}}\int_{\mathbb{R}} |K(y)|\,\widetilde{\omega_1}(y)\widetilde{\omega_2}(-y)dy \\
&= C\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_3,\upsilon_1}\right)_{a_1}}\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_3,\upsilon_2}\right)_{a_2}}\|K\|_{1,\omega}.
\end{aligned} \tag{2.5}$$

So, we find that $m(\xi,\eta) = \widehat{K(\xi-\eta)}$ defines a bilinear multiplier. If we use the inequality (2.5), we achieve

$$\|m\|_{FW} = \|B_m\| \leq C\,\|K\|_{1,\omega}.$$

□

Definition 2.6. Let $1 \leq p_i, q_i < \infty$, $a_i \in \mathbb{R}^+$ $(i = 1,2,3)$ and let ω_i, υ_i $(i = 1,2,3)$ be weight functions on $\mathbb{R}$. Suppose that ω_i, υ_i $(i = 1,2)$ are weight functions of polynomial type.

$\widetilde{M}\left[FW(p_1, q_1, \omega_1, \upsilon_1, a_1, \theta_1; p_2, q_2, \omega_2, \upsilon_2, a_2, \theta_2; p_3, p_3, \omega_3, \omega_3, a_3, \theta_3)\right]$ denotes the space of measurable functions $M : \mathbb{R} \to \mathbb{C}$ such that $m(\xi, \eta) = M(\xi - \eta) \in BM\left[FW\left(p_i, q_i, \omega_i, \upsilon_i, a_i, \theta_i\right)\right]$, that is to say

$$B_M(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(\eta)M(\xi - \eta)e^{2\pi i\langle \xi+\eta, x\rangle} d\xi d\eta$$

extends to bounded bilinear map from $\left(FW^{\theta_1, p_1 q_1}_{\omega_1, \upsilon_1}\right)_{a_1}(\mathbb{R}) \times \left(FW^{\theta_2, p_2, q_2}_{\omega_2, \upsilon_2}\right)_{a_2}(\mathbb{R})$ to $\left(FW^{\theta_3, p_3, q_3}_{\omega_3, \upsilon_3}\right)_{a_3}(\mathbb{R})$. We denote

$$\|M\|_{FW} = \|B_M\|.$$

Theorem 2.7. *Suppose that ω_3 is weight of polynomial type. Let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, $\widetilde{\omega_1} = \max\{\omega_3, \upsilon_1\}$, $\widetilde{\omega_2} = \max\{\omega_3, \upsilon_2\}$ and $\upsilon(y) = (1 + |y|)^{2s}$, $s \geq 0$. If $\mu \in M(\upsilon)$ and $m(\xi, \eta) = \widehat{\mu}(\alpha\xi + \beta\eta)$ for $\alpha, \beta \in \mathbb{R}$, then*

$$m \in BM\left[FW(p_1, q_1, \omega_3, \upsilon_1, a_1, \theta_1; p_2, q_2, \omega_3, \upsilon_2, a_2, \theta_2; p_3, p_3, \omega_3, \omega_3, a_3, \theta_3)\right].$$

Furthermore, there exists $C > 0$ such that

$$\|m\|_{FW} \leq C \|\mu\|_{\upsilon}.$$

Proof. Take any $f, g \in C^\infty_c(\mathbb{R})$. From Theorem 2.3 in [7], we can write

$$B_m(f, g)(t) = \int_{\mathbb{R}} f(t - \alpha y)g(t - \beta y)d\mu(y). \tag{2.6}$$

By [3], we have following inequalities

$$\left\|T_{\alpha y} f\right\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \leq \widetilde{\omega_1}(\alpha y) \|f\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \tag{2.7}$$

and

$$\left\|T_{\beta y} g\right\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} \leq \widetilde{\omega_2}(\beta y) \|g\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}}. \tag{2.8}$$

So using (2.6), (2.7), (2.8) and Lemma 2.3, we find $C > 0$ such that

$$\begin{aligned}
\|B_m(f, g)\|_{\left(FW^{\theta_3, p_3, p_3}_{\omega_3, \omega_3}\right)_{a_3}} &\leq \int_{\mathbb{R}} \|f(t - \alpha y)g(t - \beta y)\|_{\left(FW^{\theta_3, p_3, p_3}_{\omega_3, \omega_3}\right)_{a_3}} d|\mu|(y) \\
&\leq C \int_{\mathbb{R}} \|f(t - \alpha y)\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \|g(t - \beta y)\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} d|\mu|(y) \\
&\leq C \int_{\mathbb{R}} \widetilde{\omega_1}(\alpha y)\widetilde{\omega_2}(\beta y) \|f\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} d|\mu|(y) \\
&= C \|f\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} \int_{\mathbb{R}} \widetilde{\omega_1}(\alpha y)\widetilde{\omega_2}(\beta y) d|\mu|(y).
\end{aligned} \tag{2.9}$$

Let $|\alpha| \leq 1$ and $|\beta| \leq 1$. Since the functions $\omega_3, \upsilon_1, \upsilon_2$ are weights of polynomial type, there exist $s_i (i = 1, 2, 3) \in \mathbb{R}^+$ such that

$$\omega_3(y) = (1 + |y|)^{s_3},$$
$$\upsilon_1(y) = (1 + |y|)^{s_1}$$

and

$$\upsilon_2(y) = (1 + |y|)^{s_2}.$$

Set $max\{s_1, s_2, s_3\} = s$. Then we can write

$$\widetilde{\omega_1}(y) \leq (1 + |y|)^{s} \tag{2.10}$$

and

$$\widetilde{\omega_2}(y) \leq (1 + |y|)^s . \tag{2.11}$$

So using (2.10) and (2.11), we get

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\omega_1}(\alpha y)\widetilde{\omega_2}(\beta y) d|\mu|(y) &\leq \int_{\mathbb{R}} (1 + |\alpha y|)^s (1 + |\beta y|)^s d|\mu|(y) \\ &\leq \int_{\mathbb{R}} (1 + |y|)^s (1 + |y|)^s d|\mu|(y) \\ &= \int_{\mathbb{R}} (1 + |y|)^{2s} d|\mu|(y) = \|\mu\|_{\upsilon} ,\end{aligned} \tag{2.12}$$

where $\upsilon(y) = (1 + |y|)^{2s}$. Thus from (2.9) and (2.12), we have

$$\|B_m(f, g)\|_{\left(FW^{\theta_3, p_3, p_3}_{\omega_3, \omega_3}\right)_{a_3}} \leq c \|f\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} \|\mu\|_{\upsilon} . \tag{2.13}$$

That means $m \in BM\left[FW(p_1, q_1, \omega_3, \upsilon_1, a_1, \theta_1; p_2, q_2, \omega_3, \upsilon_2, a_2, \theta_2; p_3, p_3, \omega_3, \omega_3, a_3, \theta_3)\right]$. Hence by (2.13), we obtain

$$\|m\|_{FW} = \|B_m\| \leq C \|\mu\|_{\upsilon} .$$

Now let $|\alpha| > 1$ and $|\beta| > 1$. Then we get

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\omega_1}(\alpha y)\widetilde{\omega_2}(\beta y) d|\mu|(y) &< \int_{\mathbb{R}} (|\alpha| + |\alpha| |y|)^s (|\beta| + |\beta| |y|)^s d|\mu|(y) \\ &= |\alpha\beta|^s \int_{\mathbb{R}} \upsilon(y) d|\mu|(y) = |\alpha\beta|^s \|\mu\|_{\upsilon} ,\end{aligned} \tag{2.14}$$

where $\upsilon(y) = (1 + |y|)^{2s}$. So by (2.9) and (2.14), we find

$$\|B_m(f, g)\|_{\left(FW^{\theta_3, p_3, p_3}_{\omega_3, \omega_3}\right)_{a_3}} < c |\alpha\beta|^s \|f\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} \|\mu\|_{\upsilon} . \tag{2.15}$$

Thus we achieve

$$m \in BM\left[FW(p_1, q_1, \omega_3, \upsilon_1, a_1, \theta_1; p_2, q_2, \omega_3, \upsilon_2, a_2, \theta_2; p_3, p_3, \omega_3, \omega_3, a_3, \theta_3)\right]$$

and by (2.15)

$$\|m\|_{FW} = \|B_m\| < c |\alpha\beta|^s \|\mu\|_{\upsilon} .$$

Suppose that $|\alpha| > 1$, $|\beta| \leq 1$. We get

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\omega_1}(\alpha y)\widetilde{\omega_2}(\beta y) d|\mu|(y) &< \int_{\mathbb{R}} (|\alpha| + |\alpha| |y|)^s (|1| + |y|)^s d|\mu|(y) \\ &= |\alpha|^2 \int_{\mathbb{R}} \upsilon(y) d|\mu|(y) = |\alpha|^2 \|\mu\|_{\upsilon} .\end{aligned} \tag{2.16}$$

Hence using (2.9) and (2.16), we obtain

$$\|B_m(f, g)\|_{\left(FW^{\theta_3, p_3, p_3}_{\omega_3, \omega_3}\right)_{a_3}} < c |\alpha|^s \|f\|_{\left(FW^{\theta_1, p_1, q_1}_{\omega_3, \upsilon_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2, p_2, q_2}_{\omega_3, \upsilon_2}\right)_{a_2}} \|\mu\|_{\upsilon} . \tag{2.17}$$

Therefore, we achieve

$$m \in BM\left[FW(p_1,q_1,\omega_3,\upsilon_{1,}a_1,\theta_1;p_2,q_2,\omega_3,\upsilon_{2,}a_2,\theta_2;p_3,p_3,\omega_3,\omega_{3,}a_3,\theta_3)\right].$$

From the inequality (2.17), we have

$$\|m\|_{FW}=\|B_m\|<c\,|\alpha|^s\,\|\mu\|_\upsilon .$$

Finally let $|\alpha|\leq 1$, $|\beta|>1$. Similarly, we obtain

$$m \in BM\left[FW(p_1,q_1,\omega_3,\upsilon_{1,}a_1,\theta_1;p_2,q_2,\omega_3,\upsilon_{2,}a_2,\theta_2;p_3,p_3,\omega_3,\omega_{3,}a_3,\theta_3)\right]$$

and

$$\|m\|_{FW}=\|B_m\|\leq c\,\|\mu\|_\upsilon .$$

So the proof is completed. □

Theorem 2.8. *Let* $m\in BM\left[FW(p_i,q_i,\omega_i,v_{i,}a_i,\theta_i)\right]$. *Suppose that* $\widetilde{\omega_1}=\max\{\omega_1,v_1\}$ *and* $\widetilde{\omega_2}=\max\{\omega_2,v_2\}$. *Then* $M_{(\xi_0,\eta_0)}m\in BM\left[FW(p_i,q_i,\omega_i,v_{i,}a_i,\theta_i)\right]$ *for each* $(\xi_0,\eta_0)\in\mathbb{R}^2$ *and*

$$\left\|M_{(\xi_0,\eta_0)}m\right\|_{FW}\leq\widetilde{\omega_1}(-\xi_0)\,\widetilde{\omega_2}(-\eta_0)\,\|M\|_{FW}.$$

Proof. Let $f,g\in C_c^\infty(\mathbb{R})$ be given. By Theorem 2.4 in [7], we write

$$B_{M_{(\xi_0,\eta_0)}}m(f,g)(x)=B_m\left(T_{-\xi_0}f,T_{-\eta_0}g\right)(x). \tag{2.18}$$

On the other hand, we know that

$$\left\|T_{-\xi_0}f\right\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}}\leq\widetilde{\omega_1}(-\xi_0)\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}} \tag{2.19}$$

and

$$\left\|T_{-\eta_0}g\right\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}}\leq\widetilde{\omega_2}(-\eta_0)\,\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}}, \tag{2.20}$$

where $\widetilde{\omega_1}=\max\{\omega_1,v_1\}$ and $\widetilde{\omega_2}=\max\{\omega_2,v_2\}$ (*cf.* [3]). From the assumption $m\in BM\left[FW(p_i,q_i,\omega_i,v_{i,}a_i,\theta_i)\right]$ and by the (2.18), (2.19), (2.20), we get

$$\begin{aligned}\left\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\right\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}}&=\left\|B_m(T_{-\xi_0}f,T_{-\eta_0}g)\right\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}}\\&\leq\|B_m\|\left\|T_{-\xi_0}f\right\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}}\left\|T_{-\eta_0}g\right\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}}\\&\leq\|B_m\|\,\widetilde{\omega_1}(-\xi_0)\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}}\widetilde{\omega_2}(-\eta_0)\,\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}}\\&=\widetilde{\omega_1}(-\xi_0)\,\widetilde{\omega_2}(-\eta_0)\,\|B_m\|\,\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}}\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}}.\end{aligned} \tag{2.21}$$

Thus, we have $M_{(\xi_0,\eta_0)}m\in BM\left[FW(p_i,q_i,\omega_i,v_{i,}a_i,\theta_i)\right]$. Using the inequality (2.21), we conclude that

$$\begin{aligned}\left\|M_{(\xi_0,\eta_0)}m\right\|_{FW}=\left\|B_{M_{(\xi_0,\eta_0)}m}\right\|&\leq\widetilde{\omega_1}(-\xi_0)\,\widetilde{\omega_2}(-\eta_0)\,\|B_m\|\\&=\widetilde{\omega_1}(-\xi_0)\,\widetilde{\omega_2}(-\eta_0)\,\|M\|_{FW}.\end{aligned}$$

□

Theorem 2.9. *Let* $m\in BM\left[FW(p_i,q_i,\omega_i,v_{i,}a_i,\theta_i)\right]$. *Then* $T_{(\xi_0,\eta_0)}m\in BM\left[FW(p_i,q_i,\omega_i,v_{i,}a_i,\theta_i)\right]$ *and*

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{FW}=\|m\|_{FW}$$

for all $(\xi_0,\eta_0)\in\mathbb{R}^2$.

Proof. Assume that $f, g \in C_c^\infty(\mathbb{R})$. Then we can write the following equalities easily

$$\begin{aligned} \left\|M_{-\xi_0} f\right\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} &= \left\|e^{2\pi i(-\xi_0)(.)} f\right\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \\ &= \|f\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} . \end{aligned} \tag{2.22}$$

$$\begin{aligned} \left\|M_{-\xi_0} g\right\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} &= \left\|e^{2\pi i(-\eta_0)(.)} g\right\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} \\ &= \|g\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} . \end{aligned} \tag{2.23}$$

Moreover, it is known that by (*cf.* [7])

$$B_{T_{(\xi_0,\eta_0)}m}(f, g) = B_m\left(M_{-\xi_0} f, M_{-\eta_0} g\right). \tag{2.24}$$

Using the (2.22), (2.23), (2.24) and assumption $m \in BM\left[FW(p_i, q_i, \omega_i, \nu_i, a_i, \theta_i)\right]$, we achieve

$$\begin{aligned} \left\|B_{T_{(\xi_0,\eta_0)}m}(f, g)\right\|_{\left(FW_{\omega_3,\nu_3}^{\theta_3,p_3,q_3}\right)_{a_3}} &= \left\|B_m\left(M_{-\xi_0} f, M_{-\eta_0} g\right)\right\|_{\left(FW_{\omega_3,\nu_3}^{\theta_3,p_3,q_3}\right)_{a_3}} \\ &\le \|B_m\| \left\|M_{-\xi_0} f\right\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \left\|M_{-\xi_0} g\right\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} \\ &= \|B_m\| \|f\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \|g\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} . \end{aligned}$$

Hence $T_{(\xi_0,\eta_0)}m \in BM\left[FW(p_i, q_i, \omega_i, \nu_i, a_i, \theta_i)\right]$. Finally, we obtain

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{FW} = \|m\|_{FW} .$$

□

The following theorems will help us multiply examples of bilinear multiplier.

Theorem 2.10. *Let $m \in BM\left[FW(p_i, q_i, \omega_i, \nu_i, a_i, \theta_i)\right]$. Suppose that $\widetilde{\omega_1} = \max\{\omega_1, \nu_1\}$ and $\widetilde{\omega_2} = \max\{\omega_2, \nu_2\}$. If $\Phi \in L^1_\omega(\mathbb{R}^2)$ such that $\omega(u, v) = \widetilde{\omega_1}(u)\widetilde{\omega_2}(v)$, then $\widehat{\Phi}m \in BM\left[FW(p_i, q_i, \omega_i, \nu_i, a_i, \theta_i)\right]$ and*

$$\left\|\widehat{\Phi}m\right\|_{FW} \le \|\Phi\|_{1,\omega} \|m\|_{FW} .$$

Proof. Assume that $\Phi \in L^1_\omega(\mathbb{R}^2)$. Let $f, g \in C_c^\infty(\mathbb{R})$ be given. We know that by Proposition 2.5 in [1]

$$B_{\widehat{\Phi}m}(f, g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(u, v) B_{M_{(-u,-v)}m}(f, g)(x)\,dudv.$$

From the assumption $m \in BM\left[FW(p_i, q_i, \omega_i, \nu_i, a_i, \theta_i)\right]$ and Theorem 2.8, we have $M_{(-u,-v)}m \in BM\left[FW(p_i, q_i, \omega_i, \nu_i, a_i, \theta_i)\right]$ and

$$\left\|M_{(-u,-v)}m\right\|_{FW} \le \widetilde{\omega_1}(u)\widetilde{\omega_2}(v) \|m\|_{FW} . \tag{2.25}$$

So by (2.25), we get

$$\begin{aligned} \left\|B_{\widehat{\Phi}m}(f, g)\right\|_{\left(FW_{\omega_3,\nu_3}^{\theta_3,p_3,q_3}\right)_{a_3}} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \left\|\Phi(u, v) B_{M_{(-u,-v)}m}(f, g)\right\|_{\left(FW_{\omega_3,\nu_3}^{\theta_3,p_3,q_3}\right)_{a_3}} dudv \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\Phi(u, v)| \left\|M_{(-u,-v)}m\right\|_{FW} \|f\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \|g\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} dudv \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\Phi(u, v)| \widetilde{\omega_1}(u)\widetilde{\omega_2}(v) \|m\|_{FW} \|f\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \|g\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} dudv \\ &= \|m\|_{FW} \|f\|_{\left(FW_{\omega_1,\nu_1}^{\theta_1,p_1,q_1}\right)_{a_1}} \|g\|_{\left(FW_{\omega_2,\nu_2}^{\theta_2,p_2,q_2}\right)_{a_2}} \int_{\mathbb{R}}\int_{\mathbb{R}} |\Phi(u, v)| \widetilde{\omega_1}\widetilde{\omega_2}\,dudv \end{aligned}$$

$$= \|m\|_{FW} \|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}} \|\Phi\|_{1,\omega} . \tag{2.26}$$

Hence by (2.26), we get $\widehat{\Phi}m \in BM\left[FW(p_i, q_i, \omega_i, v_{i,}a_i, \theta_i)\right]$ and

$$\left\|\widehat{\Phi}m\right\|_{FW} \le \|\Phi\|_{1,\omega} \|m\|_{FW} .$$

□

The proof of the following corollary is easily seen from the Example 2.4 and Theorem 2.10.

Corollary 2.11. *Suppose that* $m \in BM\left[FW(p_1, q_1, \omega, v_1, a_1, \theta_1; p_2, q_2, \omega, v_2, a_2, \theta_2; p, p, \omega, \omega, a_3, \theta_3)\right]$. *If one has* $\Phi \in L^1_\omega(\mathbb{R}^2)$ *such that* $\widetilde{\omega_1} = \max\{\omega_1, v_1\}$, $\widetilde{\omega_2} = \max\{\omega_2, v_2\}$ *and* $\omega(u, v) = \widetilde{\omega_1}(u)\widetilde{\omega_2}(v)$, *then* $\widehat{\Phi} \in BM\left[FW(p_1, q_1, \omega, v_1, a_1, \theta_1; p_2, q_2, \omega, v_2, a_2, \theta_2; p, p, \omega, \omega, a_3, \theta_3)\right]$.

Theorem 2.12. *Assume that* $m \in BM\left[FW(p_i, q_i, \omega_i, v_{i,}a_i, \theta_i)\right]$. *If* $Q_1, Q_2 \subset \mathbb{R}$ *are bounded sets, then*

$$h(\xi, \eta) = \frac{1}{\mu(Q_1 \times Q_2)} \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi + u, \eta + v)\, du dv \in BM\left[FW(p_i, q_i, \omega_i, v_{i,}a_i, \theta_i)\right].$$

Proof. Take any $f, g \in C^\infty_c(\mathbb{R})$. The following equality is written by Theorem 2.9 in [7].

$$B_h(f, g)(x) = \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} B_{T_{(-u,-v)}m}(f, g)(x) du dv.$$

From Theorem 2.9, we achieve

$$\begin{aligned}
\|B_h(f, g)\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}} &= \left\| \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} B_{T_{(-u,-v)}m}(f, g)(x) du dv \right\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}} \\
&\le \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} \left\|B_{T_{(-u,-v)}m}(f, g)\right\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}} du dv \\
&\le \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} \left\|T_{(-u,-v)}m\right\|_{FW} \|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}} du dv \\
&= \|m\|_{FW} \|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}} .
\end{aligned}$$

Thus, we conclude that $h \in BM\left[FW(p_i, q_i, \omega_i, v_{i,}a_i, \theta_i)\right]$. □

Theorem 2.13. *If* $\omega_i < u_i$, $v_i < m_i$ $(i = 1, 2)$ *and* $u_3 < \omega_3$, $m_3 < v_3$, *then we have the inclusion*

$$BM\left[FW(p_i, q_i, \omega_i, v_{i,}a_i, \theta_i)\right] \subset BM\left[FW(p_i, q_i, u_i, m_{i,}a_i, \theta_i)\right].$$

Proof. Take any $m \in BM\left[FW(p_1, q_1, \omega_1, v_1, a_1, \theta_1; p_2, q_2, \omega_2, v_2, a_2, \theta_2; p_3, q_3, \omega_3, v_3, a_3, \theta_3)\right]$. Then, there exists $C_1 > 0$ such that

$$\|B_m(f, g)\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}} \le C_1 \|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}} . \tag{2.27}$$

Also using the assumptions $\omega_i < u_i, v_i < m_i$ $(i = 1, 2)$ and $u_3 < \omega_3$, $m_3 < v_3$, we know that by Theorem 2.16 in [3]

$$\left(FW^{\theta_i,p_i,q_i}_{u_i,m_i}\right)_{a_i}(\mathbb{R}) \subset \left(FW^{\theta_i,p_i,q_i}_{\omega_i,v_i}\right)_{a_i}(\mathbb{R}), \quad (i = 1, 2)$$

and

$$\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}(\mathbb{R}) \subset \left(FW^{\theta_3,p_3,q_3}_{u_3,m_3}\right)_{a_3}(\mathbb{R}).$$

Then, there exist C_i $(i = 2, 3, 4)$ such that

$$\|B_m(f,g)\|_{\left(FW^{\theta_3,p_3,q_3}_{u_3,m_3}\right)_{a_3}} \le C_2 \|B_m(f,g)\|_{\left(FW^{\theta_3,p_3,q_3}_{\omega_3,v_3}\right)_{a_3}}, \tag{2.28}$$

$$\|f\|_{\left(FW^{\theta_1,p_1,q_1}_{\omega_1,v_1}\right)_{a_1}} \le C_3 \|f\|_{\left(FW^{\theta_1,p_1,q_1}_{u_1,m_1}\right)_{a_1}} \tag{2.29}$$

and

$$\|g\|_{\left(FW^{\theta_2,p_2,q_2}_{\omega_2,v_2}\right)_{a_2}} \le C_4 \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{u_2,m_2}\right)_{a_2}}. \tag{2.30}$$

From the inequalities (2.27), (2.28), (2.29) and (2.30), we get

$$\|B_m(f,g)\|_{\left(FW^{\theta_3,p_3,q_3}_{u_3,m_3}\right)_{a_3}} \le C_0 \|f\|_{\left(FW^{\theta_1,p_1,q_1}_{u_1,m_1}\right)_{a_1}} \|g\|_{\left(FW^{\theta_2,p_2,q_2}_{u_2,m_2}\right)_{a_2}},$$

where $C_0 = C_1C_2C_3C_4$. That means

$$m \in BM\left[FW(p_1,q_1,u_1,m_1,a_1,\theta_1;p_2,q_2,u_2,m_2,a_2,\theta_2;p_3,q_3,u_3,m_3,a_3,\theta_3)\right].$$

Therefore, we obtain

$$BM\left[FW(p_i,q_i,\omega_i,v_i,a_i,\theta_i)\right] \subset BM\left[FW(p_i,q_i,u_i,m_i,a_i,\theta_i)\right].$$

□

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