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On the properties of multiplication operators in some function spaces

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Abstract

In this paper, we discuss and characterize the boundedness, compactness and closed range of the multiplication operator. Moreover, we obtain some new results about necessary and sufficient conditions for the boundedness of operator norm of multiplication operator in variable exponent amalgam spaces.

Keywords: Multiplication operators, variable exponent amalgam spaces, compactness

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1. Introduction

The amalgam spaces $(L^p, l^q)(\mathbb{R})$ (or shortly (L^p, l^q)) classify functions in point of their local and global behavior. There are several applications about time-frequency analysis, Gabor analysis and sampling theory in these spaces [18, 21, 24]. Recently, interest in variable exponential amalgam spaces $\left(L^{p(.)}, l^q\right)$ has increased and many articles on this subject have been published [3]-[7], [22, 23, 25, 28, 29].

The studies on multiplication operators have an extensive and long history. Especially, there has been published very much papers on function spaces of measurable functions [1, 2], [9]-[14], [19], [30]-[34]. These operators were also applied to problems in mathematical physics, ergodic theory, classical and statistical mechanics, dynamical systems and distribution theory. For more details, we refer to [27].

The multiplication operators under appropriate assumptions generate another class of transformations, known as the weighted composition transformations. These operators on different function spaces, such as Lebesgue, Hardy, amalgam, locally convex and other function spaces have been also studied for many years. The main goal of this paper is to investigate the boundedness and the compactness properties of multiplication operators on variable exponent amalgam spaces $\left(L^{p(.)}, \ell^q\right)$ for $1 \leq p(.), q < \infty$. Castillo et al. [13] studied the boundedness, invertibility, compactness and closedness of the range of multiplication operators on $L^{p(.)}(X)$, where (X, Σ, μ) is a $\sigma-$finite, complete measure space.

Definition 1.1. Let $p(.)$ be a measurable function from $\mathbb{R}$ into $[1, \infty)$ (called a variable exponent on $\mathbb{R}$) satisfying the condition $1 \leq p^- \leq p(.) \leq p^+ \leq \infty$, where

$$p^- = ess \inf_{x\in\mathbb{R}} p(x), \quad p^+ = ess \sup_{x\in\mathbb{R}} p(x).$$

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The variable exponent Lebesgue spaces $L^{p(.)}(\mathbb{R})$ consist of all measurable functions f such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where

$$\varrho_{p(.)}(f) = \int_{\mathbb{R}} |f(x)|^{p(x)}\, dx.$$

The space $\left(L^{p(.)}(\mathbb{R}), \|.\|_{p(.)}\right)$ is a Banach function space [16, Theorem 3.2.13]. If $p^+ < \infty$, then $f \in L^{p(.)}(\mathbb{R})$ if and only if $\varrho_{p(.)}(f) < \infty$, that is, the norm topology is equivalent to the modular topology. Also if $p(.) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$ [26]. The set of simple functions that are linear combination of indicator functions of measurable sets with finite measure is dense in $L^{p(.)}(\mathbb{R})$ under condition $p^+ < \infty$ [16, Corollary 3.4.10].

Definition 1.2. The variable exponent amalgam spaces $\left(L^{p(.)}, l^q\right)$ are defined by

$$\left(L^{p(.)}, l^q\right) = \left\{f \in L^{p(.)}_{loc}(\mathbb{R}) : \|f\|_{(p(.),q)} < \infty\right\},$$

where $\|f\|_{(p(.),q)} = \left(\sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k}\right\|^q_{p(.)}\right)^{\frac{1}{q}}$, $J_k = [k, k+1)$, $k \in \mathbb{Z}$ and $1 \leq p(.), q < \infty$.

It is known that $\left(L^{p(.)}, l^q\right)$ is a Banach function space with respect to the norm $\|.\|_{(p(.),q)}$ [6, Theorem 2]. If we write $[k, k+1)$, $[k, k+1]$ or $(k, k+1)$ for J_k, we get the same space $\left(L^{p(.)}, l^q\right)$, i.e., the amalgam space does not depend on the particular choice of J_k. Since the variable exponent amalgam spaces are the Banach function space, we can easily write the following properties from [8, Theorem 1.7].

(i) (The solid property) If $|f| \leq |g|$ μ-a.e. and $g \in \left(L^{p(.)}, l^q\right)$, then $f \in \left(L^{p(.)}, l^q\right)$.

(ii) (The Fatou property) Suppose $f_n \in \left(L^{p(.)}, l^q\right)$, $f_n \geq 0$, $(n = 1, 2, ...)$, and $f_n \uparrow f$ μ-a.e. If $f \in \left(L^{p(.)}, l^q\right)$, then $\|f_n\|_{(p(.),q)} \uparrow \|f\|_{(p(.),q)}$ whereas if $f \notin \left(L^{p(.)}, l^q\right)$, then $\|f_n\|_{(p(.),q)} \uparrow \infty$.

(iii) (Fatou's Lemma) If $f_n \in \left(L^{p(.)}, l^q\right)$, $(n = 1, 2, ...)$, $f_n \to f$ μ-a.e., and $\liminf_{n\to\infty} \|f_n\|_{(p(.),q)} < \infty$, then $f \in \left(L^{p(.)}, l^q\right)$ and

$$\|f\|_{(p(.),q)} \leq \liminf_{n\to\infty} \|f_n\|_{(p(.),q)}.$$

(iv) Every simple function belongs to $\left(L^{p(.)}, l^q\right)$.

(v) To each set E of finite measure there corresponds a constant C_E satisfying $0 < C_E < \infty$ such that

$$\int_E |f|\, d\mu \leq C_E \|f\|_{(p(.),q)}$$

for all $f \in \left(L^{p(.)}, l^q\right)$.

(vi) If $f_n \to f$ in $\left(L^{p(.)}, l^q\right)$, then $f_n \to f$ in measure on every set of finite measure; in particular, some subsequence of $\{f_n\}$ converges to f pointwise μ-a.e.

The dual space of $\left(L^{p(.)}, l^q\right)$ is isometrically isomorphic to $\left(L^{p'(.)}, l^{q'}\right)$, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Also the space $\left(L^{p(.)}, l^q\right)$ is reflexive. So this Banach function space has an absolutely continuous norm [8, Corollary 4.4]. Then it can be easily shown using [8, Proposition 3.6] that the set of simple functions is dense in $\left(L^{p(.)}, l^q\right)$ under condition $1 \leq p(.), q < \infty$.

Moreover, the spaces $L^{p(.)}(\mathbb{R})$ and $\left(L^{p(.)}, \ell^q\right)$ are not translation invariant. If we take $p(.) = p$ =const., then we have classical amalgam space $\left(L^{p(.)}, l^q\right) = (L^p, l^q)$. For detail information about these spaces see [3]-[7], [22, 23].

Definition 1.3. Let A be a measurable subset of $\mathbb{R}$. The space $\left(L^{p(.)}, l^q\right)[A]$ is defined by

$$\left(L^{p(.)}, l^q\right)[A] = \left\{f\chi_A : f \in \left(L^{p(.)}, l^q\right)\right\}.$$

Remark 1.4. We take $(H_n)_{n\in\mathbb{N}}$ that is a sequence in $\left(L^{p(.)}, l^q\right)[A]$ such that $H_n \to H$ in $\left(L^{p(.)}, l^q\right)$. Note that $H = H\chi_A + H\chi_{A^c}$. By property (vi), we achieve $H\chi_{(A_\delta(u))^c} = 0$. That means $H \in \left(L^{p(.)}, l^q\right)[A]$. Thus, we easily conclude that the space $\left(L^{p(.)}, l^q\right)[A]$ is a closed subspace of $\left(L^{p(.)}, l^q\right)$ for every measurable subset A.

Throughout this paper, we assume that $1 \leq p^- \leq p(.) \leq p^+ < \infty$, $1 \leq r^- \leq r(.) \leq r^+ < \infty$ and $1 < q, s < \infty$.

2. Multiplication operators on variable exponent amalgam spaces

Let $F(X)$ be a function space on a non-empty set X, and let $\varphi : X \to \mathbb{R}$ be a function such that $\varphi.f \in F(X)$ whenever $f \in F(X)$. Then the transformation $f \to \varphi.f$ is denoted by M_φ. In case $F(X)$ is a topological space, $M_\varphi : F(X) \to F(X)$ is called the multiplication operator induced by φ.

In the following theorem we demonstrate the boundedness of M_u in variable exponent amalgam spaces $\left(L^{p(.)}, l^q\right)$.

Theorem 2.1. *The multiplication operator $M_u : \left(L^{p(.)}, l^q\right) \to \left(L^{p(.)}, l^q\right)$ is well-defined if and only if $u \in L^\infty$. In this case M_u is bounded and*

$$\|M_u\| = \|u\|_\infty .$$

Proof. Assume that $u \in L^\infty$. Then we can write

$$|(uf)(x)| \leq \|u\|_\infty |f(x)|$$

for all $x \in \mathbb{R}$. Then we have

$$\begin{aligned} \|M_u f\|_{(p(.),q)} &= \|uf\|_{(p(.),q)} \\ &= \left(\sum_{k\in\mathbb{Z}} \left\|uf\chi_{J_k}\right\|_{p(.)}^q\right)^{\frac{1}{q}} \\ &\leq \left(\sum_{k\in\mathbb{Z}} \left\|\|u\|_\infty f\chi_{J_k}\right\|_{p(.)}^q\right)^{\frac{1}{q}} \\ &= \|u\|_\infty \|f\|_{(p(.),q)} . \end{aligned} \tag{2.1}$$

So by (2.1), we obtain

$$\|M_u\| = \sup_{f\neq 0} \frac{\|M_u f\|_{(p(.),q)}}{\|f\|_{(p(.),q)}} \leq \|u\|_\infty . \tag{2.2}$$

From the inequality (2.2), we conclude that M_u is well-defined and also bounded from $\left(L^{p(.)}, l^q\right)$ to $\left(L^{p(.)}, l^q\right)$.

Conversely, assume that M_u is well-defined $\left(L^{p(.)}, l^q\right)$ to $\left(L^{p(.)}, l^q\right)$. Also assume that $u \notin L^\infty$. Then for every $n \in \mathbb{N}$, the set

$$U_n = \{x \in \mathbb{R} : |u(x)| > n\}$$

has positive measure. If the measure is used σ-finite, then a measurable subset of U_n with finite positive measure is found. We denote this subset by V_n. Then we have $\chi_{V_n} \in \left(L^{p(.)}, l^q\right)$. Since $\left(L^{p(.)}, l^q\right)$ is solid and $\left|(u\chi_{V_n})(x)\right| \geq n\chi_{V_n}(x)$ for any $n \in \mathbb{N}$, we obtain

$$\left\|M_u\chi_{V_n}\right\|_{(p(.),q)} = \left\|u\chi_{V_n}\right\|_{(p(.),q)} \geq n\left\|\chi_{V_n}\right\|_{(p(.),q)} .$$

This contradicts the well-defined of M_u (M_u is bounded). So u must be in L^∞.

Now we will show that the operator norm of M_u is $\|u\|_\infty$. Let define the set

$$A_\varepsilon = \{x \in \mathbb{R} : |u(x)| > \|u\|_\infty - \varepsilon\}$$

which has positive measure for any $\varepsilon > 0$. Since the space $\left(L^{p(.)}, l^q\right)$ is solid, we have

$$\left\|M_u \chi_{A_\varepsilon}\right\|_{(p(.),q)} \geq \left(\|u\|_\infty - \varepsilon\right) \left\|\chi_{A_\varepsilon}\right\|_{(p(.),q)}.$$

So we find

$$\|u\|_\infty - \varepsilon \leq \|M_u\| = \sup_{f \neq 0} \frac{\|M_u f\|_{(p(.),q)}}{\|f\|_{(p(.),q)}}. \tag{2.3}$$

Due to the fact that $\varepsilon > 0$ is arbitrary and by (2.3), we achieve

$$\|u\|_\infty \leq \|M_u\|. \tag{2.4}$$

Finally combining (2.2) and (2.4), we obtain that M_u is bounded and

$$\|M_u\|_{(p(.),q)} = \|u\|_\infty .$$

□

Now, we discuss the closed range of M_u. The multiplication operator is not always one to one. For example, we take $u(x) = \chi_A(x)$ and $f(x) = \chi_B(x)$ where $A,\ B \subset \mathbb{R}$, $A \cap B = \varnothing$. Then,

$$M_u f(x) = u(x) f(x) = \chi_A(x) \chi_B(x) = 0.$$

Hence since $\mathrm{Ker}(M_u) \neq \{0\}$, M_u is not one to one.

Proposition 2.2. *M_u is one to one on $Y = \left(L^{p(.)}, l^q\right)[supp(u)]$.*

Proof. If $M_u\left(f\chi_{\mathrm{supp}(u)}\right) = 0$ with $f \in \left(L^{p(.)}, l^q\right)$, then $f(x)\chi_{\mathrm{supp}(u)}(x) = 0$ for all $x \in \mathbb{R}$. That means $\mathrm{Ker}(M_u) = \{0\}$. Hence we find that M_u is one to one on Y. □

Definition 2.3. An operator $U : X \to Y$ between normed spaces is said to be bounded below if there exists a constant $C > 0$ such that $\|Ux\| \geq C\|x\|$ for each $x \in X$.

Using the following theorem we get some results for the range of M_u.

Theorem 2.4. *Let U be a bounded linear operator $U : X \to Y$, where X and Y are Banach spaces. Then U is bounded below if and only if U is one-to-one and has closed range (see* [1]*).*

By Proposition 2.2 and Theorem 2.4, we can give the following corollary.

Corollary 2.5. *The multiplication operator $M_u : \left(L^{p(.)}, l^q\right)[supp(u)] \to \left(L^{p(.)}, l^q\right)[supp(u)]$ is bounded below if and only if M_u has closed range.*

Lemma 2.6. *If the range of an operator U in $B(X, Y)$ is closed, there is a constant K such that to each y in UX corresponds an x with $Ux = y$ and $\|x\| \leq K\|y\|$ (see* [17]*).*

Theorem 2.7. *Let $u \in L^\infty$. The multiplication operator $M_u : \left(L^{p(.)}, l^q\right) \to \left(L^{p(.)}, l^q\right)$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on supp(u).*

Proof. Assume that there exists a $\delta > 0$ for which $|u(x)| \geq \delta$ a.e. on supp(u). Then

$$\left|u(x) f(x) \chi_{supp(u)}(x)\right| \geq \delta \left|f(x) \chi_{supp(u)}(x)\right| \text{ a.e.}$$

Since the space $\left(L^{p(.)}, l^q\right)$ is solid, we have

$$\left\|M_u\left(f\chi_{supp(u)}\right)\right\|_{(p(.),q)} = \left\|uf\chi_{supp(u)}\right\|_{(p(.),q)} \geq \delta \left\|f\chi_{supp(u)}\right\|_{(p(.),q)}.$$

It implies that M_u is bounded below on $\left(L^{p(.)}, l^q\right)[supp\,(u)]$. Hence by Theorem 2.4 and Corollary 2.5, we obtain that M_u has closed range on $\left(L^{p(.)}, l^q\right)$.

Conversely assume that M_u has closed range on $\left(L^{p(.)}, l^q\right)$. Then by Lemma 2.6, M_u is bounded below on $\left(L^{p(.)}, l^q\right)[supp\,(u)]$. So there exists a $\delta > 0$ such that

$$\|M_u f\|_{(p(.),q)} \geq \delta \|f\|_{(p(.),q)} \tag{2.5}$$

for all $f \in \left(L^{p(.)}, l^q\right)[supp\,(u)]$. Now define that

$$E = \left\{x \in \text{supp}\,(u) : \ |u\,(x)| < \frac{\delta}{2}\right\}.$$

Suppose that $\mu\,(E) > 0$. Then there exists $k \in \mathbb{Z}$ such that $F = E \cap J_k$ has finite positive measure and $\chi_F \in \left(L^{p(.)}, l^q\right)[supp\,(u)]$. Also, we have

$$|u\,(x)\,\chi_F\,(x)| \leq \frac{\delta}{2}\,|\chi_F\,(x)|\,. \tag{2.6}$$

Since $\left(L^{p(.)}, l^q\right)$ is solid and by (2.6), we obtain

$$\|M_u \chi_F\|_{(p(.),q)} \leq \frac{\delta}{2}\,\|\chi_F\|_{(p(.),q)}\,. \tag{2.7}$$

Inequalities (2.5) and (2.7) contradict. Therefore $\mu\,(E) = 0$. Hence we achieve a $\delta > 0$ such that $|u\,(x)| \geq \delta$ a.e. on supp(u). □

Assume that $p'\,(.)$ and q' denote the conjugate exponents of $p\,(.)$ and q, that is, $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Now we consider the boundedness of multiplication operators from $\left(L^{p(.)}, l^q\right)$ into L^1.

Lemma 2.8 (Hölder inequality for the variable exponent amalgam spaces)**.** *If $f \in \left(L^{p(.)}, l^q\right)$ and $g \in \left(L^{p'(.)}, l^{q'}\right)$, then fg is integrable and*

$$\int_{\mathbb{R}} |fg|\,d\mu \leq C\,\|f\|_{(p(.),q)}\,\|g\|_{(p'(.),q')}$$

for some $C > 0$.

Proof. Assume that $f \in \left(L^{p(.)}, l^q\right)$ and $g \in \left(L^{p'(.)}, l^{q'}\right)$. Then we have $f\chi_{J_k} \in L^{p(.)}\,(J_k)$ and $g\chi_{J_k} \in L^{p'(.)}\,(J_k)$ for each $J_k = [k, k+1)$. On the other hand by [20], it is known that $\left(L^1, l^1\right) = L^1$. So we write

$$\int_{\mathbb{R}} |fg|\,d\mu \simeq \|fg\|_{(1,1)} = \sum_{k\in\mathbb{Z}} \left\|fg\chi_{J_k}\right\|_1 = \sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k} g\chi_{J_k}\right\|_1. \tag{2.8}$$

Applying Hölder inequality in variable Lebesque spaces [15] to (2.8), we have

$$\int_{\mathbb{R}} |fg|\,d\mu \lesssim \sum_{k\in\mathbb{Z}} C\left\|f\chi_{J_k}\right\|_{p(.)}\left\|g\chi_{J_k}\right\|_{p'(.)}. \tag{2.9}$$

We know that the constant C is independent of $k \in \mathbb{Z}$ [15, Remark 2.27]. Let us take the constant C out of the sum and apply the Hölder inequality in sequence spaces to the inequality (2.9). Thus we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |fg|\,d\mu &\lesssim C\sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k}\right\|_{p(.)}\left\|g\chi_{J_k}\right\|_{p'(.)} \\
&\leq C\left(\sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k}\right\|_{p(.)}^q\right)^{\frac{1}{q}}\left(\sum_{k\in\mathbb{Z}} \left\|g\chi_{J_k}\right\|_{p'(.)}^{q'}\right)^{\frac{1}{q'}} \\
&= C\,\|f\|_{(p(.),q)}\,\|g\|_{(p'(.),q')}\,.
\end{aligned}$$

□

Theorem 2.9. *If* $u \in \left(L^{p'(.)}, l^{q'}\right)$*, then the multiplication operator* $M_u : \left(L^{p(.)}, l^q\right) \to L^1$ *is bounded. That is*

$$\|M_u\| \leq C \|u\|_{(p'(.),q')}$$

for some $C > 0$.

Proof. We will use the Closed Graph Theorem to prove this theorem [35]. Suppose that $f_n \to f$ in $\left(L^{p(.)}, l^q\right)$ for the sequence $(f_n) \subset \left(L^{p(.)}, l^q\right)$, and $M_u(f_n) = uf_n \to h$ in $L^1(\mathbb{R})$. Then (f_n) has a subsequence (f_{n_k}) which converges pointwise almost everywhere to f by property (vi), and $uf_{n_k} \to h$ pointwise a.e. Moreover, we obtain $uf_{n_k} \to uf$ pointwise a.e. So $h = uf$ pointwise a.e. This shows that the graph of M_u is closed, and M_u is bounded from $\left(L^{p(.)}, l^q\right)$ into L^1. Using Lemma 2.8, we can write the inequality

$$\|M_u f\|_{L^1} = \|uf\|_{L^1} \leq C \|u\|_{(p'(.),q')} \|f\|_{(p(.),q)} \tag{2.10}$$

for some $C > 0$ and all $f \in \left(L^{p(.)}, l^q\right)$. Then by (2.10) we have

$$\|M_u\| = \sup_{f \neq 0} \frac{\|M_u f\|_{L^1}}{\|f\|_{(p(.),q)}} \leq C \|u\|_{(p'(.),q')} . \tag{2.11}$$

By the (2.11), we obtain that M_u is bounded from $\left(L^{p(.)}, l^q\right)$ into L^1. □

Let us investigate the boundedness of multiplication operators from $\left(L^{p(.)}, l^r\right)$ into $\left(L^{q(.)}, l^s\right)$.

Lemma 2.10 (Generalized Hölder inequality for the variable exponent amalgam spaces)**.** *Let* $\frac{1}{p(.)} + \frac{1}{t(.)} = \frac{1}{q(.)}$, $\frac{1}{r} + \frac{1}{m} = \frac{1}{s}$. *If* $f \in \left(L^{p(.)}, l^r\right)$ *and* $g \in \left(L^{t(.)}, l^m\right)$*, then* $fg \in \left(L^{q(.)}, l^s\right)$ *and*

$$\|fg\|_{(q(.),s)} \leq C \|f\|_{(p(.),r)} \|g\|_{(t(.),m)}$$

for some $C > 0$.

Proof. From Corollary 2.28, Remark 2.29 in [15] and generalized Hölder inequality for the space l^s, the proof is easily obtained with the same technique as in Lemma 2.8. □

Theorem 2.11. *Assume that* $\frac{1}{p(.)} + \frac{1}{t(.)} = \frac{1}{q(.)}$, $\frac{1}{r} + \frac{1}{m} = \frac{1}{s}$. *If* $u \in \left(L^{t(.)}, l^m\right)$*, then the multiplication operator* $M_u : \left(L^{p(.)}, l^r\right) \to \left(L^{q(.)}, l^s\right)$ *is bounded. That is*

$$\|M_u\| \leq C \|u\|_{(t(.),m)}$$

for some $C > 0$.

Proof. Let $u \in \left(L^{t(.)}, l^m\right)$. By Lemma 2.10 , there exists $C > 0$ such that

$$\|M_u f\|_{(q(.),s)} = \|uf\|_{(q(.),s)} \leq C \|u\|_{(t(.),m)} \|f\|_{(p(.),r)} . \tag{2.12}$$

Using the inequality (2.12), we get

$$\|M_u\| = \sup_{f \neq 0} \frac{\|M_u f\|_{(q(.),s)}}{\|f\|_{(p(.),r)}} \leq C \|u\|_{(t(.),m)} . \tag{2.13}$$

From the inequality (2.13), we conclude that $M_u : \left(L^{p(.)}, l^r\right) \to \left(L^{q(.)}, l^s\right)$ is bounded. □

3. Compactness of the multiplication operator on variable exponent amalgam spaces

Definition 3.1. Let $U : X \to X$ be an operator. A subspace V of X is said to be invariant under U (or U-invariant) if $U(V) \subset V$.

Lemma 3.2. *Let $U : X \to X$ be an operator. If U is compact and V is a closed U-invariant subspace of X, then $U|_V$ is compact (see* [1]*).*

Lemma 3.3. *Let A be a measurable subset of $\mathbb{R}$. If $M_u : \left(L^{p(.)}, l^q\right) \to \left(L^{p(.)}, l^q\right)$ is well-defined, then $M_u : \left(L^{p(.)}, l^q\right)[A] \to \left(L^{p(.)}, l^q\right)[A]$ is well-defined.*

Proof. Suppose that $M_u : \left(L^{p(.)}, l^q\right) \to \left(L^{p(.)}, l^q\right)$ is well-defined. Since $M_u(f\chi_A) = u(f\chi_A) = (uf)\chi_A$, we obtain that $M_u : \left(L^{p(.)}, l^q\right)[A] \to \left(L^{p(.)}, l^q\right)[A]$ is well-defined. □

Using Remark 1.4, Lemma 2.6 and Lemma 3.2, the following corollary is proved easily.

Corollary 3.4. *Let A be a measurable subset of $\mathbb{R}$. If $M_u : \left(L^{p(.)}, l^q\right) \to \left(L^{p(.)}, l^q\right)$ is well-defined, then $\left(L^{p(.)}, l^q\right)[A]$ is always a closed invariant subspace of $\left(L^{p(.)}, l^q\right)$ under M_u. Moreover if M_u is a compact operator, then we obtain that $M_u|_{\left(L^{p(.)},l^q\right)[A]} : \left(L^{p(.)}, l^q\right)[A] \to \left(L^{p(.)}, l^q\right)[A]$ is a compact operator.*

Definition 3.5. Assume that u is a real-valued essentially bounded function on $\mathbb{R}$. The set $A_\delta(u)$ is defined by

$$A_\delta(u) = \{x \in \mathbb{R} : |u(x)| \geq \delta\}$$

for $\delta > 0$.

Remark 3.6. Under same conditions, the multiplication operator $M_u|_{\left(L^{p(.)},l^q\right)[A_\delta(u)]} : \left(L^{p(.)}, l^q\right)[A_\delta(u)] \to \left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is compact, if $A = A_\delta(u)$ is taken specifically in Corollary 3.4.

Theorem 3.7. *Assume that u is a real-valued essentially bounded function on $\mathbb{R}$. Then the operator $M_u : \left(L^{p(.)}, l^q\right) \to \left(L^{p(.)}, l^q\right)$ is a compact operator if and only if the space $\left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$.*

Proof. Suppose that M_u is a compact operator. For all $x \in A_\delta(u)$, we can write

$$\left|uf\chi_{A_\delta(u)}(x)\right| \geq \delta\left|f\chi_{A_\delta(u)}(x)\right|.$$

Then, since the space $\left(L^{p(.)}, l^q\right)$ is solid, we get

$$\begin{aligned}\left\|M_u f\chi_{A_\delta(u)}\right\|_{(p(.),q)} &= \left\|uf\chi_{A_\delta(u)}\right\|_{(p(.),q)} \\ &\geq \delta\left\|f\chi_{A_\delta(u)}\right\|_{(p(.),q)}.\end{aligned}$$

So by Theorem 2.4, $M_u|_{\left(L^{p(.)},l^q\right)[A_\delta(u)]}$ has closed range and it is invertible. Also by Remark 1.4, it is known that $\left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is closed invariant subspace of the variable exponent amalgam space $\left(L^{p(.)}, l^q\right)$ under M_u. Then by Corollary 3.4, we say that $M_u|_{\left(L^{p(.)},l^q\right)[A_\delta(u)]}$ is a compact operator. Therefore from the fact that $M_u|_{\left(L^{p(.)},l^q\right)[A_\delta(u)]}$ has closed range, it is invertible and compact, we conclude that $\left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$.

Conversely, assume that $\left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$. Particularly $\left(L^{p(.)}, l^q\right)\left[A_{\frac{1}{n}}(u)\right]$ is finite dimensional for all $n \in \mathbb{N}$. Now we define $u_n : \mathbb{R} \to \mathbb{R}$ as

$$u_n(x) = \begin{cases} u(x), & |u(x)| \geq \frac{1}{n} \\ 0, & |u(x)| < \frac{1}{n} \end{cases}.$$

Then we have

$$\left|M_{u_n}f - M_u f\right| = |(u_n - u)f| \leq \|u_n - u\|_\infty |f|.$$

From the last inequality, we conclude that

$$\left\|M_{u_n}f - M_u f\right\|_{(p(.),q)} \le \|u_n - u\|_\infty \|f\|_{(p(.),q)} \le \frac{1}{n}\|f\|_{(p(.),q)}$$

due to solidness of the space $\left(L^{p(.)}, l^q\right)$. This inequality implies that M_{u_n} converges to M_u uniformly. On the other hand, since each other spaces $\left(L^{p(.)}, l^q\right)\left[A_{\frac{1}{n}}(u)\right]$ is finite dimensional, M_{u_n} is a finite rank operator. Then the operator M_{u_n} is compact. From uniform convergence, we achieve that M_u is compact operator. □

Theorem 3.8. *The set of all multiplication operators on* $\left(L^{p(.)}, l^q\right)$ *is a maximal abelian subalgebra of* $B\left(\left(L^{p(.)}, l^q\right)\right)$, *the algebra of all bounded operators on* $\left(L^{p(.)}, l^q\right)$.

Proof. Define the set H to be

$$H = \{M_u : u \in L^\infty(\mathbb{R})\}.$$

Take the operator product $M_u.M_v = M_{uv}$, where $M_u, M_v \in H$. Then $u, v \in L^\infty(\mathbb{R})$. Since $|u| \le \|u\|_\infty$ and $|v| \le \|v\|_\infty$, we have

$$\|uv\|_\infty \le \|u\|_\infty \|v\|_\infty.$$

That means this product is an inner operation. Also H is associative, commutative and distributive with respect to the sum and scalar product. Therefore we say that H is a subalgebra of $B\left(\left(L^{p(.)}, l^q\right)\right)$. Take the unit function $e : \mathbb{R} \to \mathbb{R}$ given by $e(x) = 1$ for all $x \in \mathbb{R}$. Assume that $T \in B\left(\left(L^{p(.)}, l^q\right)\right)$ is an operator which commutes with H and χ_E is the indicator function of a measurable set E. So we have

$$T(\chi_E) = T\left(M_{\chi_E}(e)\right) = M_{\chi_E}(T(e)) = \chi_E.T(e) = M_{T(e)}(\chi_E).$$

Then for any simple function s, we can write

$$T(s) = M_\omega(s), \tag{3.1}$$

where $\omega = T(e)$. Now suppose that $\omega \notin L^\infty(\mathbb{R})$. Then the set

$$W_n = \{x \in \mathbb{R} : |\omega(x)| > n\}$$

has positive measure for each $n \in \mathbb{N}$. Also since the measure is σ-finite, there exists a measurable subset of W_n with finite positive measure. We denote this subset by $\tilde{W}_n$. So we have $\chi_{\tilde{W}_n} \in \left(L^{p(.)}, l^q\right)$ for all $n \in \mathbb{N}$. Then since

$$M_\omega\left(\chi_{\tilde{W}_n}\right)(x) = \left(\omega\chi_{\tilde{W}_n}\right)(x) \ge n\chi_{\tilde{W}_n}(x)$$

for all $x \in \mathbb{R}$, we achieve

$$\left\|M_\omega\left(\chi_{\tilde{W}_n}\right)\right\|_{(p(.),q)} \ge n\left\|\chi_{\tilde{W}_n}\right\|_{(p(.),q)} \tag{3.2}$$

due to solidness of the space $\left(L^{p(.)}, l^q\right)$. By (3.1) and (3.2), we find

$$\left\|T\left(\chi_{\tilde{W}_n}\right)\right\|_{(p(.),q)} \ge n\left\|\chi_{\tilde{W}_n}\right\|_{(p(.),q)}$$

which contradicts the fact that T is a bounded operator. Therefore $\omega \in L^\infty(\mathbb{R})$ and by Theorem 2.1, M_ω is bounded. Take a non-negative function $f \in \left(L^{p(.)}, l^q\right)$. Since $p^+ < \infty$ and $q < \infty$, there exists a non-decreasing sequence $(s_n)_{n\in\mathbb{N}}$ of measurable simple functions with finite measure support such that $(s_n)_{n\in\mathbb{N}}$ converges to f a.e and norm of $\left(L^{p(.)}, l^q\right)$. So we have

$$T(f) = T\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} T(s_n) = \lim_{n\to\infty} M_\omega(s_n) = M_\omega\left(\lim_{n\to\infty} s_n\right) = M_\omega(f).$$

Then we find $T(f) = M_\omega(f)$ for all $f \in \left(L^{p(.)}, l^q\right)$. Thus we obtain $T \in H$. That means H is a maximal abelian subalgebra of $\left(L^{p(.)}, l^q\right)$. □

Corollary 3.9. *The multiplication operator M_u on $\left(L^{p(.)}, l^q\right)$ is invertible if and only if u is invertible in $L^\infty(\mathbb{R})$.*

Proof. Suppose that M_u is invertible on $\left(L^{p(.)}, l^q\right)$. Then there exists $T \in B\left(\left(L^{p(.)}, l^q\right)\right)$ such that

$$M_u T = T M_u = I. \tag{3.3}$$

Let $M_\omega \in H$. Then

$$M_\omega M_u = M_u M_\omega. \tag{3.4}$$

Combining the equalities (3.3) and (3.4), we have

$$TM_\omega = IM_\omega T = TM_\omega M_u T = TM_u M_\omega T = TM_\omega T = M_\omega T.$$

So we conclude that T commutes with H. Then by Theorem 2.11, $T \in H$, that is, there exists $\omega \in L^\infty(\mathbb{R})$ such that $T = M_\omega$. Then we have

$$M_u M_\omega = M_\omega M_u = I.$$

So we write $u\omega = \omega u = 1$ a.e. In other words, we obtain that u is invertible in $L^\infty(\mathbb{R})$.

Now we assume that u is invertible in $L^\infty(\mathbb{R})$, that is, $uu^{-1} = u^{-1}u = 1$ and $u^{-1} = \frac{1}{u} \in L^\infty(\mathbb{R})$. Then we have

$$M_u M_{u^{-1}} = M_{u^{-1}} M_u = M_{u^{-1}u} = M_1 = I.$$

That means M_u is invertible on $\left(L^{p(.)}, l^q\right)$. □

In the following theorems, we reconsider compactness of multiplication operators from $\left(L^{p(.)}, l^r\right)$ into L^1 and $\left(L^{q(.)}, l^s\right)$.

Theorem 3.10. *Assume that $u \in L^r \subset \left(L^{p'(.)}, l^{q'}\right)$ for $1 \le r < \infty$.*

(i) *If the space $\left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$, then $M_u : \left(L^{p(.)}, l^q\right) \to L^1$ is compact operator.*

(ii) *Suppose that $p(.) = 1$. Then $M_u : \left(L^1, l^q\right) \to L^1$ is compact operator if and only if the space $\left(L^1, l^q\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$.*

Proof. **(i)** From the hypothesis, $\left(L^{p(.)}, l^q\right)\left[A_{\frac{1}{n}}(u)\right]$ is finite dimensional for all $n \in \mathbb{N}$. Again take the sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n : \mathbb{R} \to \mathbb{R}$,

$$u_n(x) = \begin{cases} u(x), & |u(x)| \ge \frac{1}{n} \\ 0, & |u(x)| < \frac{1}{n} \end{cases}$$

for all $n \in \mathbb{N}$. So it is clear that $(u_n)_{n\in\mathbb{N}} \subset L^r \subset \left(L^{p'(.)}, l^{q'}\right)$, $(u_n - u)_{n\in\mathbb{N}} \subset L^\infty$, $(u_n - u)_{n\in\mathbb{N}} \subset L^r$ and $u_n - u \to 0$ in L^∞ uniformly. Also we have $|u_n| \le |u|$. Then from the Lebesgue Dominated theorem, we find $u_n - u \to 0$ in L^r. On the other hand by Theorem 2.9, $(M_{u_n})_{n\in\mathbb{N}} \subset B\left(\left(L^{p(.)}, l^q\right), L^1\right)$ and $M_u \in B\left(\left(L^{p(.)}, l^q\right), L^1\right)$. Also from Lemma 2.8 and the inclusion $L^r \subset \left(L^{p'(.)}, l^{q'}\right)$, we get

$$\begin{aligned} \left\|M_{u_n} - M_u\right\| &= \sup_{\|f\|_{(p(.),q)} \le 1} \left\|M_{u_n - u} f\right\|_1 \\ &= \sup_{\|f\|_{(p(.),q)} \le 1} \|(u_n - u) f\|_1 \le C \|u_n - u\|_r \end{aligned}$$

for some $C > 0$. Since $u_n - u \to 0$ in L^r, we find $\left\|M_{u_n} - M_u\right\| \to 0$. That means the sequence $(M_{u_n})_{n\in\mathbb{N}}$ converges to M_u in $B\left(\left(L^{p(.)}, l^q\right), L^1\right)$. Since each other spaces $\left(L^{p(.)}, l^q\right)\left[A_{\frac{1}{n}}(u)\right]$ is finite dimensional, M_{u_n} is a finite rank operator. So M_{u_n} is a compact operator for all $n \in \mathbb{N}$. Since the space of compact operators is closed in the space of bounded operators, we achieve that M_u is a compact operator.

(ii) By assumption, we have $u \in \left(L^\infty, l^{q'}\right) \subset L^\infty$. The converse of the hypothesis is clear from (i).

Now we assume that $M_u : \left(L^1, l^q\right) \to L^1$ is compact operator. For all $x \in A_\delta(u)$, we have $\left|uf\chi_{A_\delta(u)}(x)\right| \geq \delta\left|f\chi_{A_\delta(u)}(x)\right|$. Then

$$\left\|M_u f\chi_{A_\delta(u)}\right\|_1 \geq \delta\left\|f\chi_{A_\delta(u)}\right\|_1.$$

On the other hand since $L^1 = \left(L^1, l^1\right) \subset \left(L^1, l^q\right)$, there exists $C > 0$ such that

$$\left\|M_u f\chi_{A_\delta(u)}\right\|_1 \geq \delta C\left\|f\chi_{A_\delta(u)}\right\|_{(1,q)}.$$

Thus by Theorem 2.4, we find that $M_u\ |_{(L^1,l^q)[A_\delta(u)]}$has closed range and it is invertible. From Remark 1.4, it is known that $\left(L^1, l^q\right)[A_\delta(u)]$ is closed invariant subspace of the variable exponent amalgam space $\left(L^1, l^q\right)$ under M_u. So we obtain that $M_u\ |_{(L^{p(.)},l^q)[A_\delta(u)]}$ is a compact operator. Hence we achieve that $\left(L^{p(.)}, l^q\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$. □

The following theorem is easily proved similar to the proof methods in Theorem 3.10.

Theorem 3.11. *Let $u \in \left(L^{t(.)}, l^m\right)$ and let be $\frac{1}{q(.)} = \frac{1}{p(.)} + \frac{1}{t(.)}$, $\frac{1}{s} = \frac{1}{r} + \frac{1}{m}$.*

(i) *Suppose that $L^v \subset \left(L^{t(.)}, l^m\right)$ for some $1 \leq v < \infty$. If the space $\left(L^{p(.)}, l^r\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$, then $M_u : \left(L^{p(.)}, l^r\right) \to \left(L^{q(.)}, l^s\right)$ is a compact operator.*

(ii) *Assume that $\left(L^{q(.)}, l^s\right) \subset \left(L^{p(.)}, l^r\right)$. If $M_u : \left(L^{p(.)}, l^r\right) \to \left(L^{q(.)}, l^s\right)$ is a compact operator, then the space $\left(L^{p(.)}, l^r\right)[A_\delta(u)]$ is finite dimensional for each $\delta > 0$.*

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