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Characterization and Zagreb indices of the projective path graphs of order k

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Abstract

In this paper; general formulae for vertex and edge partition of projective path graphs are obtained from projective planes of order k. Then, the characterization of projective-path graphs is presented. The general formulae of additive and multiplicative Zagreb indices and Narumi-Katayama index for the projective path graphs in terms of order k are given. Finally, the vertex-adjacency matrices and energies related to projective path graphs, that are obtained from the projective planes of order $k = 2, 3, 4, 5$ are calculated.

Keywords: Graph, degree sequence, path, projective plane, topological indices, energy, spectrum, adjacency matrix

2010 MSC: 05C10, 05C31, 05C50, 05C85, 05C38

1. Introduction

A projective plane is a linear space in which "any two lines meet" and "there exists a set of four points no three of which are collinear". Projective planes are one of the most important examples of finite geometries. It is known that there are unique projective planes of orders 2,3,4,5,7 and 8. For every finite projective plane there is a interger $k \geq 2$ called order that the number of lines passing through a point, the number of points on a line, also the number of total points and total lines in the plane are calculated by using it. Recently, graphs have been obtained by the method based on the assumption that the lines of finite projective planes are taken as a "path" and some properties of the graphs that emerge in this way have also been examined. The most common feature among these investigations is the degree sequences. Topological indices are constants which have a lot of applications in Graph Theory. These indices are mostly defined in terms of vertex degrees, distances or matrices corresponding to graphs. The neighborhood (adjacency) matrix is used in many fields in graph theory and molecular chemistry. The sum of the absolute values of the eigenvalues of the neighborhood matrix gives the energy of a graph. Since the concept of energy has many chemical interpretations in graph theory, it is a very useful and widely applied concept.

The structure of this paper is planned as follows: In section 2, projective-path graphs are obtained. In section 3, general formulas of vertex and edge partition of projective path graphs are given from projective planes of order k. Then, the characterization of projective-path graphs is presented. The general formulas of additive and multiplicative Zagreb indices and Narumi-Katayama index for the projective path graphs in terms of order k are determined. In

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section 4, the vertex-adjacency matrices and energies related to projective path graphs, that are obtained from the projective planes of order $k = 2, 3, 4, 5$ are calculated.

In this paper we use the construction of a projective plane with the help of Latin squares (for more information we can refer to [1]) and we write the set of the points and the set of the lines, namely the points of each line, according to their lexicographical order.

2. Projective-path graphs

In this section, we recall the method of obtaining graphs from finite projective planes from [8, 9].

The method of obtaining graphs from finite projective planes: "Path method"

Definition 2.1. Let $\pi = (P, L)$ be a finite projective plane of order k. If l_i is a line in π then it must have $k+1$ points. We take this line as an ordered $k+1-tuple$ $l_i = (l_{i1}, l_{i2}, l_{i3}, ..., l_{ik}, l_{i(k+1)})$. It is obvious that $k \geq 2$. We establish a new set "$p(l_i)$" for any line $l_i = (l_{i1}, l_{i2}, l_{i3}, ..., l_{ik}, l_{i(k+1)})$ we define

$$l_{i,j} = \left\{ l_{ij}, l_{i(j+1)} \right\}, \quad 1 \leq j \leq k,$$

$$p(l_i) := \left\{ l_{i,j} \,|\, j = 1, 2, ..., k \right\}.$$

Theorem 2.2 (Principal Theorem). *If $\pi = (P, L)$ is a finite projective plane of order k, then we have a graph $G = (V, E)$ such that*

$$\begin{aligned} V(G) &= P, \\ E(G) &= \bigcup_{l_i \in L} p(l_i). \end{aligned}$$

Definition 2.3. The graphs obtained from a finite projective plane as Theorem 2.2 are called projective-path graphs.

3. Vertex and edge partition and some topological indices of the projective path graphs

In this section; first we give examples of the projective path graphs that are obtained from the projective plane of order $k = 2, 3, 4, 5$. Then we obtain the general formulas of the vertex and edge partition of the projective path graphs that are obtained from the projective plane of order k. Thus we calculate additive and multiplicative Zagreb indices and also Narumi-Katayama index of the projective plane of order k.

Example 3.1. The set of the points and lines of the projective plane of order 2 as follows:

$$P = \{00, 01, 10, 11, (0), (1), (\infty)\},$$

$$L = \left\{ \begin{array}{c} \{00, 11, (1)\}, \{01, 10, (1)\}, \{00, 10, (0)\}, \{01, 11, (0)\}, \\ \{00, 01, (\infty)\}, \{10, 11, (\infty)\}, \{(0), (1), (\infty)\} \end{array} \right\}.$$

The vertex and edge sets of the projective path graph that is obtained from projective plane of order 2 are as follows:

$$V(G) = \{00, 01, 10, 11, (0), (1), (\infty)\},$$

$$E(G) = \left\{ \begin{array}{c} \{00, 11\}, \{11, (1)\}, \{01, 10\}, \{10, (1)\}, \{00, 10\}, \\ \{10, (0)\}, \{01, 11\}, \{11, (0)\}, \{00, 01\}, \{01, (\infty)\}, \\ \{10, 11\}, \{11, (\infty)\}, \{(0), (1)\}, \{(1), (\infty)\} \end{array} \right\}.$$

Here $|E(G)| = 14$. Now we determine the vertex degree of each vertex: $d(00) = 3$, $d(01) = 4$, $d(10) = 5$, $d(11) = 6$, $d((0)) = 3$, $d((1)) = 4$, $d((\infty)) = 3$. As a result we get the following degree sequence as $D.S. = \{6^{(1)}, 5^{(1)}, 4^{(2)}, 3^{(3)}\}$.

Example 3.2. The set of the points and lines of the projective plane of order 3 as follows:

$$P = \{00, 01, 02, 10, 11, 12, 20, 21, 22, (0), (1), (2), (\infty)\},$$

$$L = \left\{ \begin{array}{c} \{00, 11, 22, (1)\}, \{01, 12, 20, (1)\}, \{02, 10, 21, (1)\}, \\ \{00, 12, 21, (2)\}, \{01, 10, 22, (2)\}, \{02, 11, 20, (2)\}, \\ \{00, 10, 20, (0)\}, \{01, 11, 21, (0)\}, \{02, 12, 22, (0)\}, \\ \{00, 01, 02, (\infty)\}, \{10, 11, 12, (\infty)\}, \{20, 21, 22, (\infty)\}, \\ \{(0), (1), (2), (\infty)\} \end{array} \right\}.$$

The vertex and edge sets of the projective plane path graph that is obtained from the projective plane of order 3 are as follows:

$$V(G) = \{00, 01, 02, 10, 11, 12, 20, 21, 22, (0), (1), (2), (\infty)\},$$

$$E(G) = \left\{ \begin{array}{c} \{00, 11\}, \{11, 22\}, \{22, (1)\}, \{01, 12\}, \{12, 20\} \\ , \{20, (1)\}, \{02, 10\}, \{10, 21\}, \{21, (1)\}, \{00, 12\}, \\ \{12, 21\}, \{21, (2)\}, \{01, 10\}, \{10, 22\}, \{22, (2)\}, \\ \{02, 11\}, \{11, 20\}, \{20, (2)\}, \{00, 10\}, \{10, 20\}, \\ \{20, (0)\}, \{01, 11\}, \{11, 21\}, \{21, (0)\}, \{02, 12\}, \\ \{12, 22\}, \{22, (0)\}, \{00, 01\}, \{01, 02\}, \{02, (\infty)\}, \\ \{10, 11\}, \{11, 12\}, \{12, (\infty)\}, \{20, 21\}, \{21, 22\}, \\ \{22, (\infty)\}, \{(0), (1)\}, \{(1), (2)\}, \{(2), (\infty)\} \end{array} \right\}.$$

Here $|E(G)| = 14$. Now we determine the vertex degree of each vertex: $d(00) = 3$, $d(01) = 4$, $d(10) = 5$, $d(11) = 6$, $d((0)) = 3$, $d((1)) = 4$, $d((\infty)) = 3$. As a result we get the following degree sequence as $D.S. = \{6^{(1)}, 5^{(1)}, 4^{(2)}, 3^{(3)}\}$.

Example 3.3. The set of the points and lines of the projective plane of order 4 as follows:

$$P = \left\{ \begin{array}{c} 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, \\ 23, 30, 31, 32, 33, (0), (1), (2), (3), (\infty) \end{array} \right\},$$

$$L = \left\{ \begin{array}{c} \{00, 11, 22, 33, (1)\}, \{01, 10, 23, 32, (1)\}, \{02, 13, 20, 31, (1)\}, \\ \{03, 12, 21, 30, (1)\}, \{00, 12, 23, 31, (2)\}, \{01, 13, 22, 30, (2)\}, \\ \{02, 10, 21, 33, (2)\}, \{03, 11, 20, 32, (2)\}, \{00, 13, 21, 32, (3)\}, \\ \{01, 12, 20, 33, (3)\}, \{02, 11, 23, 30, (3)\}, \{03, 10, 22, 31, (3)\}, \\ \{00, 10, 20, 30, (0)\}, \{01, 11, 21, 31, (0)\}, \{02, 12, 22, 32, (0)\}, \\ \{03, 13, 23, 33, (0)\}, \{00, 01, 02, 03, (\infty)\}, \{10, 11, 12, 13, (\infty)\}, \\ \{20, 21, 22, 23, (\infty)\}, \{30, 31, 32, 33, (\infty)\}, \{(0), (1), (2), (3), (\infty)\} \end{array} \right\}.$$

The vertex and edge sets of the projective path graph that is obtained from the projective plane of order 4 are as follows:

$$V(G) = \left\{ \begin{array}{c} 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, \\ 23, 30, 31, 32, 33, (0), (1), (2), (3), (\infty) \end{array} \right\},$$

$$E(G)=\left\{\begin{array}{c}\{00,11\},\{11,22\},\{22,33\},\{33,(1)\},\{01,10\},\\ \{10,23\},\{23,32\},\{32,(1)\},\{02,13\},\{13,20\},\\ \{20,31\},\{31,(1)\},\{03,12\},\{12,21\},\{21,30\},\\ \{30,(1)\},\{00,12\},\{12,23\},\{23,31\},\{31,(2)\},\\ \{01,13\},\{13,22\},\{22,30\},\{30,(2)\},\{02,10\},\\ \{10,21\},\{21,33\},\{33,(2)\},\{03,11\},\{11,20\},\\ \{20,32\},\{32,(2)\},\{00,13\},\{13,21\},\{21,32\},\\ \{32,(3)\},\{01,12\},\{12,20\},\{20,33\},\{33,(3)\},\\ \{02,11\},\{11,23\},\{23,30\},\{30,(3)\},\{03,10\},\\ \{10,22\},\{22,31\},\{31,(3)\},\{00,10\},\{10,20\},\\ \{20,30\},\{30,(0)\},\{01,11\},\{11,21\},\{21,31\},\\ \{31,(0)\},\{02,12\},\{12,22\},\{22,32\},\{32,(0)\},\\ \{03,13\},\{13,23\},\{23,33\},\{33,(0)\},\{00,01\},\\ \{01,02\},\{02,03\},\{03,(\infty)\},\{10,11\},\{11,12\},\\ \{12,13\},\{13,(\infty)\},\{20,21\},\{21,22\},\{22,23\},\\ \{23,(\infty)\},\{30,31\},\{31,32\},\{32,33\},\{33,(\infty)\},\\ \{(0),(1)\},\{(1),(2)\},\{(2),(3)\},\{(3),(\infty)\}\end{array}\right\}.$$

Here $|E(G)| = 14$. Now we determine the vertex degree of each vertex: $d(00) = 3$, $d(01) = 4$, $d(10) = 5$, $d(11) = 6$, $d((0)) = 3$, $d((1)) = 4$, $d((\infty)) = 3$. As a result we get the following degree sequence as $D.S. = \{6^{(1)}, 5^{(1)}, 4^{(2)}, 3^{(3)}\}$.

Example 3.4. The set of the points and lines of the projective plane of order 5 as follows:

$$P=\left\{\begin{array}{c}00,01,02,03,04,10,11,12,13,14,\\ 20,21,22,23,24,30,31,32,33,34,\\ 40,41,42,43,44,(0),(1),(2),(3),(4),(\infty)\end{array}\right\},$$

$$L=\left\{\begin{array}{c}\{00,11,22,33,44,(1)\},\{01,12,23,34,40,(1)\},\{02,13,24,30,41,(1)\},\\ \{03,14,20,31,42,(1)\},\{04,10,21,32,43,(1)\},\\ \{00,12,24,31,43,(2)\},\{01,13,20,32,44,(2)\},\{02,14,21,33,40,(2)\},\\ \{03,10,22,34,41,(2)\},\{04,11,23,30,42,(2)\},\\ \{00,13,21,34,42,(3)\},\{01,14,22,30,43,(3)\},\{02,10,23,31,44,(3)\},\\ \{03,11,24,32,40,(3)\},\{04,12,20,33,41,(3)\},\\ \{00,14,23,32,41,(4)\},\{01,10,24,33,42,(4)\},\{02,11,20,34,43,(4)\},\\ \{03,12,21,30,44,(4)\},\{04,13,22,31,40,(4)\},\\ \{00,10,20,30,40,(0)\},\{01,11,21,31,41,(0)\},\{02,12,22,32,42,(0)\},\\ \{03,13,23,33,43,(0)\},\{04,14,24,34,44,(0)\},\\ \{00,01,02,03,04,(\infty)\},\{10,11,12,13,14,(\infty)\},\{20,21,22,23,24,(\infty)\},\\ \{30,31,32,33,34,(\infty)\},\{40,41,42,43,44,(\infty)\},\\ \{(0),(1),(2),(3),(4),(\infty)\}\end{array}\right\}.$$

The vertex and edge sets of the projective path graph that is obtained from the projective plane of order 5 are as follows:

$$V(G)=\left\{\begin{array}{c}00,01,02,03,04,10,11,12,13,\\ 14,20,21,22,23,24,30,31,32,33,34,\\ 40,41,42,43,44,(0),(1),(2),(3),(4),(\infty)\end{array}\right\},$$

$$E(G)=\left\{\begin{array}{c}
\{00,11\},\{11,22\},\{22,33\},\{33,44\},\{44,(1)\},\{01,12\},\{12,23\},\\
\{23,34\},\{34,40\},\{40,(1)\},\{02,13\},\{13,24\},\{24,30\},\{30,41\},\\
\{41,(1)\},\{03,14\},\{14,20\},\{20,31\},\{31,42\},\{42,(1)\},\{04,10\},\\
\{10,21\},\{21,32\},\{32,43\},\{43,(1)\},\{00,12\},\{12,24\},\{24,31\},\\
\{31,43\},\{43,(2)\},\{01,13\},\{13,20\},\{20,32\},\{32,44\},\{44,(2)\},\\
\{02,14\},\{14,21\},\{21,33\},\{33,40\},\{40,(2)\},\{03,10\},\{10,22\},\\
\{22,34\},\{34,41\},\{41,(2)\},\{04,11\},\{11,23\},\{23,30\},\{30,42\},\\
\{42,(2)\},\{00,13\},\{13,21\},\{21,34\},\{34,42\},\{42,(3)\},\{01,14\},\\
\{14,22\},\{22,30\},\{30,43\},\{43,(3)\},\{02,10\},\{10,23\},\{23,31\},\\
\{31,44\},\{44,(3)\},\{03,11\},\{11,24\},\{24,32\},\{32,40\},\{40,(3)\},\\
\{04,12\},\{12,20\},\{20,33\},\{33,41\},\{41,(3)\},\{00,14\},\{14,23\},\\
\{23,32\},\{32,41\},\{41,(4)\},\{01,10\},\{10,24\},\{24,33\},\{33,42\},\\
\{42,(4)\},\{02,11\},\{11,20\},\{20,34\},\{34,43\},\{43,(4)\},\{03,12\},\\
\{12,21\},\{21,30\},\{30,44\},\{44,(4)\},\{04,13\},\{13,22\},\{22,31\},\\
\{31,40\},\{40,(4)\},\{00,10\},\{10,20\},\{20,30\},\{30,40\},\{40,(0)\},\\
\{01,11\},\{11,21\},\{21,31\},\{31,41\},\{41,(0)\},\{02,12\},\{12,22\},\\
\{22,32\},\{32,42\},\{42,(0)\},\{03,13\},\{13,23\},\{23,33\},\{33,43\},\\
\{43,(0)\},\{04,14\},\{14,24\},\{24,34\},\{34,44\},\{44,(0)\},\{00,01\},\\
\{01,02\},\{02,03\},\{03,04\},\{04,(\infty)\},\{10,11\},\{11,12\},\\
\{12,13\},\{13,14\},\{14,(\infty)\},\{20,21\},\{21,22\},\{22,23\},\{23,24\},\\
\{24,(\infty)\},\{30,31\},\{31,32\},\{32,33\},\{33,34\},\{34,(\infty)\},\{40,41\},\\
\{41,42\},\{42,43\},\{43,44\},\{44,(\infty)\},\{(0),(1)\},\{(1),(2)\},\{(2),(3)\},\\
\{(3),(4)\},\{(4),(\infty)\}
\end{array}\right\}.$$

Here $|E(G)| = 14$. Now we determine the vertex degree of each vertex: $d(00) = 3$, $d(01) = 4$, $d(10) = 5$, $d(11) = 6$, $d((0)) = 3$, $d((1)) = 4$, $d((\infty)) = 3$. As a result we get the following degree sequence as $D.S. = \{6^{(1)}, 5^{(1)}, 4^{(2)}, 3^{(3)}\}$.

Theorem 3.5. *The general formulas of the vertex and edge partitions of the projective path graphs that are obtained from the projective plane of order k, are given as follows:*

Vertex Partition	d_i	$\lvert d_i\rvert$
1.	$k+1$	3
2.	$k+2$	$2(k-1)$
3.	$2k+1$	$k-1$
4.	$2k+2$	$(k-1)^2$

Edge Partition	(d_i, d_j)	$\lvert(d_i, d_j)\rvert$
1.	$(k+1, k+2)$	4
2.	$(k+1, 2k+1)$	2
3.	$(k+1, 2k+2)$	$3(k-1)$
4.	$(k+2, k+2)$	$2(k-2)$
5.	$(k+2, 2k+1)$	$2(k-1)$
6.	$(k+2, 2k+2)$	$2(k-1)^2$
7.	$(2k+1, 2k+1)$	$k-2$
8.	$(2k+1, 2k+2)$	$(k-1)(2k-3)$
9.	$(2k+2, 2k+2)$	$k(k-1)(k-2)$

Now we give two corollaries for the characterization of these graphs of order k.

Corollary 3.6. *Projective path graphs of order k consists of k^2+k+1 vertices and $(k-1)\left(k^2+k+1\right)$ edges. The smallest and biggest vertex degrees of projective path graphs that are obtained from projective planes of order $k \geq 3$ are $\delta = k+1$ and $\Delta = 2k+2$ respectively.*

Corollary 3.7. *Projective path graphs of order k has the degree sequences in the following form:*

$$D.S. = \left\{(2k+2)^{(k-1)^2}, (2k+1)^{(k-1)}, (k+2)^{2(k-1)}, (k+1)^3\right\}.$$

Now we calculate the general formulas of additive and multiplicative Zagreb indices and also Narumi-Katayama index for the projective path graphs in terms of order k.

Theorem 3.8. *Let G be a projective path graph of order k, then for* $k \geq 2$, *additive and multiplicative Zagreb indices and Narumi-Katayama index for these graphs can be calculated as follows:*

$$M_1(G) = 4k^4 + 6k^3 + k^2 + 3k - 2,$$
$$M_2(G) = 4k^5 + 8k^4 - 2k^2 + 3k - 4,$$
$$M_3(G) = 8k^5 + 18k^4 + k^3 - k^2 + 4k - 6,$$
$$\Pi_1(G) = (k+1)^6 (k+2)^{4(k-1)} (2k+1)^{2(k-1)} (2k+2)^{2(k-1)^2},$$
$$\Pi_2(G) = (k+1)^{3k+3} (k+2)^{2k^2+2k-4} (2k+1)^{2k^2-k-1} (2k+2)^{2(k^3-k^2-k+1)},$$
$$NK(G) = (k+1)^3 (k+2)^{2(k-1)} (2k+1)^{(k-1)} (2k+2)^{(k-1)^2}.$$

4. Vertex-adjacency matrices and energy of the projective path graphs

In this section, we calculate the vertex-adjacency matrices and energies related to projective path graphs that are obtained from the projective planes of order $k = 2, 3, 4, 5$.

Example 4.1. The Adjacency matrix for the projective path graph of order 2 is given below:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is as follows:

$$p(x) = -(x-1)(x+1)^2(x+2)\left(x^3 - 3x^2 - 6x + 4\right).$$

Thus the spectrum of the projective path graph of order 2 is

$$\left\{1, (-1)^2, -2, -1.7466, 0.54510, 4.2015\right\}$$

and the energy of the graph is $E_G = \sum\limits_{i=1}^{n} |\lambda_i| \cong 11.4932$.

The adjacency matrix for the projective graph of order 3 is given below:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is as follows:

$$p(x) = -x^3\left(x^2-2x-4\right)\left(x^2+5x+5\right)\left(x^6-3x^5-21x^4-12x^3+36x^2+38x+8\right).$$

The spectrum of the projective path graph of order 3 is

$$\left\{\begin{array}{c} 6.40885, 0^3, -3.61803, 3.23607, -2.15345, -153128, \\ 1.44742, -1.38197, -1.23607, -0.871413, -0.300126 \end{array}\right\}$$

and the energy of the graph is $E_G = \sum_{i=1}^{n} |\lambda_i| \cong 22.184679.$
The adjacency matrix for the projective graph of order 4 is given below:

$$\begin{bmatrix}
0&1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&1&1&1&0&1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&1&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&1&0&1&1&1&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&1&1&1&0&1&0&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&1&1&1&1&1&0&1&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&1&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&1&0&1&1&1&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&1&1&0&1&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&1&0&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&1&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&1&0&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0
\end{bmatrix}.$$

The characteristic polynomial of this matrix is as follows:

$$\begin{aligned} p(x) &= -x\left(x^2-5x-3\right)\left(x^2+x+4\right)^4\left(x^2+3x+1\right) \\ &\quad \left(x^8-2x^7-57x^6-38x^5+336x^4+286x^3-423x^2-486x-121\right). \end{aligned}$$

Thus the spectrum of the projective path graph of order 4 is

$$\left\{\begin{array}{c} 8.58306, 5.54138, -5.47997, -2.61803, -2.38072, 2.00787, \\ (-1.61803)^4, 1.5744, -1.24409, (0.618034)^4, -0.610478, -0.541381, \\ -0.450074, -0.381966, 0 \end{array}\right\}$$

and the energy of the graph is $E_G = \sum_{i=1}^{n} |\lambda_i| \cong 40.357675.$
The adjacency matrix for the projective graph of order 5 is given below:

$$
\begin{pmatrix}
0&1&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&1&1&1&1&0&1&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&0&1&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&0&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&1&1&1&1&0&1&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&0&0&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&1&0&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&1&0&1&1&1&1&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&0&0&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&1&0&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&1&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&1&0&1\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0
\end{pmatrix}
$$

Figure 1. The adjacency matrix for the projective graph of order 5

The characteristic polynomial of matrix of projective path graph of order 5

$$
\begin{aligned}
p(x) &= -(x-1)^4 (x+1)^4 \left(x^2+x-1\right) \\
&\quad \begin{pmatrix} x^{21} - x^{20} - 148x^{19} - 253x^{18} + 6161x^{17} + 24950x^{16} \\ -39804x^{15} - 384504x^{14} - 558907x^{13} \\ +843362x^{12} + 3333087x^{11} + 308 \end{pmatrix}
\end{aligned}
$$

and the spectrum of the projective path graph of order 5 is

$$
\left\{ \begin{array}{c} 10.727, 7.86463, -7.44356, -4.69611, 3.90785, -2.54576, 2.27951, \\ -1.76482, -1.72531, -1.72182, -1.70787, \\ -1.70017, -1.56748, (-1)^5, (1)^5, \\ -0.782614, 0.581955, -0.486418, 0.322221, 0.155832, \\ 0.141744, 0.0920858, 0.0690535 \end{array} \right\}
$$

and the energy of the graph is $E_G = \sum\limits_{i=1}^{n} |\lambda_i| \cong 62.2838133$.

5. Conclusion

With this study; we give vertex and edge partitition of projective path graphs of order k so it is easily seen that there are some differences and similarities between the projective path graphs of order k and the projective planes of order k. Additionaly, we calculate the general formulas of additive and multiplicative Zagreb indices and also Narumi-Katayama index for the projective path graphs in terms of order k. Further, we can calculate all topological graph indices that are defined in terms of vertex degrees for projective path graphs. There are also some open problems in this subject. Also we have some results according to our new studies and previous papers. Now we are studying on the following problem: "How can we determine the topological indices in terms of distances for the projective path graphs?"

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