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Goldie ss-supplemented modules

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Abstract

In this study, it has been determined the notion of Goldie ss-supplemented modules by the help of the relation β^*_{ss}, which is defined in the form $W\beta^*_{ss}Z$, which provides conditions both of $\frac{W+Z}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{W}$ and $\frac{W+Z}{Z} \subseteq \frac{\mathrm{Soc}_s(A)+Z}{Z}$ for submodules W and Z of module A is an equivalence relation. The main features of Goldie ss-supplemented modules provided by this relation is examined. It is shown that the epimorphism $\gamma : A \longrightarrow B$ provided the relation β^*_{ss} under certain conditions and the relation β^*_{ss}, is expressed in the maximal submodules. In addition, we obtain notions of Goldie ss-lifting modules using the relation β^*_{ss} and we prove several properties of notions of these modules.

Keywords: Socle of a module, semisimple module, ss-supplement, relation β^*_{ss}

2010 MSC: 16D10, 16D60, 16D99

1. Introduction

Unless otherwise specified in this study S denotes an associative ring having an identity element and all modules are unitary left S- modules. We show the submodule representation as $B \leq A$, where B is a submodule of A. It is denoted by $Rad(A)$ the Jacobson radical of A. A submodule C of a module A is said to be *small* in A (denoted by $C \ll A$) if $B + C \neq A$ for each proper submodule B of A. A is said to be *lifting* if for every submodule B of the module A there is a direct summand C of A for $C \subseteq B$ and $\frac{B}{C} \ll \frac{A}{C}$ (*cf.* [2]). A is said to be *(ss-)supplemented* if for each submodule B of the module A, there is a submodule C of A provided that $A = B + C$, $B \cap C \ll C$ (and $B \cap C$ is semisimple) (in this case C is ss-supplement of B in A) (*cf.* [5]). A is said to be *ss-semilocal* if for each submodule B of A, there is a submodule F of A provided that $A = B + F$, $B \cap F \ll A$ and $B \cap F$ is semisimple (*cf.* [7]). H-supplemented modules were introduced in [1] as a generalization of lifting modules. According to [1] A is said to be *H-supplemented* if for each submodule G of the module A there is a direct summand D of A provided that $G + W = A$ if and only if $D + W = A$ for every submodule W of A. In [1], it is proved that A is H-supplemented if and only if for every submodule G of A there exists a direct summand D of A provided that $\frac{G+D}{D} \ll \frac{A}{D}$ and $\frac{G+D}{G} \ll \frac{A}{G}$. A is said to *have (P^*) property* (or *be a (P^*)-module*) if for each submodule B of A there is a decomposition $A = C \bigoplus C'$ provided that $C \subseteq B$ and $B \cap C' \subseteq Rad(C')$.

Let A be an S-module and $S(A) = \{W | W \leq A\}$. Then $W, Z \in S(A)$ over the set $S(A)$ under relation β^* is defined as follows.

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$$W\beta^{*}Z \Leftrightarrow \frac{W+Z}{Z} \ll \frac{A}{Z} \text{ and } \frac{W+Z}{W} \ll \frac{A}{W}.$$

Let A be an S-module. A is said to be a *Goldie*-lifting module* (respectively, Goldie*-supplemented) if for each submodule B of A, there is a direct summand D of A (respectively, a supplement D in A) provided that $B\beta^{*}D$.
In [11], Zhou and Zhang generalized the notion of socle of a module B to that of $\mathrm{Soc}_s(A)$ by considering the class of whole simple submodules of A that are small in A in place of the class of all simple submodules of A, that is,

$$\mathrm{Soc}_s(A) = \sum \{B \ll A \mid \text{B is a simple submodule of } A\}.$$

The section 2 is devoted to introducing the relation β^{*}_{ss}. We study some features of this relation and show that this relation is an equivalence relation. In section 3 we define Goldie-ss-supplemented and Goldie-ss-lifting modules. Using the study in [1] and based on the relation β^{*}_{ss}, we call A a Goldie-ss-lifting module (respectively, Goldie-ss-supplemented) if for any submodule B of A, there exists a direct summand D of A (respectively, ss-supplement D in A) so that $B\beta^{*}_{\mathrm{ss}}D$. Clearly every Goldie-$ss$-lifting module is Goldie-ss-supplemented. Let $A = E \bigoplus H$ be a distributive module. Then A is Goldie-ss-supplemented (Goldie-ss-lifting) if and only if E and H are Goldie-ss-supplemented (Goldie-ss-lifting) (Theorem 3.5). Also we prove that the factor module of a Goldie-ss-lifting modules will be Goldie-ss-lifting modules by putting some special terms. Then we give the relations between Goldie-ss-supplemented and Goldie-ss-lifting with other types of supplemented modules.

2. The relation β^{*}_{ss}

In this section, we will define the relation β^{*}_{ss} and show that it is an equivalence relation. In order to classify the equivalence classes obtained according to this defined equivalence relation, Goldie-ss-supplemented modules will be defined in the next section.

Definition 2.1. Let W, Z be submodules of a module M. We define W and Z are equivalent by the relation β^{*}_{ss} , $W\beta^{*}_{\mathrm{ss}}Z$, if and only if $\frac{W+Z}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{W}$ and $\frac{W+Z}{Z} \subseteq \frac{\mathrm{Soc}_s(A)+Z}{Z}$.

Lemma 2.2. *The above relation β^{*}_{ss} is an equivalence relation.*

Proof. β^{*}_{ss} is reflexive: For every $W \leq A$ then, $\frac{W}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{W}$. So $W\beta^{*}_{\mathrm{ss}}W$.

β^{*}_{ss} is Symmetric: Let's take the elements $W\beta^{*}_{\mathrm{ss}}Z$ then, $\frac{W+Z}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{W}$ and $\frac{W+Z}{Z} \subseteq \frac{\mathrm{Soc}_s(A)+Z}{Z}$ are held. It follows that $W + Z \subseteq \mathrm{Soc}_s(A) + W$ and $W + Z \subseteq \mathrm{Soc}_s(A) + Z$. Here $\frac{Z+W}{Z} \subseteq \frac{\mathrm{Soc}_s(A)+Z}{Z}$ and $\frac{Z+W}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{W}$. Hence $Z\beta^{*}_{\mathrm{ss}}W$ is obtained.

β^{*}_{ss} is transitive: Take $W\beta^{*}_{\mathrm{ss}}Z$ and $Z\beta^{*}_{\mathrm{ss}}X$. Then we get

$$\frac{W+Z}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{\mathrm{W}}$$

and

$$\frac{W+Z}{Z} \subseteq \frac{\mathrm{Soc}_s(A)+Z}{Z},$$

$$\frac{Z+X}{Z} \subseteq \frac{\mathrm{Soc}_s(A)+Z}{Z}$$

and

$$\frac{Z+X}{X} \subseteq \frac{\mathrm{Soc}_s(A)+X}{X}.$$

It follows from

$$W + Z \subseteq \mathrm{Soc}_s(A) + W$$

and

$$W + Z \subseteq \mathrm{Soc}_s(A) + Z,$$

$$Z + X \subseteq \mathrm{Soc}_s(A) + Z$$

and

$$Z + X \subseteq \mathrm{Soc}_s(A) + X \tag{2.1}$$

can be found,

$$W + X \subseteq \mathrm{Soc}_s(A) + W$$

and

$$W + X \subseteq \mathrm{Soc}_s(A) + X.$$

Thus $W\beta^*_{ss}X$. □

Proposition 2.3. *Let $\gamma : A \longrightarrow B$ be an epimorphism. The following statements satisfy:*

(1) *If $W, Z \leq A$ and $W\beta^*_{ss}Z$, then $\gamma(W)\beta^*_{ss}\gamma(Z)$.*

(2) *If $W, Z \leq B$ and $W\beta^*_{ss}Z$, then $\gamma^{-1}(W)\beta^*_{ss}\gamma^{-1}(Z)$.*

(3) *If $W \leq A$, $W \subseteq Soc_s(A)$, $C \leq B$ and $\gamma(W)\beta^*_{ss}C$, then $W\beta^*_{ss}\gamma^{-1}(C)$.*

Proof. (1) Suppose that $W\beta^*_{ss}Z$ for $W, Z \leq A$. Then we have $W + Z \subseteq \mathrm{Soc}_s(A) + W$ and $W + Z \subseteq \mathrm{Soc}_s(A) + Z$. So $\gamma(W) + \gamma(Z) \subseteq \mathrm{Soc}_s(B) + \gamma(W)$ and $\gamma(W) + \gamma(Z) \subseteq \mathrm{Soc}_s(B) + \gamma(Z)$ implies that $\gamma(W)\beta^*_{ss}\gamma(Z)$.

(2) Let $W\beta^*_{ss}Z$ for $W, Z \leq B$. Then $W + Z \subseteq \mathrm{Soc}_s(B) + W$ and $W + Z \subseteq \mathrm{Soc}_s(B) + Z$. Since γ is an epimorphism $\gamma^{-1}(W) + \gamma^{-1}(Z) \subseteq \mathrm{Soc}_s(A) + W$ and $\gamma^{-1}(W) + \gamma^{-1}(Z) \subseteq \mathrm{Soc}_s(A) + Z$, it follows that $\gamma^{-1}(W)\beta^*_{ss}\gamma^{-1}(Z)$.

(3) Suppose that $\gamma(W)\beta^*_{ss}C \subseteq \mathrm{Soc}_s(B)$, $C \leq B$. We have $\gamma(W)+C \subseteq \mathrm{Soc}_s(B)+\gamma(W)$ and $\gamma(W)+C \subseteq \mathrm{Soc}_s(B)+C$. It follows that $\gamma^{-1}(C) + W \subseteq \mathrm{Soc}_s(A) + \gamma^{-1}(C)$ and $\gamma^{-1}(C) + W \subseteq \mathrm{Soc}_s(A) + W$, because γ is an epimorphism and $W \subseteq \mathrm{Soc}_s(A)$. Therefore, $W\beta^*_{ss}\gamma^{-1}(C)$. □

Proposition 2.4. *For submodules W, $J_1 \leq A$, $Soc_s(A) \subseteq J_2 < A$ such that, $J_1 + J_2 = A$ and $W\beta^*_{ss}J_1$, then $W \nsubseteq J_2$.*

Proof. Let's admit that $W \subseteq J_2$. Since $\mathrm{Soc}_s(A) \leq J_2$, we have $W + J_2 = A$. By assumption, $J_2 = A$ contradiction then $W \nsubseteq J_2$. □

Proposition 2.5. *Let $W \leq C$ where C is a maximal submodule of A. If $W\beta^*_{ss}Z$ then $Z \subseteq C$.*

Proof. Let's admit that $Z \nsubseteq C$. Then $Z + C = A$. As $W\beta^*_{ss}Z$ and $\mathrm{Soc}_s(A) \subseteq C$, we get $C + W = A$. But $W \subseteq C$ implies that $C = A$, a contradiction. □

Proposition 2.6. *Let $W_1, W_2, Z_1, Z_2 \leq A$ so that $W_1\beta^*_{ss}Z_1$ and $W_2\beta^*_{ss}Z_2$. Then*

$$(W_1 + W_2)\beta^*_{ss}(Z_1 + Z_2)$$

and

$$(W_1 + Z_2)\beta^*_{ss}(Z_1 + W_2).$$

Proof. Let's admit that $W_1\beta^*_{ss}Z_1$ and $W_2\beta^*_{ss}Z_2$. Then

$$W_1 + Z_1 \subseteq \mathrm{Soc}_s(A) + W_1$$

and

$$W_1 + Z_1 \subseteq \mathrm{Soc}_s(A) + Z_1,$$
$$W_2 + Z_2 \subseteq \mathrm{Soc}_s(A) + W_2$$

and

$$W_2 + Z_2 \subseteq \mathrm{Soc}_s(A) + Z_2.$$

By using these inclusions, it can be easily seen that

$$(W_1 + W_2)\beta^*_{ss}(Z_1 + Z_2)$$

and

$$(W_1 + Z_2)\beta^*_{ss}(Z_1 + W_2).$$

□

Corollary 2.7. *Let $W, Z \leq A$ and $C \subseteq Soc_s(A)$. Then $W\beta^*_{ss}Z$ if and only if $W\beta^*_{ss}(Z + C)$.*

Proof. (⇒) This implication follows from Proposition 2.6. and the fact that $0\beta^*_{\text{ss}}C$.
(⇐) Since $C \subseteq \text{Soc}_s(A)$, we have $Z\beta^*_{\text{ss}}(Z + C)$. Now this means that transitivity of the relation β^*_{ss}. □

Corollary 2.8. *Let $W, Z_1, \ldots, Z_n \leq A$. If $W\beta^*_{ss}Z_i$ for $i = 1, \ldots, n$. Then $W\beta^*_{ss} \sum_{i=1}^{n} Z_i$.*

Corollary 2.9. *Let A be an S-module and $W, Z \leq A$. If $W\beta^*_{ss}Z$, for each submodule E of A so that $W + Z + E = A$ then $W + E = A$ and $X + E = A$.*

Proof. Let $W\beta^*_{ss}Z$. Let's take any submodule E of A with $W+Z+E = A$. Then we have $W+Z+E = A$, $\frac{W+Z}{Z} + \frac{E+Z}{Z} = \frac{A}{Z}$. Since $\frac{W+Z}{Z} \ll \frac{A}{Z}$, we get $Z + E = A$ by the hypothesis. Similarly, it can be shown that $W + E = A$. □

Corollary 2.10. *If $W, Z \leq A$ submodules which are semisimple and small in A and they are semisimple. Then $W\beta^*_{ss}Z$.*

Proof. We shall prove that $\frac{W+Z}{Z} \subseteq Soc_s\left(\frac{A}{Z}\right)$, $\frac{W+Z}{Z} \subseteq Soc_s\left(\frac{A}{W}\right)$. By [1, Theorem 2.3] $\frac{W+Z}{Z} \ll \frac{A}{Z}$ and $\frac{W+Z}{W} \ll \frac{A}{W}$. $\eta_1 : A \to \frac{A}{W}$ and $\eta_2 : A \to \frac{A}{Z}$ be natural homomorphisms. Since $\eta_1(W) = \frac{W+Z}{W}$ and $\eta_2(Z) = \frac{W+Z}{Z}$ is a homomorphic images of semisimple submodules $W, Z \leq A$, then $\frac{W+Z}{W}$ and $\frac{W+Z}{Z}$ are semisimple modules by [4, 8.1.5], as desired. □

3. Goldie-*ss*-supplemented modules

In this section, Goldie-*ss*-supplemented modules, which are defined as an intensification of Goldie*-supplemented modules defined in [1], will be studied and Goldie-*ss*-lifting modules, which are an intensification of Goldie-*ss*-supplemented modules, will be defined and their algebraic properties will be presented.

Definition 3.1. Let A be a module;

(1) We say A *Goldie-ss-supplemented* if for each submodule B of A, there is an *ss*-supplement submodule T in A provided that $B\beta^*_{\text{ss}}T$.

(2) We say A a *Goldie-ss-lifting module* if for each submodule B of A, there is a direct summand D of A provided that $B\beta^*_{\text{ss}}D$.

Proposition 3.2. *Let A be a Goldie-ss-supplemented module. Then it is necessary and sufficient there is a ss-supplement submodule T of A provided that $T + Soc_s(A) = W + Soc_s(A)$ for each $W \leq A$.*

Proof. (⇒) Let A be a Goldie-*ss*-supplemented module and $W \leq A$. Then, there is an *ss*-supplement submodule T of A so that $W + T \subseteq \text{Soc}_s(A) + W$ and $W + T \subseteq \text{Soc}_s(A) + T$. Then $T+ \text{Soc}_s(A) \subseteq X + \text{Soc}_s(A)$ and $W + \text{Soc}_s(A) \subseteq T + \text{Soc}_s(A)$. It follows that $T + \text{Soc}_s(A) = W + \text{Soc}_s(A)$.
(⇐) It is obvious. □

Proposition 3.3. *Let A be a module where is a ss-supplement submodule T of A and a $Q \subseteq Soc_s(A)$, provided that $W = T + Q$ for every $W \leq A$. Then A is Goldie-ss-supplemented.*

Proof. We want to show that $W\beta^*_{\text{ss}}T$. By using the following two inclusions

$$W + T = T + Q \subseteq \text{Soc}_s(A) + T + Q = \text{Soc}_s(A) + W$$

and

$$W + T = T + Q + T \subseteq \text{Soc}_s(A) + T,$$

we get

$$\frac{W + T}{W} \subseteq \frac{\text{Soc}_s(A) + X}{W}$$

and

$$\frac{W + T}{T} \subseteq \frac{\text{Soc}_s(A) + T}{T},$$

as required. □

Proposition 3.4. *Let A be a Goldie-ss-supplemented module. Then there is an equality $W = T + Q$ is provided where T is ss-supplement in A and $Q \subseteq Soc_s(A)$ with $Soc_s(A) \subseteq W$ for each $W \leq A$.*

Proof. Let $W \leq A$ with $\mathrm{Soc}_s(A) \subseteq W$. According to acceptance there is a ss-supplement submodule T of A so that $W\beta^*_{ss}T$. Then $T \subseteq W$ and $W = \mathrm{Soc}_s(A) + (T \cap W) = \mathrm{Soc}_s(A) + T$. This indicates that it is completed the proof. □

Recall that a module A is *distributive* if the intersection is distributed over the sum for its submodules of the module, equivalently $B \cap (C + F) = (B \cap C) + (B \cap F)$ for submodules $C; F; N$ of B.

Theorem 3.5. *Let $A = E \oplus H$ be a module with A is a distributive module. Then A is a Goldie-ss-supplemented (Goldie-ss-lifting) module if and only if both of E and H are a Goldie-ss-supplemented (Goldie-ss-lifting) module.*

Proof. (⇒) Let $W \leq E$. Then there are submodules T, F of A provided that $T + F = A$ and $T \cap F \subseteq \mathrm{Soc}_s(A)$ and $W\beta^*_{ss}T$. We must show that $W\beta^*_{ss}(E \cap T)$. As $W\beta^*_{ss}T$, we have $W + T \subseteq \mathrm{Soc}_s(A) + W$ and $W + T \subseteq \mathrm{Soc}_s(A) + T$. It follows that $W+(E\cap T) \subseteq \mathrm{Soc}_s(E)+W$ and $W+(E\cap T) \subseteq (Soc_s(E)+(E\cap T)+(H\cap T)+(Soc_s(H)) \cap E$ because $W \subseteq E$. By modularity, $W + (E \cap T) \subseteq \mathrm{Soc}_s(E) + W$ and $W + (E \cap T) \subseteq \mathrm{Soc}_s(E) + (E \cap T)$. Thus $W\beta^*_{ss}(E \cap T)$. for $C, F, B \leq A$, according to acceptance $(E \cap T) + (E \cap F) = E$ and $(E \cap T) \cap (E \cap F) = E \cap T \cap F \subseteq \mathrm{Soc}_s(E \cap T) \oplus \mathrm{Soc}_s(H \cap T)$. This implies that $E \cap T \cap F \subseteq \mathrm{Soc}_s(E \cap T)$. So $(E \cap T)$ is an ss-supplement of $(E \cap F)$ in E. And so, E is Goldie-ss-supplemented. Similarly, H is Goldie-ss-supplemented.

(⇐) Let $L \leq A$, $L_1 = E \cap L$ and $L_2 = H \cap L$. There is $F_1, T_1 \leq E$ provided that $F_1\beta^*_{ss}T_1$, $F_1 + T_1 = E$ and $F_1 \cap T_1 \subseteq \mathrm{Soc}_s(T_1)$. Then we get $F_2, T_2 \leq H$ with $L_2\beta^*_{ss}T_2$, $F_2 + T_2 = H$ and $F_2 \cap T_2 \subseteq \mathrm{Soc}_s(T_2)$. By Proposition 2.6, $L\beta^*_{ss}(T_1 + T_2)$. In addition, $T_1 + T_2 + F_1 + F_2 = A$ and $(T_1 + T_2) \cap (F_1 + F_2) = (T_1 \cap F_1) + (T_2 \cap F_2) \subseteq \mathrm{Soc}_s(T_1) + \mathrm{Soc}_s(T_2) \subseteq \mathrm{Soc}_s(T_1 + T_2)$. This means that, $T_1 + T_2$ is an ss-supplement submodule in A. Thus A is Goldie-ss-supplemented. The proof for E and H being Goldie-ss-lifting module is similar. □

Example 3.6. Let's get the ring $S = R[[a\ ,\ b]]$ of formal power series over a field R with indeterminates a and b. Then S is a commutative noetherian local domain with maximal ideal $J = Sa + Sb$. It follows that the ring S is semiperfect. Since S is a domain, J is a uniform S-module. Then we get J is indecomposable. Now consider that J is a Goldie-ss-lifting module with $B \subsetneq J$ provided that that $B \nsubseteq \mathrm{Soc}_s\left({}_SJ\right)$. Then $B\beta^*_{ss}0$ or $N\beta^*_{ss}J$. So $B \subseteq \mathrm{Soc}_s\left({}_SJ\right)$ or $N = J$. Consequently, J is not a Goldie-ss-lifting module.

Proposition 3.7. *Let A_0 be a direct summand of a module A for each decomposition $A = B \oplus C$ of A' and B' of B and C' of C be submodules provided that $A = A_0 \oplus B' \oplus C'$. If A is a Goldie-ss-lifting module, then $\frac{A}{A_0}$ is a Goldie-ss-lifting module.*

Proof. Let $\frac{W}{A_0} \leq \frac{A}{A_0}$. As A is a Goldie-ss-lifting module, there is a decomposition $A = B \bigoplus C$ provided that $W\beta^*_{ss}B$. Then $\frac{W+B}{B} \subseteq \frac{\mathrm{Soc}_s(A)+B}{B}$ and $\frac{W+B}{W} \subseteq \frac{\mathrm{Soc}_s(A)+W}{W}$. By hypothesis, $A = A_0 \oplus B' \oplus C'$ for $B' \leq B$ and $C' \leq C$. It's obvious to see that $\frac{A_0 \oplus B'}{A_0}$ is a Goldie-ss-lifting module of $\frac{W}{A_0}$ in $\frac{A}{A_0}$. □

Proposition 3.8. *Let A be a module. Then the following conditions are equivalent:*

(1) A is a Goldie-ss-lifting module;

(2) A is ss-semilocal and each direct summand of $\frac{A}{Soc_s(A)}$ lifts to a direct summand of A.

Proof. (1) ⇒ (2) It is clear that every Goldie-ss-lifting module is ss-semilocal. Now we're just proving the last one statement. Let $\frac{B}{\mathrm{Soc}_s(A)} \leq \frac{A}{\mathrm{Soc}_s(A)}$. Since A is a Goldie-ss-lifting module, there is a direct summand $D \leq A$ provided that $B\beta^*_{ss}D$, i.e. $\frac{B+D}{B} \subseteq \frac{\mathrm{Soc}_s(A)+B}{B}$ and $\frac{B+D}{D} \subseteq \frac{\mathrm{Soc}_s(A)+D}{D}$. So $D \subseteq B$. It follows from $\frac{B}{\mathrm{Soc}_s(A)} = \frac{D+\mathrm{Soc}_s(A)}{\mathrm{Soc}_s(A)}$ that $\frac{B}{\mathrm{Soc}_s(A)}$ lifts to D.

(2) ⇒ (1) Let $B \leq A$. Then according to acceptance, $\frac{B+\mathrm{Soc}_s(A)}{\mathrm{Soc}_s(A)} = \overline{B}$ is a direct summand of $\frac{A}{\mathrm{Soc}_s(A)} = \overline{A}$. Therefore by (2), $\overline{B} = \overline{F}$ such that $A = F \oplus C$. The remaining part of the proof is seen by taking F is a Goldie-ss-lifting module of B in A. □

Proposition 3.9. *Every amply ss-supplemented module is Goldie-ss-supplemented.*

Proof. Let A be amply ss-supplemented and $W \leq A$. Let $W \subseteq \mathrm{Soc}_s(A)$. it is clear that condition $W\beta^*_{ss}0$ has been achieved. So assume that $W \nsubseteq \mathrm{Soc}_s(A)$. As A is ss-semilocal, there is a submodule F of A provided that $W + F = A$ and $W \cap F \subseteq \mathrm{Soc}_s(A)$. According to acceptance, there is an ss-supplement T of F in A containing in W. So $A = T + F$ and $T \cap F \subseteq \mathrm{Soc}_s(A)$. We get $W = T + (F \cap W) \subseteq \mathrm{Soc}_s(A) + T$ as $T \subseteq W$. It follows that $W\beta^*_{ss}T$. Therefore, A is Goldie-ss-supplemented. □

Since π-projective ss-supplemented modules are amply ss-supplemented, we have the following corollary by using Proposition 3.9.

Corollary 3.10. *Every π-projective ss-supplemented module is Goldie-ss-supplemented.*

4. Conclusion

This article proposes a different equivalence relation in mathematics. The article has two main sections. In the first part, the classification of the defined equivalence relation with submodules is focused on. In the second part, the definition of two new classes of modules is provided by associating this equivalence relation with set-theoretic formulations with module structures. The article has been prepared to present the standard explanation of the mathematical concept of equivalence, which covers the concepts of equivalence relation and equivalence class, in module theory in a way that enriches it.

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