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Truncation error upper bounds in derivative Whittaker–type plane sampling reconstruction

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Abstract

A survey is presented on the author's mean square and almost sure Whittaker–type derivative sampling theorems obtained for the class $L^\alpha(\Omega, \mathfrak{F}, \mathsf{P})$; $0 \le \alpha \le 2$ of stochastic processes having spectral representation, with the aid of the Weierstraß σ function. Processes of this class are represented by interpolation series. The results are valid for harmonizable and weakly stationary (or in the Hinčin sense stationary) processes ($\alpha = 2$) as well. The formulæ are interpreted in the α–mean and also almost sure sense when the input function and its derivatives are sampled at the points of the integer lattice $\mathbb{Z}^2$. The circular truncation error is introduced and used in the truncation error analysis and related sampling sum convergence rates are shown.

Keywords: Circular truncation error, derivative sampling, Leont'ev spaces of entire functions, Piranashvili–type stochastic processes, truncation error upper bounds, Weierstraß sigma–function, Whittaker–type sampling, (p, q)–order weighted differential operator

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1. Introduction

Consider the Weierstraß σ-function

$$\sigma(z) = z \prod_{(m,n)\in\mathbb{Z}^2}{}' \left(1 - \frac{z}{m+\mathrm{i}n}\right) \exp\left(\frac{z}{m+n\mathrm{i}} + \frac{z^2}{2(m+n\mathrm{i})^2}\right), \qquad z \in \mathbb{C},$$

where the dashed product means that the zero term is omitted. Then the following growth result can be exposed for the Weierstraß σ function.

Theorem 1.1 (*cf.* [14, Theorem 1, p. 158])**.** *Let* $\delta_z := \inf|z - \mathbb{Z}^2|$. *Then for* $z \in \mathbb{C}$ *we have the bilateral bounding inequality*

$$\delta_z\, K_1(\delta_z) \le |\sigma(z)| \exp\left\{-\frac{\pi}{2}|z|^2\right\} \le \delta_z\, K_2(\delta_z), \tag{1.1}$$

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where

$$K_1(\delta_z) = \left(1 - \frac{\pi^4\delta_z^4}{90}\right)\left(1 - \frac{\pi^2\delta_z^4}{6}\Big(\boldsymbol{G} - \frac{\pi^2}{15}\Big)\right)^2 e^{-\frac{\pi}{2}\delta_z^2},$$
$$K_2(\delta_z) = \exp\left\{\frac{\pi\delta_z^2}{2}\Big(\frac{\pi\boldsymbol{G}}{3}\,\delta_z^2 - 1\Big)\right\},$$

and $\boldsymbol{G}$ *stands for the Catalan constant.*

Remark 1.2. The Catalan's constant is nothing else than the Dirichlet's beta function's value $\beta(2)$, i.e.

$$\boldsymbol{G} = \beta(2) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2}\,.$$

Its numerical value is approximately $\boldsymbol{G} \approx 0.915966$.

The functions $K_j(\delta_z)$, $j = 1, 2$ monotone decrease on $\big[0, 1/\sqrt{2}\big]$, so

$$\begin{aligned}
\min K_1(\delta_z) &= K_1(1/\sqrt{2}) = \Big(1 - \tfrac{\pi^4}{360}\Big)\Big(1 - \tfrac{\pi^2}{24}\big(\boldsymbol{G} - \tfrac{\pi^2}{15}\big)\Big)^2 e^{-\frac{\pi}{4}} := \mathbf{K}_1\\
\max K_1(\delta_z) &= K_1(0) = 1\\
\min K_2(\delta_z) &= \exp\left\{\frac{\pi}{48}(\pi\boldsymbol{G} - 1)\right\}\\
\max K_2(\delta_z) &= K_2(0) = 1\,.
\end{aligned}$$

The first result of this kind known by the author was given in the celebrated monograph [8] by Hurwitz and Courant in which it is proved that

$$\ln M_\sigma(r) \sim \pi r^2/2 \qquad r \to \infty,$$

also see [18, Chapter 4, §1, Problem 49].

Walter Hayman deduced that for certain absolute $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_1 \leq \mathbf{K}_2$ there holds [3, pp. 436–437]

$$\delta_z\,\mathbf{K}_1 \;\leq |\sigma(z)|e^{-\frac{A}{2}\,|z|^2} \leq\; \delta_z\,\mathbf{K}_2, \qquad z \in \mathbb{C}.$$

In turn, the value $A = \frac{\pi}{2}$ is erroneously given in [3], the exact value is $A = \pi$, which means that the type of $\sigma(z)$ is $\frac{\pi}{2}$, see [13].

Finally, exact numerical values of Hayman's constants are derived by the author in [14, Corollary 1.1., p. 161] where the constants

$$A_1 = \mathbf{K}_1 \approx 0.26574548; \quad A_2 = \mathbf{K}_2 = 1.$$

Exhaustive treatment of this problem is given in [14, Remark 2, p. 161], where another form guard–band functions are proposed for the functional bounds (1.1), which include the sine and sine hyperbolic functions' second order powers for the lower and upper bound, respectively. In turn, we point out that these bounds are less efficient than the here used ones $K_j(z)$.

2. Whittaker–type derivative sampling

Let f be entire function for which

$$\limsup_{r\to\infty} \frac{\ln M_f(r)}{r^2} < \frac{q\pi}{2s^2}, \qquad q \in \mathbb{N},\ s > 0,$$

and generalizing J. M. Whittaker's result [20], Pogány [13, 14] proved the qth order derivative Whittaker–type sampling reconstruction series representation

$$f(z) = \sigma^q(z/s) \sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{f^{(q-1-j-k)}(s(m+n\mathrm{i}))}{j!(q-1-j-k)!} \frac{R^q_{mnj}}{(z - s(m+n\mathrm{i}))^{k+1}},$$

which converges uniformly on all compact subsets of $\mathbb{C}$, where

$$R^q_{mnj} = s^{q-j} \lim_{w\to m+n\mathrm{i}} \Big(\frac{\mathrm{d}}{\mathrm{d}w}\Big)^j \left(\frac{w-m-n\mathrm{i}}{\sigma(w)}\right)^q .$$

The Leont'ev function spaces $\mathfrak{S}_{[\rho,\psi)}$, that is $\mathfrak{S}_{[\rho,\psi]}$ are collections of entire functions having order less than ρ; when its order is equal to ρ, the type of the function is less than ψ, $\leq \psi$, respectively, (*cf.* [10, 15]).

The qth order derivative, uniformly spaced Whittaker–type sampling formula contains the sampled values of all derivatives $f^{(j)}(m+n\mathrm{i})$ up to $j = q-1$ whilst $(m,n) \in \mathbb{Z}^2$. So, for all $f \in \mathfrak{S}_{[2,\frac{\pi q}{2})}$ and $s = 1$ the previous formula one reduces to

$$f(z) = \sigma^q(z) \sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-j-1} \frac{f^{(q-j-k-1)}(m+n\mathrm{i}) R^q_{mnj}}{j!(q-j-k-1)!(z-m-n\mathrm{i})^{k+1}} \tag{2.1}$$

uniformly on compact subsets of $\mathbb{C}$, where

$$R^q_{mnj} = \lim_{w\to m+n\mathrm{i}} \Big(\frac{\mathrm{d}}{\mathrm{d}w}\Big)^j \left(\frac{w-m-n\mathrm{i}}{\sigma(w)}\right)^q .$$

This result is due to J. M. Whittaker for $q = 1$ (*cf.* [20]); Higgins in [4] gave no explicit formula, but the general qth order formula could be synthesized from his article; Pogány has given explicit formulæ for $q = 2, 3, 4$, see [13].

The *Whittaker derivative sampling series truncated to* $\mathscr{Z}(r)$ is the finite sum

$$\mathcal{I}_r(z; f; \sigma; q) = \sum_{(m,n)\in\mathscr{Z}(r)} \sum_{j=0}^{q-1} \sum_{k=0}^{q-j-1} \frac{\sigma^q(z) f^{(q-j-k-1)}(m+ni) R^q_{mnj}}{j!(q-j-k-1)!(z-m-ni)^{k+1}}, \tag{2.2}$$

where $\mathscr{Z}(r) := \{(m,n) |\, |m+ni| < r\}$ and under the *circular truncation error* we mean the difference

$$\epsilon_r(f; z; q) := f(z) - \mathcal{I}_r(z; f; \sigma; q).$$

Further, for all $f \in \mathfrak{S}_{[2,\frac{\pi q}{2}\vartheta]}$, $\vartheta \in [0,1)$ and

$$N \geq \left\lfloor \frac{|z|}{\sqrt{2(1-\vartheta)}} + \frac{1}{2} \right\rfloor + 1, \quad z \in \mathbb{C}$$

we have [14, Eq. (16), p. 162]

$$|\epsilon_{\sqrt{2}(N+1/2)}(f; z; q)| \leq \frac{A_f (2N+1)(4N+3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q (2(1-\sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})}.$$

Here A_f characterizes $f \in \mathfrak{S}_{[2,\frac{\pi q}{2}\vartheta]}$ in the way that

$$|f(z)| \leq A_f \exp\left\{\frac{\pi q \vartheta}{2} |z|^2\right\}, \qquad z \in \mathbb{C}.$$

Finally, the exact sampling reconstruction formula (2.1) turns out to be [14, Theorem 2, p. 163]

$$\lim_{N\to\infty} \mathcal{I}_{\sqrt{2}(N+1/2)}(z; f; \sigma; q) = f(z),$$

uniformly in $z \in \mathbb{C}$.

In this limiting procedure the convergence rate estimate will be [14, p. 164]

$$|\epsilon_{\sqrt{2}(N+1/2)}(f; z; q)| = \mathscr{O}\left(N^q e^{-2\pi q(1-\vartheta)N}\right).$$

Obviously, the convergence rate's magnitude is exponentially decresing, therefore

$$\lim_{N\to\infty} \mathcal{I}_{\sqrt{2}(N+1/2)}(z; f; \sigma; q) = f(z)$$

uniformly in $z \in \mathbb{C}$ whenever $q \geq \lfloor \frac{2\pi}{\psi} \rfloor + 1$.

Naturally arises the question about how many derivatives we have to have that the accuracy of approximation is below a prescribed approximation error level ε. Let z be fixed and $\epsilon > 0$ be already given. The the minimal solution in N of the constraint inequality [14, Eq. (19), p. 163]

$$\frac{A_f(2N+1)(4N+3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q(2(1-\sqrt{1-\vartheta})N+1+\sqrt{1-\vartheta})} < \epsilon,$$

becomes the optimal/minimal N to hold the derivative Whittaker sampling approximation procedure.

3. Derivative sampling of $\mathbb{D}_\sigma^{(p,q)}[f](z)$

The here presented definitions and the related results belong to the recent publication (*cf.* [17]). These results were found in the author's cooperation with Z. A. Piranashvili, see the book chapter [12] and the appropriate references therein.

Let f be an entire function coming from Leont'ev functions space $\mathfrak{S}_{[2,\pi\psi/2]}$; $\psi > 0$. For such input funtion f define the (p,q)–order weighted differential operator for $p \in \mathbb{N}_0, q \in \mathbb{N}$ with respect to the Weierstraß σ–function as [17, p. 5]

$$\mathbb{D}_\sigma^{(p,q)}[f](z) = \frac{(-1)^p\, \sigma^{p+q}(z)}{\Gamma(p+1)} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^p \frac{f(z)}{\sigma^q(z)}\,.$$

Accordingly to the above used notations, the Whittaker type q order derivative sampling reconstruction series for $\mathbb{D}_\sigma^{(p,q)}[f](z)$ reads [17, Eq. (5), p. 5]

$$\mathbb{D}_\sigma^{(p,q)}[f](z) = \sum_{(m,n)\in\mathbb{Z}^2} \sum_{l=0}^{q-1} \sum_{k=0}^{q-1-l} \frac{\binom{p+k}{k}\sigma^{p+q}(z)\, B_{qkl}(z_{mn}) f^{(l)}(z_{mn})}{l!(q-1-l-k)!\,(z-z_{mn})^{p+1+k}},$$

where $z_{mn} = m + n\mathrm{i}$ and

$$B_{qkl}(z_{mn}) = \lim_{\zeta\to z_{mn}} \left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^{q-1-l-k} \left(\frac{\zeta - z_{mn}}{\sigma(\zeta)}\right)^q.$$

The related truncated sampling series with respect to the already introduced index set $\mathscr{Z}(r)$ becomes

$$\mathcal{I}_r(z; \mathbb{D}_\sigma^{(p,q)}[f]) = \sum_{(m,n)\in\mathscr{Z}(r)} \sum_{l=0}^{q-1} \sum_{k=0}^{q-1-l} \frac{\binom{p+k}{k}\sigma^{p+q}(z)\, B_{qkl}(z_{mn}) f^{(l)}(z_{mn})}{l!(q-1-l-k)!\,(z-z_{mn})^{p+1+k}},$$

associated sampling truncation error as the difference

$$\epsilon_r(z; \mathbb{D}_\sigma^{(p,q)}[f]) = \mathbb{D}_\sigma^{(p,q)}[f](z) - \mathcal{I}_r(z; \mathbb{D}_\sigma^{(p,q)}[f])\,.$$

One of the first main results follows.

Theorem 3.1 (*cf.* [17, Theorem 1, p. 6])**.** *Let $f \in \mathfrak{S}_{[2,\pi\psi/2]}$ and $0 < \psi < \theta q - (1-\theta)p,\ \theta \in (0,1]$ for all $p, q \in \mathbb{N}$. Then we have*

$$\left|\epsilon_r(\mathbb{D}_\sigma^{(p,q)}[f]; z)\right| \leq \frac{A_f\ \exp\left\{-\frac{\pi}{2}\left(\theta q - (1-\theta)p - \psi\right)r^2\right\}}{\sqrt{2}^{\,p+q}\ r^p\ (H(r)\,\mathbf{K}_1)^q\ (1-\sqrt{1-\theta})^{p+1}}\,.$$

Moreover, the series expansion

$$\mathbb{D}_\sigma^{(p,q)}[f](z) = \sigma^{p+q}(z) \sum_{(m,n)\in\mathscr{Z}^2} \sum_{l=0}^{q-1} \sum_{k=0}^{q-1-l} \frac{\binom{p+k}{k} B_{qkl}(z_{mn}) f^{(l)}(z_{mn})}{l!(q-1-l-k)!\,(z-z_{mn})^{p+1+k}},$$

holds uniformly for all compact z–sets from $\mathbb{C}$.

4. The $L^\alpha(\Omega, \mathfrak{F}, \mathsf{P})$ class of stochastic processes

Consider zero–mean stochastic processes

$$\xi(t) = \int_\Lambda f(t,\lambda) Z_\xi(d\lambda), \qquad (t \in \mathbb{C})$$

on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, where $Z_\xi(d\lambda)$ is a random measure on Λ such that $\mathsf{E}Z(\Delta)Z^*(\Delta') = F(\Delta, \Delta')$, see e.g. the result by Z. A. Piranashvili [11, pp. 710–712]. Our aim is to establish analogues of (2.1) for ξ in the α–mean and almost surely.

Definition 4.1 (*cf.* [16, Definition 1, p. 386])**.** Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a probability space. For $0 \le \alpha \le 2$ let $L^\alpha(\Omega, \mathfrak{F}, \mathsf{P}) := L^\alpha(\Omega)$ be the corresponding space of complex-valued random variables equipped for $0 < \alpha \le 2$ with the quasi-norm $(\mathsf{E}|\cdot|^\alpha)^{1/\alpha} := \|\cdot\|_\alpha$. On $L^0(\Omega)$ the topology is the one induced by convergence in probability.

A stochastic process $\xi(t) \in L^\alpha(\Omega)$ is a family of complex valued random variables $\{\xi(t) \colon\ t \in \mathbb{C}\}$ with continuous complex t.

Let $\{\xi(t) \colon t \in \mathbb{C}\}$ be a subset of $L^2(\Omega)$ defined on fixed $(\Omega, \mathfrak{F}, \mathsf{P})$. Let $B_\xi(t,s)$ the autocorrelation function of ξ and

$$B_\xi(t, s^*) = \int_\Lambda \int_\Lambda f(t,\lambda) f^*(s^*, \mu) F_\xi(d\lambda, d\mu), \tag{4.1}$$

where $\Lambda \subseteq \mathbb{C}$. Here $F_\xi(\cdot,\cdot)$ is positive definite complex set function, additive with respect both of its arguments and

$$\int_\Lambda \int_\Lambda |F_\xi(d\lambda, d\mu)| < \infty.$$

one of our main tools is Karhunen–Cramér theorem, which reads as follows, see [21, p. 156].

Theorem 4.2 (Karhunen–Cramér). *Assume that the correlation function $B_\xi(t,s)$ of a stochastic process $\xi(t)$ possesses a double integral representation in the form*

$$B_\xi(t,s) = \int_\Lambda \int_\Lambda f(t,\lambda)\, f^*(s,\mu) F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu)\,, \qquad \Lambda \subseteq \mathbb{R};\ t, s \in \mathbb{R}, \tag{4.2}$$

where $F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu)$ is a positive definite kernel of finite variation and

$$\int_\Lambda \int_\Lambda f(t,\lambda)\, f^*(s,\mu) F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu) < \infty.$$

Then $\xi(t)$ has a spectral representation and

$$\xi(t) = \int_\Lambda f(t,\lambda) Z_\xi(\mathrm{d}\lambda), \qquad t \in T, \tag{4.3}$$

where Z_ξ is a stochastic measure satisfying

$$F_\xi(S_1, S_2) = \mathsf{E}Z_\xi(S_1) Z_\xi^*(S_2), \qquad S_1, S_2 \in \sigma(\Lambda)\,; \tag{4.4}$$

here $\sigma(\Lambda)$ stands for the appropriate σ-field.

Conversely, if the stochastic process $\xi(t)$ is expressive in the form (4.3) *in which the spectral measure Z_ξ satisfies* (4.4) *and F_ξ is of bounded variation, then the correlation function B_ξ possesses the spectral representation* (4.2)*.*

Now, by the Karhunen–Cramér theorem it holds

$$\xi(t) = \int_\Lambda f(t,\lambda) Z_\xi(d\lambda), \qquad t \in \mathbb{C}, \tag{4.5}$$

in the mean–square as a Lebesgue integral, where $Z_\xi(d\lambda)$ is a random measure on Λ such that $\mathsf{E}Z(\Delta)Z^*(\Delta') = F_\xi(\Delta, \Delta')$. Certain special cases of the correlation function (4.1) associated with the stochastic process (4.5) are:

- If F_ξ in (4.1) concentrates of diagonal $\lambda = \mu$, then the resulting autocorrelation is called of *Karhunen class*, and B_ξ becomes
$$B_\xi(t, s^*) = \int_\Lambda f(t,\lambda) f^*(s^*,\lambda) F_\xi(d\lambda).$$
The spectral representation of the resulting *Karhunen process* $\xi(t)$ remains (4.5) and the assumed second order finiteness condition holds:
$$|B_\xi(t,s^*)|^2 \le B_\xi(t,t) B_\xi(s^*,s^*) \le \sup_{u\in\mathbb{C}} B_\xi^2(u,u) := \mathfrak{B}_\xi^2 < \infty\ .$$
- Assume that the considered $\xi(t) \in L^\alpha(\Omega)$ possesses spectral representation like (4.5), i.e.
$$\xi(t) = \int_\Lambda f(t,\lambda) Z_\xi(d\lambda), \qquad t \in \mathbb{C},$$
for any $\Lambda \in \mathscr{B}(\mathbb{R})$ such that $\operatorname{supp}(Z_\xi) = \Lambda \subseteq \mathbb{R}$.

Now, we are interested in the α–mean and almost sure P convergence of the Whittaker–type sampling series to the initial process $\xi \in L^\alpha(\Omega)$. To realize this goal we need a dilation type result.

Theorem 4.3. *Let $\{\xi(t)\colon\ t \in \mathbb{C}\}$ be $L^\alpha(\Omega)$–stochastic process with random measure Z_ξ, $\alpha \in [0,2]$. Then there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{\mathsf{P}})$ with $L^2(\Omega) \subset L^2(\widetilde{\Omega})$, a Karhunen process $\{\eta(t)\colon\ t \in \mathbb{C}\} \subset L^2(\widetilde{\Omega})$ and a random variable $\Xi \in L^{2\alpha/(2-\alpha)}(\Omega)$ such that*
$$\xi(t) = \Xi \mathcal{P} \eta(t), \qquad t \in \mathbb{C},$$
where $\mathcal{P}$ is the orthogonal projection from $L^2(\widetilde{\Omega})$ to $L^2(\Omega)$.

The proof can be synthetized with the help of the L^2–case result by Rao [19] and its L^α–case real time–parameter result by Christian Houdré [6, 7]. Next, the derivative sampling approach needs the concept of α–mean derivative of the stochastic process $\xi(t) \in L^\alpha(\Omega)$.

Definition 4.4 (*cf.* [2])**.** If there exists a random variable $\xi^{[1]_\alpha}(t)$ such that
$$\lim_{h\to 0} \left\| \frac{\xi(t+h) - \xi(t)}{h} - \xi^{[1]_\alpha}(t) \right\|_\alpha = 0,$$
for $\alpha \in [0,2]$, it is called the first α-mean derivative of the stochastic process $\xi(t) \in L^\alpha(\Omega)$. The case $\alpha = 0$, for some $\varepsilon > 0$, means
$$\lim_{h\to 0} \mathsf{P}\left\{ \left| \frac{\xi(t+h) - \xi(t)}{h} - \xi^{[1]_0}(t) \right| > \varepsilon \right\} = 0.$$

Remark 4.5. The case $\alpha = 2$ defines the mean–square, in another words "in medio" derivative. So the signification l.i.m. for the mean–square limit as well.

Theorem 4.6 (*cf.* [16, Lemma 3, p. 388])**.** *Let $\frac{\partial}{\partial t} f(t,\lambda) \in L^2(F_\eta; \Lambda)$. Then*
$$\xi^{[1]_\alpha}(t) = \int_\Lambda \frac{\partial}{\partial t} f(t,\lambda) Z_\xi(d\lambda),$$
and $\xi^{[1]_\alpha}(t) \in L^\alpha(\Omega)$.

Finally, the circular truncation error we defines in the α–mean sense
$$\tau_r(\xi; t; \alpha) := \|\xi(t) - \mathcal{I}_r(t; \xi; \sigma; q)\|_\alpha^\alpha,$$
where $\mathcal{I}_r(t;\xi;\sigma;q)$ denotes the Whittaker–type q^{th}–order derivative plane–sampling reconstruction sum (2.2) applied to the process $\xi(t) \in L^\alpha(\Omega)$ truncated to the terms corresponding to the index set $\mathscr{Z}(r)$, that is
$$\mathcal{I}_r(t;\xi;\sigma;q) = \sum_{(m,n)\in\mathscr{Z}(r)} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{\xi^{[q-1-j-k]_\alpha}(m+in)}{j!(q-1-j-k)!} \frac{\sigma^q(t)\, R_{mnj}^q}{(t-m-ni)^{k+1}}\ .$$

5. Derivative Whittaker sampling in $L^\alpha(\Omega, \mathfrak{F}, \mathsf{P})$

The second set of main results follow.

Theorem 5.1 (*cf.* [16, Lemma 4, p. 389]). *Let $\{\xi(t)\colon t \in \mathbb{C}\}$ be a $L^\alpha(\Omega)$–type stochastic process. If $f(t,\lambda)$ appears as the kernel function in the spectral representation* (4.5)*, there holds in the α-mean:*

$$\mathcal{I}_r(t;\xi;\sigma;q) = \int_\Lambda \mathcal{I}_r(t;f;\sigma;q) Z_\xi(d\lambda).$$

Theorem 5.2 (*cf.* [16, Theorem 1, p. 389]). *Let $\xi(t) \in L^\alpha(\Omega)$ possesses representation* (4.5) *and $f(\cdot,\lambda)$ is of type ψ that $\sup_{\lambda\in\Lambda}\sup_{t\in\mathbb{C}} |f(t,\lambda)| = A_f < \infty$. If $\xi(t)$ has $q-1, q \in \mathbb{N}$ α-mean derivatives, then*

$$\tau_r(\xi;t;\alpha) \le \left(\frac{A_f \|\Xi\|_{2\alpha/(2-\alpha)}\sqrt{\mathfrak{B}_\eta}}{(\sqrt{2}\,\mathbf{K}_1)^q}\right)^\alpha \left(\frac{r\, e^{r\psi - \pi q(r^2-|t|^2)/2}}{(r-|t|)H^q(r)}\right)^\alpha,$$

where

$$H(r) := \frac{1 - |1 - 2(r^2 - [r^2])|}{4r + \sqrt{2}}.$$

Corollary 5.3 (*cf.* [16, Corollary 1, p. 391]). *For $r = \sqrt{2}(N+1/2)$, $|t| = r\sqrt{1-\theta}$, $\theta \in (0,1)$ define $\Delta_\theta := |\pi q\theta - \psi\sqrt{2}|$. Then*

$$\sup_{\lambda\in\Lambda} |\epsilon_{\sqrt{2}(N+1/2)}(f;t;\lambda)| \le \begin{cases} \mathbf{C}_1 N^q e^{-\pi q\theta N^2} & \Delta_\theta = 0, \\ \mathbf{C}_2 N^q e^{-\Delta_\theta N} & \Delta_\theta \ne 0, \end{cases}$$

where

$$\mathbf{C}_1 = \frac{A_f e^{\pi q\theta/2}}{1-\sqrt{1-\theta}}\left(\frac{7}{\mathbf{K}_1}\right)^q, \quad \mathbf{C}_2 = \frac{A_f e^{\psi/\sqrt{2} - \pi q\theta/4 + \Delta_\theta^2/(\pi q\theta)}}{1-\sqrt{1-\theta}}\left(\frac{7}{\mathbf{K}_1}\right)^q.$$

Theorem 5.4 (*cf.* [16, Theorem 2, p. 391]). *Let the situation be the same as in the previous theorem. Then the q^{th} order Whittaker–type derivative sampling reconstruction formula*

$$\xi(t) = \sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{\sigma^q(t)\, \xi^{[q-1-j-k]_\alpha}(m+in)}{j!(q-1-j-k)!} \frac{R^q_{mnj}}{(t-m-ni)^{k+1}}, \tag{5.1}$$

holds true in α-mean, and for $0 < \alpha \le 2$ it holds a.s. P for all fixed $t \in \mathbb{C}$.

Comment. For fixed $t \in \mathbb{C}$ and $\varepsilon > 0$ by the Markov–inequality we have

$$\mathsf{P}_N = \mathsf{P}\left\{|\xi(t) - \mathcal{I}_{\sqrt{2}(N+1/2)}(t;\xi;\sigma;q)| \ge \varepsilon\right\} \le \varepsilon^{-\alpha}\tau_{\sqrt{2}(N+1/2)}(\xi;t;\alpha).$$

So

$$\sum_{N\ge1} \mathsf{P}_N = \sum_{N\ge1} \mathscr{O}\left(N^{\alpha q} e^{-\alpha\Delta_\theta N}\right) < \frac{C\cdot\Gamma(\alpha q+1)}{(\alpha\Delta_\theta)^{\alpha q+1}}.$$

With the Borel–Cantelli lemma we deduce that (5.1) is valid with probability one.

Theorem 5.5 (*cf.* [16, Theorem 3, pp. 391–392]). *There exists a positive integer $N(\omega)$, $\omega \in \Omega$ such that*

$$\mathsf{P}\{|\xi(t) - \mathcal{I}_n(t;\xi;\sigma;q)| < \varrho(n) \text{ for all } n \ge N(\omega)\} = 1,$$

when both series

$$\sum_{n\ge1} \left(\varrho(n) n^{-q} e^{\Delta_\theta n}\right)^{-\alpha}, \qquad \Delta_\theta \ne 0,$$
$$\sum_{n\ge1} \left(\varrho(n) e^{\pi\theta q n}\, n^{-q}\right)^{-\alpha}, \qquad \Delta_\theta = 0.$$

converge.

6. Sampling of Piranashvili–processes

Our next goal is to present the above exposed results' counterparts when the input stochastic processes are of α–Piranashvili type.

Definition 6.1 (*cf.* [16, Eq. (31), p. 392]). Let $\xi(t) \in L^\alpha(\Omega)$ has integral form (4.5) with the kernel $f(t,\lambda)$ which satisfies

$$|f(t,\lambda)| \le \widetilde{L}_f(\lambda)(1+|t|^m)e^{c^\star(\lambda)|\Im\{t\}|} \tag{6.1}$$

for certain fixed $m \in \mathbb{N}$, $\sup_{\lambda\in\Lambda}\widetilde{L}_f(\lambda) := \mathbf{L}_f < \infty$ and $\psi = \sup_{\lambda\in\Lambda} c^\star(\lambda)$ is finite. Such a process will be called Piranashvili α–process in the sequel.

Theorem 6.2 (*cf.* [16, Theorem 4, p. 392]). *Let $\xi(t) \in L^\alpha(\Omega)$ be a Piranashvili α–process represented by* (9) *with a kernel function* (18). *Then for a fixed $m \in \mathbb{N}$ and for all $t \in \operatorname{int}(\{z\colon\ |z| = r\})$ it is*

$$\begin{aligned}
|\epsilon_r(f;t;\lambda)| &= |f(t) - \mathcal{I}_r(t;f;\sigma;q)| \\
&\le \frac{\mathbf{L}_f\ r(1+|t|^m)\sup_{t\in\mathbb{C}}|f(t,\lambda)|}{(r-|t|)(\sqrt{2}\,\mathbf{K}_1 H(r))^q}\exp\left\{c^\star(\lambda)r - \frac{\pi q}{2}(r^2-|t|^2)\right\} \\
\tau_r(\xi;t;\alpha) &\le \left(\frac{A_f\mathbf{L}_f\|\Xi\|_{2\alpha/(2-\alpha)}\sqrt{\mathfrak{B}_\eta}}{(\sqrt{2}\,\mathbf{K}_1)^q}\right)^\alpha \left(\frac{r(1+|t|^m)\ e^{r\psi-\pi q(r^2-|t|^2)/2}}{(r-|t|)H^q(r)}\right)^\alpha .
\end{aligned} \tag{6.2}$$

Finally, for any $q \in \mathbb{N}$ the formula

$$\xi(t) = \sum_{(m,n)\in\mathbb{Z}^2}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1-j}\frac{\sigma^q(t)\ \xi^{[q-1-j-k]_\alpha}(m+in)R^q_{mnj}}{j!(q-1-j-k)!(t-m-ni)^{k+1}}$$

holds in both the α-mean and almost surely P *on compact t - subsets of* $\mathbb{C}$.

In deriving the truncation error upper bound (6.2) we did not assume and use any bandlimitedness (BL) of $\xi(t) \in L^\alpha(\Omega)$ (and 'a fortiori' $\eta(t)$ also is not necessarily bandlimited!), there are no reasons to restrict our considerations to the class of *conventionally* BL stochastic processes, i.e. Λ could coincide with $\mathbb{R}$, see the (w)-BL processes which were extensively studied by Zakai, Cambanis and Masry, and Lee, (see, e.g. [1, 9]). These processes possess a spectral representation like (4.5) if the kernel function satisfies a condition slightly different than (6.1), see Piranashvili's approach [11, pp. 709, 717].

7. Illustrative examples

The following three examples consist the section [16, Examples, pp. 393–394].

Example 7.1. Let $\xi(t)$ a Piranashvili α-process, $\alpha \in [0,2]$, with first derivative in the α-mean, i.e. $\frac{\partial}{\partial t}f(t,\lambda) \in L^2(F_\eta,\Lambda)$. Then

$$\xi(t) = \sigma^2(t)\sum_{(m,n)\in\mathbb{Z}^2}\left\{\left(\frac{1}{t-m-ni} - 2\pi(m-ni)\right)\frac{\xi(m+in)}{t-m-ni} + \frac{\xi^{[1]_\alpha}(m+in)}{t-m-ni}\right\}e^{-\pi(m^2+n^2)}$$

in the α-mean and a.s. P (only for $\alpha > 0$), uniformly on all compact t - sets in $\mathbb{C}$.

For a deterministic, entire function of order 2 and with type less than $\pi/2$ ($q = 2,\ s = \sqrt{2}$) this formula appears in Higgins' works [4, Theorem 3, p.169], [5, Problem 9.4, p.101].

The following two results, Example 7.2 (in which $q = 3$) and Example 7.3 (in which $q = 4$) are novelties in the stochastic setting; their deterministic variants are given in [13] for signals from the Leont'ev–type function space $\mathfrak{S}_{[2,\frac{\pi q}{2s^2})}$.

Example 7.2. Let $\xi(t)$ be a twice differentiable Piranashvili α-process, $\alpha \in [0,2]$, in the α-mean sense, i.e. $\frac{\partial}{\partial t} f(t,\lambda)$, $\frac{\partial^2}{\partial t^2} f(t,\lambda) \in L^2(F_\eta, \Lambda)$. Then

$$\begin{aligned}\xi(t) = \sum_{(m,n)\in\mathbb{Z}^2} (-1)^{m+n+mn} \Bigg\{ &\left(\frac{(3\pi z^*_{mn})^2}{2} + \frac{3\pi z^*_{mn}}{t - z_{mn}} + \frac{1}{(t-z_{mn})^2} \right) \frac{\xi(z_{mn})}{t - z_{mn}} \\ &+ \left(\frac{1}{t-z_{mn}} - 3\pi z^*_{mn} \right) \frac{\xi^{[1]_\alpha}(z_{mn})}{t - z_{mn}} + \frac{\xi^{[2]_\alpha}(z_{mn})}{2(t-z_{mn})} \Bigg\} e^{-\frac{3\pi}{2}|z_{mn}|^2}\, \sigma^3(t),\end{aligned}$$

where $z_{mn} = m + in$. This formula is valid in the α-mean and a.s. P (only for $\alpha > 0$), uniformly in t, where t belongs to a compact subset of $\mathbb{C}$.

Example 7.3. When $\xi(t)$ is threefold differentiable Piranashvili α-process, $\alpha \in [0,2]$ in the α-mean sense, i.e. $\frac{\partial}{\partial t} f(t,\lambda)$, $\frac{\partial^2}{\partial t^2} f(t,\lambda)$, $\frac{\partial^3}{\partial t^3} f(t,\lambda) \in L^2(F_\eta, \Lambda)$, then

$$\begin{aligned}\xi(t) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{\sigma^4(t)}{6(t-\zeta_{mn})} \Big\{ \Big(&\frac{6}{(t-\zeta_{mn})^3} - \frac{24\pi\zeta^*_{mn}}{(t-\zeta_{mn})^2} + \frac{3(4\pi\zeta^*_{mn})^2}{t-\zeta_{mn}} \\ &- (4\pi\zeta^*_{mn})^3 \Big) \xi(\zeta_{mn}) + \Big(\frac{6}{(t-\zeta_{mn})^2} - \frac{24\pi\zeta^*_{mn}}{t-\zeta_{mn}} + 48(\pi\zeta^*_{mn})^2 \Big) \xi^{[1]_\alpha}(\zeta_{mn}) \\ &+ \Big(\frac{3}{t-\zeta_{mn}} - 12\pi\zeta^*_{mn} \Big) \xi^{[2]_\alpha}(\zeta_{mn}) + \xi^{[3]_\alpha}(\zeta_{mn}) \Big\} e^{-2\pi|\zeta_{mn}|^2},\end{aligned}$$

where $z_{mn} = m + in$. The above series representation formula holds true in the α–mean and a.s. P (only for $\alpha > 0$), uniformly in t, on all compact t–sets in $\mathbb{C}$.

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