Montes Taurus J. Pure Appl. Math. 5 (1), 71–77, 2023

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Variational approach for a class of delay second-order differential equations

Fatima Dib [a], Mustapha Yebdri [b], Naima Daoudi-Merzagui [c]

[a] *Department of Sciences and Technology, University Center of Maghnia, 13300 Maghnia, Algeria - Laboratory of Dynamical Systems and Applications, University of Tlemcen*
[b] *Laboratory of Dynamical Systems and Applications, University of Tlemcen*
[c] *Department of Mathematics, University of Tlemcen, 13000 Tlemcen, Algeria*

Abstract

We consider a class of nonautonomous second-order delay differential equations. By means of variational approach, we prove the existence of at least one periodic solution.

Keywords: 34B15, 34C15, 34C25, 58E30

2010 MSC: Differential equation, periodic solution, variational method

1. Introduction

Partial differential equations and functional differential equations, delay ones in particular, are the most mathematical tools used in the study of infinite dimensional phenomena (*cf.* [2, 6, 8]). Variational methods provide a solid basis for the existence theory of partial differential equations (*cf.* [15]). However it is not the case in delay differential equations. To our knowledge, there is a few research in this direction (*cf.* [1, 13]).

As it is known variational methods consist in transforming a differential equation in an integral problem, where the differential operator can be formulated as the first variation derivative, of an appropriate functional energy, on a suitable Banach space. This technic identifies an important class of problems which can be solved using relatively simple techniques from nonlinear functional analysis (*cf.* [3, 10, 11, 15]). The advantage of this new formulation is that solving such problem (at least weakly) is equivalent to finding the critical points of the functional on the suitable Banach space. One of the most studied aspects in delay differential equations is the existence of periodic solutions (*cf.* [4, 5, 9, 14, 16]).

The delay differential equation

$$x''(t) = -f(x(t-r))$$

under periodic conditions was investigated by Guo and Guo [7], existence and multiplicity results were obtained using the critical point theory and S^1-index theory.

†*Article ID:* MTJPAM-D-21-00038

Email addresses: fatimadib1967@yahoo.fr (Fatima Dib), yebdri@yahoo.com (Mustapha Yebdri), nmerzagui@yahoo.fr (Naima Daoudi-Merzagui)

Received:3 June 2021, Accepted:1 June 2023, Published:21 July 2023

⋆*Corresponding Author:* Fatima Dib

This paper is devoted to the study of the existence of nonconstant $2r-$periodic solutions of the following nonlinear problem

$$\begin{cases} x''(t) = -f(t, x(t), x(t-r)), \\ x(0) - x(2r) = x'(0) - x'(2r) = 0, \\ \int_0^{2r} x(t)dt = 0, \end{cases} \tag{1.1}$$

where $r \in \mathbb{R}^+$ is a given constant, $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous $r-$periodic in t and satisfies some conditions specified in the following.

By the application of a variational approach based on a direct minimization, we obtain the existence of at least one nonconstant $2r-$periodic solution of problem (1.1).

The problem

$$\begin{cases} -x''(t) + \lambda x(t-r) = h(t, x(t), \\ x(0) - x(2r) = x'(0) - x'(2r) = 0, \\ \int_0^{2r} x(t)dt = 0 \end{cases} \tag{1.2}$$

is treated as a particular case of (1.1).

The rest of this paper is organized as follows, in section 2 we give some necessary preliminaries, which will be used to prove our main result. In section 3, we show the existence of at least one nonconstant $2r-$periodic solution of problem (1.1), in section 4, we study the particular case (1.2) and at last, we give an example to illustrate the applicability of our result.

2. Preliminaries

In this section, we recall some basic definitions, notations and few results from functional analysis that we shall use in the remainder of the paper.

Throughout this work, we note: $I = [0, 2r]$; for $1 \leq p < \infty$, $L^p(I)$ is the classical Lebesgue space of measurable functions $x : I \to \mathbb{R}$ which are p-integrable functions.

For $x \in L^p(I)$, we define its norm by

$$\|x\|_{L^p} = \left(\int_0^{2r} |x(t)|^p \, dt \right)^{\frac{1}{p}} .$$

Furthermore, $L^\infty(I)$ denote the space of essentially bounded functions $x : I \to \mathbb{R}$, equipped with the norm

$$\|x\|_{L^\infty} = ess \sup_{t \in I} |x(t)| .$$

Let $\|x\|_\infty = \sup_{t \in I} |x(t)|$ denote the norm of $x \in C(I)$, the space of real-valued continuous functions.

For all $x \in L^p(I)$, we denote by $x_r(t) := x(t - r)$ for $t \in \mathbb{R}$.

We have that if x is a $T-$periodic function, then $\int_0^T x(t)dt = \int_a^{T+a} x(t)dt$, for all $a \in \mathbb{R}$. Hence, for all $x \in L^p(I)$,

$$\|x_r\|_{L^p} = \|x\|_{L^p} . \tag{2.1}$$

Let

$$H^1_{2r} = \left\{ x : \mathbb{R} \to \mathbb{R} \;/\; x \in L^2(I),\ x' \in L^2(I);\ x(0) = x(2r) \text{ and } x'(0) = x'(2r) \right\},$$

provided with the inner product $(.,.)$ defined by

$$(x, y) = \int_0^{2r} \left[x(t)y(t) + x'(t)y'(t) \right] dt, \quad for\ all\ x, y \in H^1_{2r},$$

which induces the norm

$$\|x\|_{H^1_{2r}} = \left(\int_0^{2r} |x(t)|^2 \, dt + \int_0^{2r} \left| x'(t) \right|^2 dt \right)^{\frac{1}{2}}, \quad for\ all\ x \in H^1_{2r}.$$

Then H^1_{2r} is a Banach space (in fact is a Hilbert space).

For all $x \in H^1_{2r}$, there is constant c such that

$$\|x\|_{L^1} \leq c\,\|x\|_{H^1_{2r}}\,. \tag{2.2}$$

We are looking for nonconstant $2r$−periodic solutions, for this, we consider the weakly closed subspace X of H^1_{2r} defined by

$$X = \left\{ x \in H^1_{2r},\ \int_0^{2r} x(t)dt = 0 \right\}.$$

The subspace X has some important properties:

Proposition 2.1. *Let $x \in X$, then*

$$\int_0^{2r} |x(t)|^2\, dt \leq \frac{r^2}{\pi^2} \int_0^T \left|x'(t)\right|^2 dt, \tag{2.3}$$

(Wirtinger inequality) and

$$\|x\|_\infty^2 \leq \frac{r}{6} \int_0^{2r} \left|x'(t)\right|^2 dt,$$

(Sobolev inequality).

With the help of the Wirtinger inequality in Proposition 2.1, on the subspace X, we can obtain the equivalent norm

$$\|x\| = \left(\int_0^{2r} \left|x'(t)\right|^2 dt \right)^{\frac{1}{2}}.$$

Definition 2.2. A function $x \in X$ is a solution of problem (1.1) if the function x satisfies the equation $x''(t) = -f(t, x(t), x(t-r))$.

The following result plays an important role in the proof of our main result.

Theorem 2.3 (*cf.* [11, Theorem 1.1])**.** *If E is a reflexive Banach space, $X \subset E$ weakly closed, and $\varphi : X \to \mathbb{R}$ is weakly lower semi continuous, then φ has a minimum over X if and only if it has a bounded minimizing sequence in X.*

Remark 2.4. φ has a bounded minimizing sequence in X if φ is coercive in X, i.e.

$$\varphi(x) \to \infty \text{ when } \|x\|_X \to \infty.$$

3. Main results

In this section, we state and prove our main result, for this, we consider the problem (1.1) under the following assumptions:

- **(H1)** $f(t, x_1, x_2) \in C(\mathbb{R}^3, \mathbb{R})$ and $\frac{\partial f}{\partial t}(t, x_1, x_2) \neq 0$.
- **(H2)** $f(t + r, x_1, x_2) = f(t, x_1, x_2)\ \forall (x_1, x_2) \in \mathbb{R}^2$.
- **(H3)** **(i)** There exists a function $g(t, x, y) \in C(\mathbb{R}^3, \mathbb{R})$ such that $\frac{\partial g}{\partial x}$ is well defined, continuous and

$$f(t, x_1, x_2) = g(t, x_2, x_1) + \int_0^{x_2} g'_x(t, x_1, \omega)d\omega. \tag{3.1}$$

  (ii) g is bounded above i.e. there exists a constant $k \in \mathbb{R}$ such that $g(t, x, y) \leq k$ for all $(t, x, y) \in \mathbb{R}^3$.

The main result is presented as follows.

Theorem 3.1. *Assume that* (H1)-(H3) *hold.*

Then problem (1.1)*, has at least one nonconstant* $2r-$*periodic solution.*

Proof. We consider the functional

$$\psi : X \to \mathbb{R},$$

defined by

$$\psi(x) = \int_0^{2r} \left[\frac{1}{2} \left| x'(t) \right|^2 - \int_0^{x(t-r)} g(t, x(t), \omega) d\omega \right] dt. \tag{3.2}$$

The proof will be based on several claims.

Claim 1. The functional $\psi : X \to \mathbb{R}$ defined by (3.2) is continuous, differentiable, and moreover, the critical points of ψ are weak solutions of problem (1.1).

Using the continuity of g we obtain the continuity and differentiability of ψ.

Then, for all $x, y \in X$ and $\varepsilon > 0$, we have

$$\begin{aligned}
\psi(x+\varepsilon y) &= \int_0^{2r} \left[\frac{1}{2} \left| (x+\varepsilon y)'(t) \right|^2 - \int_0^{(x+\varepsilon y)(t-r)} g(t, (x+\varepsilon y)(t), \omega) d\omega \right] dt \\
&= \frac{1}{2} \int_0^{2r} (x'(t) + \varepsilon y'(t))^2 dt - \int_0^{2r} \left(\int_0^{x(t-r)+\varepsilon y(t-r)} g(t, x(t)+\varepsilon y(t), \omega) d\omega \right) dt \\
&= \frac{1}{2} \int_0^{2r} \left[(x'(t))^2 + \varepsilon^2 (y'(t))^2 + 2\varepsilon x'(t) y'(t) \right] dt \\
&\quad - \int_0^{2r} \left(\int_0^{x(t-r)+\varepsilon y(t-r)} g(t, x(t), \omega) d\omega + \varepsilon \left(\int_0^{x(t-r)+\varepsilon y(t-r)} g'_x(t, x(t)+\varepsilon\theta(t) y(t), \omega) d\omega \right) y(t) \right) dt, \qquad (3.3)
\end{aligned}$$

where $0 \leq \theta(t) \leq 1$. From (3.3), we find

$$\begin{aligned}
\psi(x+\varepsilon y) &= \psi(x) + \varepsilon \int_0^{2r} x'(t) y'(t) dt - \varepsilon \int_0^{2r} \left(\int_0^{x(t-r)+\varepsilon y(t-r)} g'_x(t, x(t)+\theta(t)\varepsilon y(t), \omega) d\omega \right) y(t) dt \\
&\quad - \int_0^{2r} \left(\int_{x(t-r)}^{x(t-r)+\varepsilon y(t-r)} g(t, x(t), \omega) d\omega \right) dt + \frac{\varepsilon^2}{2} \int_0^{2r} (y'(t))^2 dt, \qquad (3.4)
\end{aligned}$$

by a classic result of analysis and the periodicity of g, x and y we have

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{1}{\varepsilon} \int_{x(t-r)}^{x(t-r)+\varepsilon y(t-r)} g(t, x(t), \omega) d\omega &= \lim_{\epsilon \to 0} \frac{1}{\varepsilon} \int_{x(t+r)}^{x(t+r)+\varepsilon y(t+r)} g(t+r, x(t), \omega) d\omega \\
&= g(t+r, x(t), x(t+r)) y(t+r) \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{2r} g(t+r, x(t), x(t+r)) y(t+r) dt &= \int_r^{3r} g(t, x(t-r), x(t)) y(t) dt \\
&= \int_0^{2r} g(t, x(t-r), x(t)) y(t) dt, \qquad (3.6)
\end{aligned}$$

by (3.5) and (3.6) we obtain

$$\begin{aligned}
\langle \psi'(x), y \rangle &= \lim_{\epsilon \to 0} \frac{1}{\varepsilon} \left(\psi(x+\varepsilon y) - \psi(x) \right) \\
&= \int_0^{2r} x'(t) y'(t) dt - \int_0^{2r} \left(\int_0^{x(t-r)} g'_x(t, x(t), \omega) d\omega + g(t, x(t-r), x(t)) \right) y(t) dt. \qquad (3.7)
\end{aligned}$$

Hence, using integration by parts we arrive at

$$\langle \psi'(x), y\rangle = \int_0^{2r}\left[-x''(t) - \int_0^{x(t-r)} g'_x(t, x(t), \omega)d\omega + g(t, x(t-r), x(t))\right]y(t)dt. \tag{3.8}$$

Therefore, the Euler equation corresponding to the function $\psi(x)$ is as follows:

$$x''(t) + \int_0^{x(t-r)} g'_x(t, x(t), \omega)d\omega + g(t, x(t-r), x(t)) = 0, \tag{3.9}$$

by the assumption (H2)(i), (3.9) is equivalent to

$$x''(t) + f(t, x(t), x(t-r)) = 0.$$

Then, we may get $2r$−periodic weak solutions of the problem (1.1), by seeking critical points of the functional $\psi(x)$.
Claim 2. The functional ψ is weakly lower semi continuous.
Let (x_n) be a weakly convergent sequence to x in X. Then

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|, \tag{3.10}$$

and (x_n) converges uniformly to x on I, so $x_n \to x$ on $L^2(I)$, we have

$$\begin{aligned}
\liminf_{n\to\infty} \psi(x_n) &= \liminf_{n\to\infty}\left[\frac{1}{2}\|x_n\|^2 - \int_0^{2r}\left(\int_0^{x_n(t-r)} g(t, x_n(t), \omega)d\omega\right)dt\right]\\
&\ge \frac{1}{2}\|x\|^2 - \int_0^{2r}\left(\int_0^{x(t-r)} g(t, x(t), \omega)d\omega\right)dt\\
&= \psi(x).
\end{aligned}$$

Claim 3. The functional ψ is coercive.
By using the assumption (H3), (2.1) and (2.2) we have

$$\begin{aligned}
\psi(x) &= \|x\|^2 - \int_0^{2r}\left(\int_0^{x(t-r)} g(t, x(t), \omega)d\omega\right)dt\\
&\ge \|x\|^2 - \int_0^{2r} kx(t-r)dt\\
&\ge \|x\|^2 - |k|\int_0^{2r} |x(t-r)|\, dt \qquad (3.11)\\
&= \|x\|^2 - |k|\,\|x\|_{L^1}\\
&\ge \|x\|^2 - c_1\|x\|,
\end{aligned}$$

for some $c_1 > 0$. This implies that $\lim_{\|x\|\to\infty} \psi(x) = +\infty$, and ψ is coercive.
Hence, by Theorem 2.3 ψ has a minimum, which is a critical point of ψ. Then, the problem (1.1) has at least one nonconstant $2r$−periodic solution. The proof of the main result is complete. □

4. Particular case

We next consider the nonlinear delay problem (1.2)

$$\begin{cases} -x''(t) - \lambda x(t-r) = h(t, x(t)),\\ x(0) - x(2r) = x'(0) - x'(2r) = 0,\\ \int_0^{2r} x(t)dt = 0, \end{cases}$$

where $r \in \mathbb{R}^+$ a given constant, $\lambda \in \mathbb{R}^+$, and h satisfies:

(h1) $h(t,s) \in C(\mathbb{R}^2, \mathbb{R})$ such that $\frac{\partial h}{\partial t}(t,s) \neq 0$.

(h2) $h(t+r,s) = h(t,s)\ \forall s \in \mathbb{R}$.

(h3) h is bounded above i.e. there exists $M \in \mathbb{R}$ such that $h(t,s) \leq M$, for all $(t,s) \in \mathbb{R}^2$.

We can now state the following result.

Theorem 4.1. *Assume that* (h1)-(h3) *hold, and* $\lambda < \frac{\pi^2}{r^2}$.
Then, problem (1.2) *has at least one nonconstant* $2r$*−periodic solution.*

Proof. The problem (1.2) is a particular case of the initial problem (1.1) where,

$$f(t, x(t), x(t-r)) = \lambda x(t-r) + h(t, x(t)), \tag{4.1}$$

that is

$$f(t, x_1, x_2) = \lambda x_2 + h(t, x_1). \tag{4.2}$$

For proof this result, one needs to verify all the assumptions (H1)-(H3) of Theorem 3.1.
(H1) is verified by the assumption (h1).
(H2) is verified by the assumption (h2).
For verify (H3), just take

$$g(t,x,y) = \frac{\lambda}{2}x + h(t,y), \tag{4.3}$$

indeed, by (H1)

$$\frac{\partial g}{\partial x} = \frac{\lambda}{2},$$

is well defined and continuous. Then we have

$$\begin{aligned} f(t,x_1,x_2) &= g(t,x_2,x_1) + \int_0^{x_2} g'_x(t,x_1,\omega)d\omega \\ &= \frac{\lambda}{2}x_2 + h(t,x_1) + \int_0^{x_2} \frac{\lambda}{2} d\omega \\ &= \lambda x_2 + h(t,x_1). \end{aligned} \tag{4.4}$$

For (1.1), we obtain the coercivity of the functional ψ from the assumption (H3), in our particular case, $g(t,x,y) = \frac{\lambda}{2}x + h(t,y)$ is unbounded above, then to obtain the coercivity of ψ, we used the fact that h is bounded above and $\lambda < \frac{\pi^2}{r^2}$.
Indeed, by (4.3) we have

$$\begin{aligned} \psi(x) &= \int_0^{2r}\left[\frac{1}{2}\left|x'(t)\right|^2 - \int_0^{x(t-r)}\left(\frac{\lambda}{2}x(t) + h(t,\omega)\right)d\omega\right]dt \\ &= \frac{1}{2}\|x\|^2 - \frac{\lambda}{2}\int_0^{2r} x(t)x(t-r)dt - \int_0^{2r}\left(\int_0^{x(t-r)} h(t,\omega)d\omega\right)dt \\ &\geq \frac{1}{2}\|x\|^2 - \frac{\lambda}{2}\int_0^{2r} |x(t)x(t-r)|\, dt - \int_0^{2r}\left(\int_0^{x(t-r)} h(t,\omega)d\omega\right)dt. \end{aligned} \tag{4.5}$$

From (4.5) , and by using the Hölder inequality, (h3), Proposition 2.1 and (2.1), we obtain,

$$\begin{aligned} \psi(x) &\geq \frac{1}{2}\|x\|^2 - \frac{\lambda}{2}\|x\|_{L^2}\|x_r\|_{L^2} - M\|x\|_{L^1} \\ &= \frac{1}{2}\|x\|^2 - \frac{\lambda}{2}\|x\|^2_{L^2} - M\|x\|_{L^1} \\ &\geq \frac{1}{2}\|x\|^2 - \frac{\lambda}{2}\frac{r^2}{\pi^2}\|x\|^2 - M\|x\|_{L^1} \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda r^2}{\pi^2}\right)\|x\|^2 - c_2\|x\|, \end{aligned}$$

for some $c_2 > 0$.
Hence by Theorem 2.3, the problem (1.2) has at least one nonconstant $2r-$periodic solution. The proof of this second result is complete. □

Example 4.2. Let $r = \frac{\pi}{2}$, $\lambda < 4$. By applying Theorem 4.1, the following nonlinear problem

$$\begin{cases} -x''(t) + \lambda x(t - \frac{\pi}{2}) = x^2(t) + \sin 4t, \\ x(0) - x(\pi) = x'(0) - x'(\pi) = 0, \\ \int_0^\pi x(t)dt = 0 \end{cases}$$

is solvable, that is has at least one nonconstant $\pi-$periodic solution.

Acknowledgments

The authors are grateful for the referee's careful reviewing and helpful comments.

Author Contributions: Fatima Dib: Conceptualization, Methodology, Review and editing.
Mustapha Yebdri: Conceptualization, Investigation, Supervision.
Naima Daoudi Merzagui: Investigation, Supervision.

Conflict of Interest: The authors declare no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

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