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Kenmotsu manifolds with quarter-symmetric non-metric connections

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Abstract

We categories Kenmotsu manifold with quarter-symmetric non-metric connections. In relation to this relationship, we examine Ricci soliton on such manifolds. A last example is shown.

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1. Introduction

Boothby and Wang [5] investigated odd dimensional manifolds with contact and almost contact structures from a topological standpoint in 1958. Sasaki and Hatakeyama [23] explored them further using tensor calculus in 1961. Tano [26] identified linked almost contact metric manifolds whose automorphism groups had the greatest dimension. For such a manifold $\mathcal{M}^n$, the sectional curvature of plane sections containing ζ is a constant, say c. If $c > 0$, $\mathcal{M}^n$ is a homogeneous Sasakian manifold with constant sectional curvature. If c=0, $\mathcal{M}^n$ is the product of a line or a circle and a Khler manifold with constant holomorphic sectional curvature. If $c < 0$, $\mathcal{M}^n$ is warped product space $\mathfrak{R} \times_f C^n$.
Kenmotsu developed a class of contact Riemannian manifolds in 1972 and named them Kenmotsu manifold (*cf.* [14]). He proved that if Kenmotsu manifold satisfies the condition $\mathcal{R}(\varrho_1, \varrho_2) \cdot \mathcal{R}$=0, then the manifold is of negative curvature -1, where $\mathcal{R}$ is the Riemannian curvature tensor of type $(1, 3)$ and $\mathcal{R}(\varrho_1, \varrho_2)$ denotes the derivation of the tensor algebra at each point of the tangent space. Recently, Kenmotsu manifolds have been studied by several authors (*cf.* [2], [7]-[9], [13], [15]-[17], [19, 28]).
Hamilton [10] introduce the theory of Ricci flow to establish a canonical metric on a smooth manifold in 1982. The Ricci flow is a Riemannian manifold evolution equation defined as follows:

$$\frac{\partial}{\partial t} g(t) = -\mathcal{R}(t) g(t).$$

A Ricci soliton $(g, \mathcal{V}, \lambda)$ on a Riemannian manifold $(\mathcal{M}, g)$ is a generalization of an Einstein metric such that it satisfies the following condition (*cf.* [11]):

$$\mathfrak{L}_{\mathcal{V}} g + 2\mathcal{S} + \lambda g = 0, \tag{1.1}$$

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where $\mathcal{S}$ is the Ricci tensor, $\mathfrak{L}_{\mathcal{V}}$ is the Lie derivative operator along the vector field $\mathcal{V}$ on $(\mathcal{M}, g)$ and λ is a real number. The Ricci soliton is considered to be decreasing, stable, or growing depending on whether λ is negative, zero, or positive. It became especially crucial after Perelman used Ricci solitons to answer the long-standing Poincare conundrum stated in 1904. Several geometers (*cf.* [1, 3, 6, 12, 20, 24, 27, 29, 30]) and others have investigated Ricci soliton.

Patra et al. [18] define a quarter symmetric non-metric connection on pseudo-symmetric Kenmotsu manifold and prove its existence. Motivated by this study, in this paper we consider a quarter symmetric non-metric connection on a Kenmotsu manifold. After introduction section 2, we give a brief account of Kenmotsu manifolds. In section 3, we recall the expression for curvature tensor and some basic results. Section 4 deals with the study of Ricci soliton on a Kenmotsu manifold with such connection. Section 5 deals with locally ϕ-symmetric Kenmotsu manifold in view of connection $\widetilde{\nabla}$. ϕ-recurrent Kenmotsu manifold admitting connection $\widetilde{\nabla}$ are studied in section 6. Section 7 contain locally projective ϕ-symmetric Kenmotsu manifold with respect to the connection $\widetilde{\nabla}$. Section 8 is devoted to study of ϕ-projectively flat Kenmotsu manifold with respect to such connection $\widetilde{\nabla}$. In section 9, we derive criteria for a vector field in a Kenmotsu manifold to be Killing with respect to the quarter symmetric non-metric connection $\widetilde{\nabla}$.

2. Preliminaries

Let $\mathcal{M}^{2n+1}(\phi, \zeta, \eta, g)$ be an almost contact Riemannian manifold, where ϕ is a $(1,1)$-tensor field, η is a 1-form and g is a Riemannian metric, then following relations hold (*cf.* [4, 32]):

$$\phi\zeta = 0, \quad \eta(\phi\varrho_1) = 0, \quad \eta(\zeta) = 1, \tag{2.1}$$

$$\phi^2(\varrho_1) = -\varrho_1 + \eta(\varrho_1)\zeta, \tag{2.2}$$

$$g(\varrho_1, \zeta) = \eta(\varrho_1),$$

$$g(\phi\varrho_1, \phi\varrho_2) = g(\varrho_1, \varrho_2) - \eta(\varrho_1)\eta(\varrho_2), \tag{2.3}$$

$$\left(\nabla_{\varrho_1}\phi\right)(\varrho_2) = -\eta(\varrho_2)\phi(\varrho_1) - g(\varrho_1, \phi\varrho_2)\zeta, \tag{2.4}$$

$$\nabla_{\varrho_1}\zeta = \varrho_1 - \eta(\varrho_1)\zeta, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g, then $(\mathcal{M}^{2n+1}, \phi, \zeta, \eta, g)$ is called an almost Kenmotsu manifold (*cf.* [14]). On a Kenmotsu manifold, it can be shown that

$$\left(\nabla_{\varrho_1}\eta\right)\varrho_2 = g(\phi\,\varrho_1, \phi\,\varrho_2), \tag{2.6}$$

$$\mathcal{F}(\varrho_1, \varrho_2) = -\mathcal{F}(\varrho_2, \varrho_1), \tag{2.7}$$

where $\mathcal{F}(\varrho_1, \varrho_2)$=$g(\phi\varrho_1, \varrho_2)$ is a fundamental 2-form.
In Kenmotsu manifolds, we have (*cf.* [14]):

$$\eta(\mathcal{R}(\varrho_1, \varrho_2)\varrho_3) = \eta(\varrho_2)g(\varrho_1, \varrho_3) - \eta(\varrho_1)g(\varrho_2, \varrho_3),$$

$$\mathcal{R}(\varrho_1, \varrho_2)\zeta = \eta(\varrho_1)\varrho_2 - \eta(\varrho_2)\varrho_1, \tag{2.8}$$

$$\mathcal{R}(\zeta, \varrho_1)\varrho_2 = \eta(\varrho_2)\varrho_1 - g(\varrho_1, \varrho_2)\zeta, \quad \mathcal{R}(\zeta, \varrho_1)\zeta = \varrho_1 - \eta(\varrho_1)\zeta,$$

$$\mathcal{S}(\phi\varrho_1, \phi\varrho_2) = \mathcal{S}(\varrho_1, \varrho_2) + 2n\eta(\varrho_1)\eta(\varrho_2),$$

$$\mathcal{S}(\varrho_1, \zeta) = -2n\eta(\varrho_1), \tag{2.9}$$

for any vector fields $\varrho_1, \varrho_2, \varrho_3$ on $\mathcal{M}^{2n+1}$, where $\mathcal{R}$ is the Riemannian curvature tensor and $\mathcal{S}$ is the Ricci tensor.

3. Quarter-symmetric non-metric connection on $\left(\mathcal{M}^{2n+1}, \phi, \zeta, \eta, g\right)$

Let $\mathcal{M}^{2n+1}$ be a Kenmotsu manifold with Levi-Civita connection ∇. We define a linear connection $\widetilde{\nabla}$ on $\mathcal{M}^{2n+1}$ as

$$\widetilde{\nabla}_{\varrho_1}\varrho_2 = \nabla_{\varrho_1}\varrho_2 + \eta(\varrho_2)\phi(\varrho_1). \tag{3.1}$$

Then a linear connection $\widetilde{\nabla}$ is said to be quarter symmetric connection if the torsion tensor $\widetilde{\mathcal{T}}$ with respect to the connection $\widetilde{\nabla}$ satisfies

$$\widetilde{\mathcal{T}}(\varrho_1, \varrho_2) = \eta(\varrho_2)\phi\varrho_1 - \eta(\varrho_1)\phi\varrho_2, \tag{3.2}$$

where as a linear connection $\widetilde{\nabla}$ is said to be non-metric connection if $\widetilde{\nabla}g \neq 0$.

Using (2.7), we have

$$\left(\widetilde{\nabla}_{\varrho_1}g\right)(\varrho_2, \varrho_3) = -\Big\{\eta(\varrho_2)\, g(\phi\varrho_1, \varrho_3) + \eta(\varrho_3)\, g(\phi\varrho_1, \varrho_2)\Big\}. \tag{3.3}$$

A linear connection $\widetilde{\nabla}$ is said to be a quarter-symmetric non-metric connection if satisfies (3.1), (3.2) and (3.3).

Thus, a relation between the curvature tensor $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ of $\mathcal{M}^{2n+1}$ is given by

$$\widetilde{\mathcal{R}}(\varrho_1, \varrho_2)\varrho_3 = \mathcal{R}(\varrho_1, \varrho_2)\varrho_3 + 2\eta(\varrho_3)g(\phi\varrho_1, \varrho_2)\zeta + g(\varrho_1, \varrho_3)\phi\varrho_2 - g(\varrho_2, \varrho_3)\phi\varrho_1. \tag{3.4}$$

Also from (3.4), we obtain

$$\widetilde{\mathcal{S}}(\varrho_1, \varrho_2) = \mathcal{S}(\varrho_1, \varrho_2) + g(\phi\varrho_1, \varrho_2), \tag{3.5}$$

$$\widetilde{\mathcal{R}}(\zeta, \varrho_1)\varrho_2 = \eta(\varrho_2)\varrho_1 - g(\varrho_1, \varrho_2)\zeta + \eta(\varrho_2)\phi\varrho_1, \tag{3.6}$$

$$\eta\left(\widetilde{\mathcal{R}}(\varrho_1, \varrho_2)\varrho_3\right) = g(\varrho_1, \varrho_3)\eta(\varrho_2) - g(\varrho_2, \varrho_3)\eta(\varrho_1) + 2g(\phi\varrho_1, \varrho_2)\eta(\varrho_3),$$

$$\widetilde{\mathcal{R}}(\varrho_1, \varrho_2)\zeta = \eta(\varrho_1)\varrho_2 - \eta(\varrho_2)\varrho_1 - \eta(\varrho_2)\phi\varrho_1 + \eta(\varrho_1)\phi\varrho_2 + 2g(\phi\varrho_1, \varrho_2)\zeta,$$

$$\widetilde{\mathcal{S}}(\varrho_1, \zeta) = -2n\eta(\varrho_1), \tag{3.7}$$

$$\widetilde{Q}\varrho_1 = Q\varrho_1 + \phi\varrho_1,$$

$$\widetilde{r} = r,$$

$$g(\widetilde{\mathcal{R}}(\varrho_4, \varrho_1)\varrho_2, \varrho_3) + g(\widetilde{\mathcal{R}}(\varrho_4, \varrho_1)\varrho_3, \varrho_2) = 4g(\varrho_2, \varrho_3)g(\phi\varrho_4, \varrho_1), \tag{3.8}$$

where $\widetilde{\mathcal{S}}$ and $\mathcal{S}$ are the Ricci tensor, $\widetilde{r}$ and r the scalar curvature with respect to the connection $\widetilde{\nabla}$ and ∇ respectively.

4. Ricci solitons on $\left(\mathcal{M}^{2n+1}, \phi, \zeta, \eta\right)$ with $\mathcal{QSNMC}$ $\widetilde{\nabla}$

We recollect the idea of Ricci soliton (g, ζ, λ) on $\mathcal{M}^{2n+1}$ with respect to $\widetilde{\nabla}$. Then from (1.1), we have

$$\left(\widetilde{\mathfrak{L}}_\zeta g\right)(\varrho_1, \varrho_2) + 2\widetilde{\mathcal{S}}(\varrho_1, \varrho_2) + \lambda g(\varrho_1, \varrho_2) = 0. \tag{4.1}$$

With the help of (2.5) and (3.1), we get

$$\left(\widetilde{\mathfrak{L}}_\zeta g\right)(\varrho_1, \varrho_2) = 2\Big\{g(\varrho_1, \varrho_2) - \eta(\varrho_1)\eta(\varrho_2)\Big\}. \tag{4.2}$$

Using (3.5) and (4.2) in (4.1), we obtain

$$\mathcal{S}(\varrho_1, \varrho_2) = -(1+\lambda)g(\varrho_1, \varrho_2) + \eta(\varrho_1)\eta(\varrho_2) - g(\phi\varrho_1, \varrho_2). \tag{4.3}$$

Setting ϱ_2=ζ in (4.3) and using (2.9), we yield

$$\lambda = 2n. \tag{4.4}$$

This leads to the result:

Theorem 4.1. *A Ricci soliton on a Kenmotsu manifold $\mathcal{M}^{2n+1}$ with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ is always expanding.*

Corollary 4.2. *A Ricci soliton on an η-Einstein Kenmotsu manifold $\mathcal{M}^{2n+1}$ in terms of $\mathcal{QSNMC}$ $\widetilde{\nabla}$ is always expanding.*

Let $(g, \mathcal{V}, \lambda)$ be a Ricci soliton on $\mathcal{M}^{2n+1}$ in terms of the $\mathcal{QSNMC}$ $\widetilde{\nabla}$ such that $\mathcal{V}$ is pointwise collinear with ζ, i.e., $\mathcal{V}=\mu\zeta$ where μ is a function. Then (1.1) implies that

$$\mu g(\widetilde{\nabla}_{\varrho_1}\zeta, \varrho_2) + (\varrho_1\mu)\eta(\varrho_2) + \mu g(\varrho_1, \widetilde{\nabla}_{\varrho_2}\zeta) + (\varrho_2\mu)\eta(\varrho_1) + 2\widetilde{\mathcal{S}}(\varrho_1, \varrho_2) + \lambda g(\varrho_1, \varrho_2) = 0. \tag{4.5}$$

Replacing $\varrho_2=\zeta$ in (4.5) and using (2.5), (3.1) and (3.7), we get

$$(\varrho_1\mu) + (\zeta\mu)\eta(\varrho_1) - 4n\eta(\varrho_1) + 2\lambda\eta(\varrho_1) = 0. \tag{4.6}$$

Again setting $\varrho_1=\zeta$ in (4.6) we have

$$(\zeta\mu) = 2n - \lambda. \tag{4.7}$$

Equations (4.6) and (4.7), yield

$$d\mu = (2n - \lambda)\eta. \tag{4.8}$$

Applying d on (4.8) we yield

$$(2n - \lambda)d\eta = 0. \tag{4.9}$$

Since $d\eta \neq 0$, then (4.9) implies that $\lambda=2n$. Consequently from (4.8) we get $d\mu=0$, that is, μ is constant. So we have:

Theorem 4.3. *If $(g, \mathcal{V}, \lambda)$ be a Ricci soliton on a Kenmotsu manifold $\mathcal{M}^{2n+1}$ in reference to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ such that $\mathcal{V}=\mu\zeta$, then $\mathcal{V}$ is a constant multiple of ζ and the Ricci soliton is always expanding.*

5. Locally ϕ-symmetric Kenmotsu manifold in terms of $\mathcal{QSNMC}$ $\widetilde{\nabla}$

Definition 5.1 (*cf.* [25])**.** An $(2n + 1)$-dimensional Kenmotsu manifold $\mathcal{M}^{2n+1}$ is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_{\varrho_4}\mathcal{R})(\varrho_1, \varrho_2)\varrho_3) = 0, \tag{5.1}$$

for all vector fields $\varrho_1, \varrho_2, \varrho_3$ and ϱ_4 orthogonal to ζ.

As per the definition (5.1) we define a locally ϕ-symmetric Kenmotsu manifold in terms of $\mathcal{QSNMC}$ $\widetilde{\nabla}$ by

$$\phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1, \varrho_2)\varrho_3\right) = 0, \tag{5.2}$$

for all vector fields $\varrho_1, \varrho_2, \varrho_3$ and ϱ_3 orthogonal to ζ.
In view of (3.1) and (3.4), we have

$$\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1, \varrho_2)\varrho_3 = \left(\nabla_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1, \varrho_2)\varrho_3 + \eta\left(\widetilde{\mathcal{R}}(\varrho_1, \varrho_2)\varrho_3\right)\phi(\varrho_4).$$

Taking covariant differentiation of (3.4) with respect to ϱ_4 and using (2.4), (2.5) and (2.6), we get

$$\begin{aligned}\left(\nabla_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1, \varrho_2)\varrho_3 = \left(\nabla_{\varrho_4}\mathcal{R}\right)(\varrho_1, \varrho_2)\varrho_3 + 2\left\{\begin{array}{l} g(\phi\varrho_4, \phi\varrho_3)g(\phi\varrho_1, \varrho_2)\zeta - g(\varrho_4, \phi\varrho_1)\eta(\varrho_2)\eta(\varrho_3)\zeta \\ +\eta(\varrho_3)g(\phi\varrho_1, \varrho_2) - \eta(\varrho_3)\eta(\varrho_4)g(\phi\varrho_1, \varrho_2)\zeta \end{array}\right\} \\ -g(\varrho_4, \phi\varrho_2)g(\varrho_1, \varrho_3)\zeta - \eta(\varrho_2)g(\varrho_1, \varrho_3)\phi\varrho_4 + g(\varrho_2, \varrho_3)g(\varrho_4, \phi\varrho_1) + g(\varrho_2, \varrho_3)\eta(\varrho_1)\phi\varrho_4.\end{aligned} \tag{5.3}$$

Taking the inner product of (3.4) with ζ, we have

$$\eta\left(\widetilde{\mathcal{R}}(\varrho_1, \varrho_2)\varrho_3\right) = g(\varrho_1, \varrho_3)\eta(\varrho_2) - g(\varrho_2, \varrho_3)\eta(\varrho_1) + 2\eta(\varrho_3)g(\varrho_1, \varrho_2). \tag{5.4}$$

Using (2.1), (5.3) and (5.4), equation (5.2) reduces to

$$\begin{aligned}&\phi^2\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3=\phi^2(\nabla_{\varrho_4}\mathcal{R})(\varrho_1,\varrho_2)\varrho_3+2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\phi^2(\varrho_4)\\&-\eta(\varrho_2)g(\varrho_1,\varrho_3)\phi^2(\phi\varrho_4)+\eta(\varrho_1)g(\varrho_2,\varrho_3)\phi^2(\varrho_4)+g(\varrho_1,\varrho_2)\eta(\varrho_2)\phi^2(\phi\varrho_4)\\&-\eta(\varrho_1)g(\varrho_2,\varrho_3)\phi^2(\phi\varrho_4)+2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\phi^2(\phi\varrho_4).\end{aligned}\tag{5.5}$$

Taking $\varrho_1,\varrho_2,\varrho_3$ and ϱ_4 are orthogonal to ζ, equations (5.5) yield

$$\phi^2\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}(\varrho_1,\varrho_2)\varrho_3\right)=\phi^2\left(\nabla_W\mathcal{R}(\varrho_1,\varrho_2)\varrho_3\right).$$

As a result, we may assert the following:

Theorem 5.2. *A Kenmotsu manifold $\mathcal{M}^{2n+1}$ in relation to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ is locally ϕ-symmetric if and only if the Levi-Civita connection ∇ is so.*

6. ϕ-recurrent Kenmotsu manifold with $\mathcal{QSNMC}$ $\widetilde{\nabla}$

Definition 6.1 (*cf.* [25]). An $(2n+1)$-dimensional Kenmotsu manifold $\mathcal{M}^{2n+1}$ is said to be ϕ-recurrent if their exist non-zero 1-form A such that

$$\phi^2\left((\nabla_{\varrho_4}\mathcal{R})(\varrho_1,\varrho_2)\varrho_3\right)=A(\varrho_4)\mathcal{R}(\varrho_1,\varrho_2)\varrho_3,\tag{6.1}$$

for arbitrary vector fields $\varrho_1,\varrho_2,\varrho_3$ and ϱ_4.

If $\varrho_1,\varrho_2,\varrho_3$ and ϱ_4 are orthogonal to ζ then $\mathcal{M}^{2n+1}$ is called locally ϕ-recurrent manifold. If the 1-form A vanishes, then $\mathcal{M}^{2n+1}$ is reduces to ϕ-symmetric manifolds.

Analogous to the definition (6.1), we define ϕ-recurrent with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ if there exist a non-zero 1-form A such that

$$\phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3\right)=A(\varrho_4)\widetilde{\mathcal{R}}(\varrho_1,\varrho_2)\varrho_3,\tag{6.2}$$

for vector fields $\varrho_1,\varrho_2,\varrho_3$ and ϱ_4.

Suppose $\mathcal{M}^{2n+1}$ is ϕ-recurrent with $\mathcal{QSNMC}$ $\widetilde{\nabla}$, then from (2.2) and (6.2), we can write

$$-g\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3,\varrho_5\right)+\eta\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3\right)\eta(\varrho_5)=A(\varrho_4)g\left(\widetilde{\mathcal{R}}(\varrho_1,\varrho_2)\varrho_3,\varrho_5\right).\tag{6.3}$$

By virtue of (4.4) and (4.6), equation (6.3) reduces to

$$-g\left(\left(\nabla_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3,\varrho_5\right)+\eta\left(\left(\nabla_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3\right)\eta(\varrho_5)=A(\varrho_4)g\left(\widetilde{\mathcal{R}}(\varrho_1,\varrho_2)\varrho_3,U\right).\tag{6.4}$$

Using (4.5) and (3.4), in (6.4), we have

$$\begin{aligned}&-\;g\left(\left(\nabla_{\varrho_4}\mathcal{R}\right)(\varrho_1,\varrho_2)\varrho_3,\varrho_5\right)-2g(\phi\varrho_4,\phi\varrho_3)g(\phi\varrho_1,\varrho_2)\eta(\varrho_5)+g(\varrho_4,\phi\varrho_1)\eta(\varrho_2)\eta(\varrho_3)\eta(\varrho_5)-2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)g(\varrho_4,\varrho_5)\\&+\;2\eta(\varrho_3)\eta(\varrho_5)\eta(\varrho_4)g(\phi\varrho_1,\varrho_2)+g(\varrho_4,\phi\varrho_2)g(\varrho_1,\varrho_3)\eta(\varrho_5)-g(\varrho_2,\varrho_3)g(\varrho_4,\phi\varrho_1)\eta(\varrho_5)+\eta\left((\nabla_{\varrho_4}R)(\varrho_1,Y)\varrho_3\right)\eta(\varrho_5)\\&+\;2g(\phi\varrho_4,\phi\varrho_3)g(\phi\varrho_1,\varrho_2)\eta(\varrho_5)-2g(\varrho_4,\phi\varrho_1)\eta(\varrho_2)\eta(\varrho_5)\eta(\varrho_3)-g(\varrho_4,\phi\varrho_2)g(\varrho_1,\varrho_3)\eta(\varrho_5)+g(\varrho_2,\varrho_3)g(\varrho_4,\phi\varrho_1)\eta(\varrho_5)\\&=\;A(\varrho_4)\Big\{g(\mathcal{R}(\varrho_1,\varrho_2)\varrho_3,\varrho_5)+2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\eta(\varrho_5)+g(\varrho_1,\varrho_3)g(\phi\varrho_2,\varrho_5)-g(\varrho_2,\varrho_3)g(\phi\varrho_1,\varrho_5)\Big\}.\end{aligned}\tag{6.5}$$

Substituting $\varrho_2=\zeta$ in (6.5), using (2.1), it yield

$$\begin{aligned}&-g\left(\left(\nabla_{\varrho_4}\mathcal{R}\right)(\varrho_1,\varrho_2)\zeta,\varrho_5\right)-2g(\phi\varrho_1,\varrho_2)g(\varrho_4,\varrho_5)+2\eta(\varrho_5)\eta(\varrho_4)g(\phi\varrho_1,\varrho_2)+\eta\left(\left(\nabla_{\varrho_4}\mathcal{R}\right)(\varrho_1,\varrho_2)\zeta\right)\eta(\varrho_5)\\&=A(\varrho_4)\Big\{g(\mathcal{R}(\varrho_1,\varrho_2)\zeta,\varrho_5)+2g(\phi\varrho_1,\varrho_2)\eta(\varrho_5)+\eta(\varrho_1)g(\phi\varrho_2,\varrho_5)-\eta(\varrho_2)g(\phi\varrho_1,\varrho_5)\Big\}.\end{aligned}\tag{6.6}$$

Now, putting $\varrho_1=\varrho_5=\zeta$ in (6.6) and taking summation over i, $1\le i\le 2n+1$, we yield

$$-\left(\nabla_{\varrho_4}\mathcal{S}\right)(\varrho_2,\zeta)+\sum_{i=1}^{n}g\left(\left(\nabla_{\varrho_4}\mathcal{R}\right)(e_i,\varrho_2)\zeta,\zeta\right)g(e_i,\zeta)=A(\varrho_4)\mathcal{S}(\varrho_2,\zeta).\tag{6.7}$$

From (6.7), we consider the second term

$$g\left(\left(\nabla_{\varrho_4}\mathcal{R}\right)(e_i,\varrho_2)\zeta,\zeta\right) = g\left(\nabla_{\varrho_4}\mathcal{R}(e_i,\varrho_2)\zeta,\zeta\right) - g\left(\mathcal{R}\left(\nabla_{\varrho_4}e_i,\varrho_2\right)\zeta,\zeta\right) - g\left(\mathcal{R}(e_i,\nabla_{\varrho_4}\varrho_2)\zeta,\zeta\right) - g\left(\mathcal{R}(e_i,\varrho_2)\nabla_{\varrho_4}\zeta,\zeta\right), \quad (6.8)$$

at $p \in \mathcal{M}$. In local co-ordinates $\nabla_{\varrho_4}e_i{=}\varrho_4^j\Gamma_{ji}^h e_h$, where Γ_{ji}^h are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij}{=}\delta_{ij}, \delta_{ij}$ is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_{\varrho_4}e_i{=}0$. Since $\mathcal{R}$ is skew-symmetric, also we have

$$g\left(\mathcal{R}(e_i,\nabla_{\varrho_4}\varrho_2)\zeta,\zeta\right) = 0. \quad (6.9)$$

By the use of (6.9) and $\nabla_{\varrho_4}e_i{=}0$, equation (6.8) yield

$$g\left((\nabla_{\varrho_4}\mathcal{R})(e_i,\varrho_2)\zeta,\zeta\right) = g\left(\nabla_{\varrho_4}\mathcal{R}(e_i,\varrho_2(\zeta,\zeta\right) - g\left(\mathcal{R}(e_i,\varrho_2)\nabla_{\varrho_4}\zeta,\zeta\right).$$

With reference to $g\left(\mathcal{R}(e_i,\varrho_2)\zeta,\zeta\right) = -g(\mathcal{R}(\zeta,\zeta)e_i,\varrho_2) = 0$ and $\nabla_{\varrho_4}g = 0$, we have

$$g\left(\nabla_{\varrho_4}\mathcal{R}(e_i,\varrho_2)\zeta,\zeta\right) - g\left(\mathcal{R}(e_i,\varrho_2)\zeta,\nabla_{\varrho_4}\zeta\right) = 0$$

which implies that

$$g\left((\nabla_{\varrho_4}\mathcal{R})(e_i,\varrho_2)\zeta,\zeta\right) = -g\left(\mathcal{R}(e_i,\varrho_2)\zeta,\nabla_{\varrho_4}\zeta\right) - g\left(\mathcal{R}(e_1,\varrho_2)\nabla_{\varrho_4}\zeta,\zeta\right) \quad (6.10)$$

and

$$g\left((\nabla_{\varrho_4}\mathcal{R})(e_i,\varrho_2)\zeta,\zeta\right) = 0. \quad (6.11)$$

On the other hand, we have

$$\left(\nabla_{\varrho_4}\mathcal{S}\right)(\varrho_2,\zeta) = -2ng(\phi\varrho_4,\phi\varrho_2) - \mathcal{S}(\varrho_2,\varrho_4) - 2n\eta(\varrho_2)\eta(\varrho_4). \quad (6.12)$$

In view of (6.11) and (6.12), equation (6.7) reduce to

$$\mathcal{S}(\varrho_2,\varrho_4) + 2ng(\varrho_2,\varrho_4) = -2n\eta(\varrho_2)A(\varrho_4).$$

Replacing ϱ_2 and ϱ_4 by $\phi\varrho_2$ and $\phi\varrho_4$ in above equations and using (2.3) and (2.9), we get

$$\mathcal{S}(\varrho_2,\varrho_4) = -2ng(\varrho_2,\varrho_4). \quad (6.13)$$

Thus we state the result:

Theorem 6.2. *A ϕ-recurrent Kenmotsu manifold $\mathcal{M}^{2n+1}$ with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ is an Einstein manifold.*

Again, in view of (6.13), equation (3.5) can be written as

$$\widetilde{\mathcal{S}}(\varrho_2,\varrho_4) = -2ng(\varrho_2,\varrho_4) + g(\phi\varrho_2,\varrho_4). \quad (6.14)$$

By virtue of (6.14) we have from (4.1) that

$$g\left(\widetilde{\nabla}_{\varrho_2}\zeta,\varrho_4\right) + g\left(\varrho_2,\widetilde{\nabla}_{\varrho_4}\zeta\right) + 2(\lambda-2n)g\left(\varrho_2,\varrho_4\right) + 2g\left(\phi\varrho_2,\varrho_4\right) = 0. \quad (6.15)$$

Replacing $\varrho_2{=}\varrho_4{=}\zeta$ in (6.15) and using (2.5) and (3.1), we get $g(\nabla_\zeta\zeta,\zeta){=}{-}(\lambda-2n)$. But $g(\nabla_{\varrho_2}\zeta,\zeta){=}0$ for any vector field ϱ_2 on $\mathcal{M}^{2n+1}$. Since ζ has a constant term. Hence we get $\lambda{=}2n$. As per this consequence (6.15) take the form

$$g\left(\widetilde{\nabla}_{\varrho_2}\zeta,\varrho_4\right) + g\left(\varrho_2,\widetilde{\nabla}_{\varrho_4}\zeta\right) + 2g\left(\phi\varrho_2,\varrho_4\right) = 0.$$

Again, putting $\varrho_2{=}\zeta$ in (6.10), using (2.5) and (3.1), we have $g\left(\nabla_\zeta\zeta,\varrho_4\right){=}0$ for any vector field ϱ_4 on $\mathcal{M}^{2n+1}$, and then $\nabla_\zeta\zeta{=}0$, i.e., ζ is geodesic vector field. So we have:

Theorem 6.3. *If (g,ζ,λ) be a Ricci soliton on a ϕ-recurrent Kenmotsu manifold $\mathcal{M}^{2n+1}$ with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ then*

(i) *Ricci soliton (g,ζ,λ) is always expanding.*

(ii) *ζ is a geodesic vector field.*

7. Locally projective ϕ-symmetric Kenmotsu manifold with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$

Definition 7.1. An $(2n+1)$-dimensional dimensional Kenmotsu manifold is said to be locally projective ϕ-symmetric in term of $\mathcal{QSNMC}$ if

$$\phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3\right)=0 \tag{7.1}$$

for all vector fields $\varrho_1,\varrho_2,\varrho_3,\varrho_4$ orthogonal to ζ.

where $\widetilde{\mathcal{P}}$ is the projective curvature tensor with respect to such connection defined as

$$\widetilde{\mathcal{P}}(\varrho_1,\varrho_2)\varrho_3=\widetilde{\mathcal{R}}(\varrho_1,\varrho_2)\varrho_3-\frac{1}{n-1}\Big\{\widetilde{\mathcal{S}}(\varrho_2,\varrho_3)\varrho_1-\widetilde{\mathcal{S}}(\varrho_1,\varrho_3)\varrho_2\Big\}, \tag{7.2}$$

where $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ are the Riemannian curvature tensor and Ricci tensor with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$.
In view of equations (3.1), we can write

$$\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3=\left(\nabla_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3+\eta\left(\widetilde{\mathcal{P}}(\varrho_1,\varrho_2)\varrho_3\right)\phi\varrho_4. \tag{7.3}$$

Now, differentiating (7.2) with respect to ϱ_4, we get

$$\left(\nabla_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3=\left(\nabla_{\varrho_4}\widetilde{\mathcal{R}}\right)(\varrho_1,\varrho_2)\varrho_3-\frac{1}{n-1}\Big\{\left(\nabla_{\varrho_4}\widetilde{\mathcal{S}}\right)(\varrho_2,\varrho_3)\varrho_1-\left(\nabla_{\varrho_4}\widetilde{\mathcal{S}}\right)(\varrho_1,\varrho_3)\varrho_2\Big\}. \tag{7.4}$$

Using (3.7) and (4.5) in (7.4), we yield

$$\begin{aligned}\left(\nabla_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3 =& \left(\nabla_{\varrho_4}\mathcal{R}\right)(\varrho_1,\varrho_2)Z+2g(\phi W,\phi Z)g(\phi\varrho_1,\varrho_2)\zeta-2g(W,\phi\varrho_1)\eta(\varrho_2)\eta(\varrho_3)\zeta+2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\varrho_4\\ &-2g(\phi\varrho_1,\varrho_2)\eta(\varrho_3)\eta(\varrho_4)\zeta-g(\varrho_4,\phi\varrho_2)g(\varrho_1,\varrho_3)\zeta-\eta(\varrho_2)g(\varrho_1,\varrho_3)\phi\varrho_4\\ &+g(\varrho_2,\varrho_3)g(\varrho_4,\phi\varrho_1)\zeta+g(\varrho_2,\varrho_3)\eta(\varrho_1)\phi\varrho_4\\ &-\tfrac{1}{n-1}\left\{\begin{array}{l}\left(\nabla_{\varrho_4}\mathcal{S}\right)(\varrho_2,\varrho_3)\eta(\varrho_1)-g(\varrho_4,\phi\varrho_2)\eta(\varrho_3)-g(\varrho_3,\phi\varrho_4)\eta(\varrho_2)\\ -\left(\nabla_{\varrho_4}\mathcal{S}\right)(\varrho_1,\varrho_3)\eta(\varrho_2)+g(\varrho_4,\phi\varrho_1)\eta(\varrho_3)+\eta(\varrho_1)g(\phi\varrho_4,\varrho_3)\end{array}\right\},\end{aligned} \tag{7.5}$$

which on using (7.1), reduces to

$$\begin{aligned}\left(\nabla_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3 =& \left(\nabla_{\varrho_4}\mathcal{P}\right)(\varrho_1,\varrho_2)\varrho_3+2g(\phi\varrho_4,\phi\varrho_3)g(\phi\varrho_1,\varrho_2)\zeta-2g(\varrho_4,\phi\varrho_1)\eta(\varrho_2)\eta(\varrho_3)\zeta\\ +& 2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\varrho_4-2g(\phi\varrho_1,\varrho_2)\eta(\varrho_3)\eta(\varrho_4)\zeta-g(\varrho_4,\phi\varrho_2)g(\varrho_1,\varrho_3)\zeta\\ -& \eta(\varrho_2)g(\varrho_1,\varrho_3)\phi\varrho_4+g(\varrho_2,\varrho_3)g(\varrho_4,\phi\varrho_1)\zeta+g(\varrho_2,\varrho_3)\eta(\varrho_1)\phi\varrho_4.\end{aligned} \tag{7.6}$$

Now, using equation (3.4) and (3.5) in (7.2), we get

$$\widetilde{\mathcal{P}}(\varrho_1,\varrho_2)\varrho_3=\mathcal{P}(\varrho_1,\varrho_2)\varrho_3+2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\zeta+g(\varrho_1,\varrho_3)\phi\varrho_2-g(\varrho_2,\varrho_3)\phi\varrho_1-\frac{1}{n-1}\Big\{g(\phi\varrho_2,\varrho_3)\varrho_1-g(\phi\varrho_1,\varrho_3)\varrho_2\Big\}. \tag{7.7}$$

Taking inner product of (7.7) with ζ and using (2.1), (2.8) and (2.9), we get

$$\eta\left(\widetilde{\mathcal{P}}(\varrho_1,\varrho_2)\varrho_3\right)=g(\varrho_1,\varrho_3)\eta(\varrho_2)-g(\varrho_2,\varrho_3)\eta(\varrho_1)-\frac{1}{n-1}\Big\{g(\phi\varrho_2,\varrho_3)\eta(\varrho_1)-g(\phi\varrho_1,\varrho_3)\eta(\varrho_2)\Big\}. \tag{7.8}$$

Now using (2.2), (7.6) and (7.8), equation (7.3) reduces to

$$\begin{aligned}\phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3\right) =& \phi^2\left(\left(\nabla_{\varrho_4}\mathcal{P}\right)(\varrho_1,\varrho_2)\varrho_3\right)+2g(\phi\varrho_4,\phi\varrho_3)g(\phi\varrho_1,\varrho_2)\phi^2\zeta-2g(\varrho_4,\phi\varrho_1)\eta(\varrho_2)\eta(\varrho_3)\phi^2\zeta\\ +& 2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\phi^2\varrho_4-2g(\phi\varrho_1,\varrho_2)\eta(\varrho_3)\eta(\varrho_4)\phi^2\zeta-g(\varrho_4,\phi\varrho_2)g(\varrho_1,\varrho_3)\phi^2\zeta\\ -& \eta(\varrho_2)g(\varrho_1,Z)\phi^2(\phi\varrho_4)+g(\varrho_2,\varrho_3)g(\varrho_4,\phi\varrho_1)\phi^2\zeta+g(\varrho_2,\varrho_3)\eta(\varrho_1)\phi^2(\phi\varrho_4)\\ +& \left\{g(\varrho_1,\varrho_3)\eta(\varrho_2)-g(\varrho_2,\varrho_3)\eta(\varrho_1)-\frac{1}{n-1}\left\{g(\phi\varrho_2,\varrho_3)\eta(\varrho_1)-g(\phi\varrho_1,\varrho_3)\eta(\varrho_2)\right\}\right\}\phi^2(\phi\varrho_4).\end{aligned}$$

By taking ϱ_1, ϱ_2, ϱ_3 and ϱ_4 orthogonal to ζ, above equations reduces to

$$\phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3\right)=\phi^2\left(\left(\nabla_{\varrho_4}\mathcal{P}\right)(\varrho_1,\varrho_2)\varrho_3\right).$$

Hence, we state the outcome:

Theorem 7.2. *An $(2n+1)$-dimensional Kenmotsu manifold is locally projective ϕ-symmetric with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ if and only if it is locally projective ϕ-symmetric with respect to the Levi-Civita connection ∇.*

Again from (2.2), (7.3), (7.5) and (7.8), we have

$$\begin{aligned} \phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3\right) &= \phi^2\left(\left(\nabla_{\varrho_4}\mathcal{R}\right)(\varrho_1,\varrho_2\varrho_2)\varrho_3\right) + 2g(\phi\varrho_4,\phi\varrho_3)g(\phi\varrho_1,\varrho_2)\phi^2\zeta - 2g(\varrho_4,\phi\varrho_1)\eta(\varrho_2)\eta(\varrho_3)\phi^2\zeta \\ &+ 2\eta(\varrho_3)g(\phi\varrho_1,\varrho_2)\phi^2(\varrho_4) - 2g(\phi\varrho_1,\varrho_2)\eta(\varrho_3)\eta(\varrho_4)\phi^2\zeta - g(\varrho_4,\phi\varrho_2)g(\varrho_1,\varrho_3)\phi^2\zeta \\ &- \eta(\varrho_2\varrho_2)g(\varrho_1,\varrho_3)\phi^2(\phi\varrho_4) + g(\varrho_2,\varrho_3)g(\varrho_4,\phi\varrho_1)\phi^2\zeta + g(\varrho_2,\varrho_3)\eta(\varrho_1)\phi^2(\phi\varrho_4) \\ &- \frac{1}{n-1}\left\{\begin{array}{l} \left(\nabla_{\varrho_4}\mathcal{S}\right)(\varrho_2,\varrho_3)\eta(\varrho_1) - g(\varrho_4,\phi\varrho_2)\eta(\varrho_3) - g(\varrho_3,\phi\varrho_4)\eta(\varrho_2) \\ -\left(\nabla_{\varrho_4}\mathcal{S}\right)(\varrho_1,\varrho_3)\eta(\varrho_2) + g(\varrho_4,\phi\varrho_1)\eta(\varrho_3) + \eta(\varrho_1)g(\phi\varrho_4,\varrho_3) \end{array}\right\} \\ &+ \left\{g(\varrho_1,\varrho_3)\eta(\varrho_2) - g(\varrho_2,\varrho_3)\eta(\varrho_1) - \frac{1}{n-1}\left\{g(\phi\varrho_2,\varrho_3)\eta(\varrho_1) - g(\phi\varrho_1,\varrho_3)\eta(\varrho_2)\right\}\right\}\phi^2(\phi\varrho_4). \end{aligned} \tag{7.9}$$

Taking ϱ_1, ϱ_2, ϱ_3 and ϱ_4 orthogonal to ζ, equation (7.9) reduces to

$$\phi^2\left(\left(\widetilde{\nabla}_{\varrho_4}\widetilde{\mathcal{P}}\right)(\varrho_1,\varrho_2)\varrho_3\right) = \phi^2\left(\left(\nabla_{\varrho_4}\mathcal{P}\right)(\varrho_1,\varrho_2)\varrho_3\right).$$

Therefore, we have:

Theorem 7.3. *A ϕ-symmetric Kenmotsu manifold admitting $\mathcal{QSNMC}$ $\widetilde{\nabla}$ is locally projective ϕ-symmetric if and only if it is locally ϕ-symmetric with respect to the Levi-Civita connection ∇.*

8. ϕ-projectively flat Kenmotsu manifold with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$

Definition 8.1. An $(2n+1)$-dimensional Kenmotsu manifold $\mathcal{M}^{2n+1}$ is said to be ϕ-projectively flat with respect to quarter-symmetric non-metric connection if it satisfies

$$\phi^2\left(\widetilde{\mathcal{P}}(\phi\varrho_1,\phi\varrho_2)\phi\varrho_3\right) = 0$$

for all vector fields ϱ_1, ϱ_2,ϱ_3 and ϱ_4 on $\mathcal{M}^{2n+1}$.

Let $\mathcal{M}^{2n+1}$ be a ϕ-projectively flat Kenmotsu manifold with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$, then $\phi^2\left(\widetilde{\mathcal{P}}(\phi\varrho_1,\phi\varrho_2)\phi\varrho_3\right) = 0$ holds if and only if

$$g\left(\widetilde{\mathcal{P}}(\phi\varrho_1,\phi\varrho_2)\phi\varrho_3,\phi\varrho_4\right) = 0,$$

for any $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in \mathcal{M}^{2n+1}$.
So in view of (7.2), ϕ-projectively flat means

$$g\left(\widetilde{\mathcal{R}}(\phi\varrho_1,\phi\varrho_2)\phi\varrho_3,\phi\varrho_4\right) = \frac{1}{n-1}\left\{\widetilde{\mathcal{S}}(\phi\varrho_2,\phi\varrho_3)g(\phi\varrho_1,\phi\varrho_4) - \widetilde{\mathcal{S}}(\phi\varrho_1,\phi\varrho_3)g(\phi\varrho_2,\phi\varrho_4)\right\} \tag{8.1}$$

which on using (3.4) and (3.5) equation (8.1) reduces to

$$\begin{aligned} &g\left(\mathcal{R}(\phi\varrho_1,\phi\varrho_2)\phi\varrho_3,\phi\varrho_4\right) - g(\phi\varrho_1,\phi\varrho_3)g(\varrho_2,\phi\varrho_4) + g(\phi\varrho_2,\phi\varrho_3)g(\phi\varrho_2,\phi\varrho_3)g(\phi\varrho_1,\phi\varrho_4) \\ &= \frac{1}{n-1}\left\{\mathcal{S}(\phi\varrho_2,\phi\varrho_3)g(\phi\varrho_1,\phi\varrho_4) - g(\varrho_2,\phi\varrho_3)g(\phi\varrho_1,\phi\varrho_4) - \mathcal{S}(\phi\varrho_1,\phi\varrho_3)g(\phi\varrho_2,\phi\varrho_4) - g(\varrho_1,\phi\varrho_3)g(\phi\varrho_2,\phi\varrho_4)\right\}. \end{aligned} \tag{8.2}$$

Let $\{e_1, e_2, \ldots, e_{2n-1}, \zeta\}$ be a local orthonormal basis of the vector fields in $\mathcal{M}^{2n+1}$. Using the fact that $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n-1}, \zeta\}$ is also local orthonormal basis. Putting ϱ_1=ϱ_4=e_i in equations (8.2) and summing over i, we get

$$\begin{aligned} &\sum_{i=1}^{n-1}\left\{g\left(\mathcal{R}(\phi e_i,\phi\varrho_2)\phi\varrho_3,\phi e_i\right) - g(\phi e_i,\phi\varrho_3)g(\varrho_2,\phi e_i) + g(\phi\varrho_2,\phi\varrho_3)g(\phi\varrho_2,\phi\varrho_3)g(\phi e_i,\phi e_i)\right\} \\ &= \frac{1}{n-1}\sum_{i=1}^{n-1}\left\{\mathcal{S}(\phi\varrho_2,\phi\varrho_3)g(\phi e_i,\phi e_i) - g(\varrho_2,\phi\varrho_3)g(\phi e_i,\phi e_i) - \mathcal{S}(\phi e_i,\phi\varrho_3)g(\phi\varrho_2,\phi e_i) - g(e_i,\phi\varrho_3)g(\phi\varrho_2,\phi e_i)\right\}. \end{aligned} \tag{8.3}$$

Also, we notice that (*cf.* [19]):

(a) $\sum_{i=1}^{n-1} g(\mathcal{R}(\phi e_i, \phi\varrho_2)\phi\varrho_3, \phi e_i) = \mathcal{S}(\phi\varrho_2, \phi\varrho_3) + g(\phi\varrho_2, \phi\varrho_3)$.

(b) $\sum_{i=1}^{n-1} g(\phi e_i, \phi\varrho_3)\mathcal{S}(\phi\varrho_2, \phi e_i) = \mathcal{S}(\phi\varrho_2, \phi\varrho_3)$.

(c) $\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1$.

(d) $\sum_{i=1}^{n-1} g(\phi e_i, \phi\varrho_3)g(\phi\varrho_2, \phi e_i) = g(\phi\varrho_2, \phi\varrho_3)$.

Therefore from (d) of the above equation and (8.3), we have

$$\mathcal{S}(\phi\varrho_2, \phi\varrho_3) = 2n\left\{-2n - \frac{1}{2n} - 1\right\} g(\phi\varrho_2, \phi\varrho_3). \tag{8.4}$$

Using (2.3) and (2.9), equation (8.4) takes the form

$$\mathcal{S}(\varrho_2, \varrho_3) = 2n\left\{-2n - \frac{1}{2n} - 1\right\} g(\varrho_2, \varrho_3) + \left\{2n - 2n\left(-2n - \frac{1}{2n} - 1\right)\right\} \eta(\varrho_2)\eta(\varrho_3).$$

Thus, in this situation we have:

Theorem 8.2. *A ϕ-projectively flat Kenmotsu manifold admitting $\mathcal{QSNMC}$ $\widetilde{\nabla}$ is an η-Einstein manifold with respect to the Levi-Civita connection ∇.*

Also, in view of Theorem 8.2, $\mathcal{M}^{2n+1}$ is an η-Einstein manifold. In the same fashion as the proof of Theorem 6.2, we state that

Theorem 8.3. *If (g, ζ, λ) is a Ricci soliton on ϕ-projectively flat Kenmotsu manifold $\mathcal{M}^{2n+1}$ with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$ then*

(i) *Ricci soliton (g, ζ, λ) is always expanding.*

(ii) *ζ is a geodesic vector field.*

By virtue of Theorem 6.2 and Theorem 8.2, we state that

Corollary 8.4. *If (g, ζ, λ) is a Ricci soliton on Kenmotsu manifold then ϕ-recurrent and ϕ-projectively flat are equivalent with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$.*

9. Conditions for a vector field in Kenmotsu manifold to be Killing with respect to $\mathcal{QSNMC}$ $\widetilde{\nabla}$

Definition 9.1 (*cf.* [31])**.** A vector field ϱ_1 on a Kenmotsu manifold is said to be conformal Killing vector field if

$$\mathfrak{L}_{\varrho_1} g = \rho\, g,$$

where ρ is a function on the manifold. If ρ=0, then the vector field ρ is said to be a Killing vector field.

Let the vector field ϱ_1 be a conformal Killing vector field on $\mathcal{M}^{2n+1}$ with respect to connection $\widetilde{\nabla}$. Then for a function ρ, we have

$$\left(\widetilde{\mathfrak{L}}_{\varrho_1} g\right)(\varrho_2, \varrho_3) = \rho g(\varrho_2, \varrho_3). \tag{9.1}$$

From (2.5) and (3.1) we get $\widetilde{\nabla}_\zeta\zeta$=0. So the integral curves are geodesics then from (9.1) we have by setting ϱ_2=ϱ_3=ζ

$$\rho = \left(\widetilde{\mathfrak{L}}_{\varrho_1} g\right)(\zeta, \zeta).$$

On the other hand

$$\left(\widetilde{\mathfrak{L}}_{\varrho_1} g\right)(\zeta, \zeta) = 2g\left(\widetilde{\nabla}_\zeta \varrho_1, \zeta\right)$$

and

$$2\widetilde{\nabla}_\zeta\left(g(\varrho_1, \zeta)\right) = 2g\left(\widetilde{\nabla}_\zeta \varrho_1, \zeta\right).$$

Therefore we have

$$\rho = \left(\widetilde{\mathfrak{L}}_{\varrho_1} g\right)(\zeta, \zeta) = 2\widetilde{\nabla}_\zeta(g(\varrho_1, \zeta)) = 2g\left(\widetilde{\nabla}_\zeta \varrho_1, \zeta\right).$$

If ϱ_1 is orthogonal to ζ, ρ=0 and hence $\left(\widetilde{\mathfrak{L}}_{\varrho_1} g\right)$=0, that is, ϱ_1 is a Killing vector field.

Thus we state:

Theorem 9.2. *If a conformal Killing vector field ϱ_1 on $\mathcal{M}^{2n+1}$ with respect to $QSNMC$ $\widetilde{\nabla}$ is orthogonal to ζ, then ϱ_1 is Killing.*

Corollary 9.3. *If a conformal Killing vector field ϱ_1 on $\mathcal{M}^{2n+1}$ with respect to $QSNMC$ $\widetilde{\nabla}$ is orthogonal to ζ, then Ricci soliton (g, ζ, λ) is always expanding.*

Let ϱ_5 be a vector field on $\mathcal{M}^{2n+1}$ such that $\widetilde{\mathfrak{L}}_{\varrho_5}\widetilde{R}$=0. In view of (3.8), we have

$$\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)\left(\widetilde{\widetilde{R}}(\varrho_4, \varrho_1)\varrho_2, \varrho_3\right) + \left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)\left(\widetilde{R}(\varrho_4, \varrho_1)Z\varrho_3, \varrho_2\right) = 4\left\{\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\varrho_2, \varrho_3)g(\phi\varrho_4, \varrho_1) + g(\varrho_2, \varrho_3)\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\phi\varrho_4, \varrho_1)\right\}. \quad (9.2)$$

Taking ϱ_4=ϱ_2=ϱ_3=ζ in (9.2) and using (3.6), we get

$$\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\varrho_1, \zeta) = \eta(\varrho_1)\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\zeta, \zeta) + \left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\phi\varrho_1, \zeta). \quad (9.3)$$

Again putting ϱ_4=ϱ_2=ζ in (9.2) we have

$$\begin{aligned}&\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\varrho_1, \varrho_3) - \eta(\varrho_1)\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\zeta, \varrho_3) + \left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\phi\varrho_1, \varrho_3) + \left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\varrho_1, \zeta)\eta(\varrho_3) \\ &-g(\varrho_1, \varrho_3)\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\zeta, \zeta) + \eta(\varrho_3)\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\varrho_1, \zeta) = 0.\end{aligned} \quad (9.4)$$

In view of (9.3) and (9.4), we yield

$$\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\varrho_1, \varrho_3) = g(\varrho_1, \varrho_3)\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\zeta, \zeta).$$

Since $\widetilde{S}(\zeta, \zeta)$=-2$n$, we apply the Lie-derivative on it and using $\widetilde{\mathfrak{L}}_{\varrho_5}\widetilde{\mathcal{R}}$=0 we get $\widetilde{\mathfrak{L}}_{\varrho_5}\widetilde{S}$=0, that is, $\widetilde{S}\left(\widetilde{\mathfrak{L}}_{\varrho_5}\zeta, \zeta\right)$=0. But $\widetilde{S}(\zeta, \zeta)$=-2$n$. So $\widetilde{\mathfrak{L}}_{\varrho_5}\zeta$=0 and hence $g\left(\widetilde{\mathfrak{L}}_{\varrho_5}\zeta, \zeta\right)$=0 therefore $\left(\widetilde{\mathfrak{L}}_{\varrho_5} g\right)(\zeta, \zeta)$=0. So, in view of (9.1), we get ρ=0 this implies that vector field ϱ_5 is Killing vector field.

Thus state the following:

Theorem 9.4. *If a vector field ϱ_5 on $\mathcal{M}^{2n+1}$ with respect to $QSNMC$ $\widetilde{\nabla}$ leaves the curvature tensor invariant, then ϱ_5 is Killing vector field.*

Corollary 9.5. *If a vector field ϱ_5 on $\mathcal{M}^{2n+1}$ with respect to $QSNMC$ $\widetilde{\nabla}$ leaves the curvature tensor invariant, then Ricci soliton (g, ζ, λ) is always expanding.*

10. An example

We consider a 3-dimensional manifold, that is,

$$\mathcal{M}^3 = \left\{(x_1, x_2, x_3) \in R^3, (x_1, x_2, x_3) \neq (0, 0, 0)\right\},$$

where (x_1, x_2, x_3) are standard coordinates in $\mathcal{R}^3$. We define the vector fields

$$e_1 = x_3\frac{\partial}{\partial x_1}, \quad e_2 = x_3\frac{\partial}{\partial x_2}, \quad e_3 = -x_3\frac{\partial}{\partial x_3}$$

are linearly independent at each point of $\mathcal{M}^3$. The non-metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \ \ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(\varrho_3) = g(\varrho_3, e_3)$ for any $\varrho_3 \in \chi(\mathcal{M})$. Also we define a (1, 1)-tensor field ϕ as

$$\phi(e_1) = -e_2, \ \ \phi(e_3) = e_1, \ \ \phi(e_3) = 0.$$

Using the linearity property of ϕ and g we have

$$\eta(e_3) = 1, \ \ \phi^2\varrho_3 = -\varrho_3 + \eta(\varrho_3)e_3, \ \ g(\phi\varrho_3, \phi\varrho_4) = g(\varrho_3, \varrho_4) - \eta(\varrho_3)\eta(\varrho_4),$$

for any $\varrho_3, \varrho_4 \in \chi(\mathcal{M}^3)$. Thus for e_3=ζ, the structure (ϕ, ζ, η, g) define an almost contact metric structure on M^3 Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2.$$

Now we recall the Koszul's formula

$$2g(\nabla_{\varrho_1}\varrho_2, \varrho_3) = \varrho_1 g(\varrho_2, \varrho_3) + \varrho_2(\varrho_3, \varrho_3) - \varrho_3(\varrho_1, \varrho_2) - g(\varrho_1, [\varrho_2, \varrho_3]) - g(\varrho_2, [\varrho_1, \varrho_3]) + g(\varrho_3, [\varrho_1, \varrho_2]).$$

Using the above formula we can calculate

$$\nabla_{e_1}e_1 = 0, \ \ \nabla_{e_1}e_2 = 0, \ \ \nabla_{e_1}e_3 = e_1.$$

$$\nabla_{e2}e_1 = 0, \ \ \nabla_{e_2}e_2 = 0, \ \ \nabla_{e2}e_3 = e_2.$$

$$\nabla_{e_3}e_1 = 0, \ \ \nabla_{e3}e_2 = 0, \ \ \nabla_{e_3}e_3 = 0.$$

Thus, the manifold satisfies $\nabla_{\varrho_1}\zeta$=-ϱ_1+$\eta(\varrho_1)e_3$, for ζ=e_3. With the help of (3.1), we calculate $\tilde{\nabla}$ as follows

$$\tilde{\nabla}_{e_1}e_1 = 0, \ \ \tilde{\nabla}_{e_1}e_2 = 0, \ \ \tilde{\nabla}_{e_1}e_3 = e_1 - e_2.$$

$$\tilde{\nabla}_{e2}e_1 = 0, \ \ \tilde{\nabla}_{e_2}e_2 = 0, \ \ \tilde{\nabla}_{e2}e_3 = e_2 + e_1.$$

$$\tilde{\nabla}_{e_3}e_1 = 0, \ \ \tilde{\nabla}_{e_3}e_2 = 0, \ \ \tilde{\nabla}_{e_3}e_3 = 0.$$

In view of (3.2), the torsion tensor $\widetilde{\mathcal{T}}$ in term of $\mathcal{QSNMC}$ $\tilde{\nabla}$ as follows:

$$\widetilde{\mathcal{T}}(e_i, e_j) = 0, \ \ \forall i = 1, 2, 3 \text{ and } \tilde{\mathrm{T}}(e_1, e_2) = 0, \ \ \widetilde{\mathcal{T}}(e_1, e_3) = -e_2, \quad \widetilde{\mathcal{T}}(e_2, e_3) = e_1.$$

Also we have

$$(\tilde{\nabla}_{e_1}g)(e_2, e_3) = 1, \ \ (\tilde{\nabla}_{e_2}g)(e_3, e_1) = -1, \ \ (\tilde{\nabla}_{e3}g)(e_1, e_2) = 0.$$

Thus $\mathcal{M}^3$ is a 3-dimensional Kenmotsu manifold with $\mathcal{QSNMC}$ $\tilde{\nabla}$.

We also calculate

$$\begin{cases} \widetilde{\mathcal{R}}(e_1, e_2)e_3 = 0, \ \ \widetilde{\mathcal{R}}(e_2, e_3)e_3 = -(e_2 + e_1), \ \ \widetilde{\mathcal{R}}(e_1, e_2)e_2 = 0, \\ \widetilde{\mathcal{R}}(e_1, e_3)e_3 = -e_1, \ \ \widetilde{\mathcal{R}}(e_2, e_3)e_2 = 0, \ \ \widetilde{\mathcal{R}}(e_1, e_3)e_2 = 0, \\ \widetilde{\mathcal{R}}(e_1, e_2)e_1 = 0, \ \ \widetilde{\mathcal{R}}(e_2, e_3)e_1 = 0, \ \ \widetilde{\mathcal{R}}(e_1, e_3)e_1 = 0 \end{cases}$$

and the Ricci tensor as follows:

$$\begin{cases} \widetilde{\mathcal{S}}(e_1, e_1) = 1, \quad \widetilde{\mathcal{S}}(e_2, e_2) = 1, \quad \widetilde{\mathcal{S}}(e_3, e_3) = 0, \\ \widetilde{\mathcal{S}}(e_1, e_2) = 0, \quad \widetilde{\mathcal{S}}(e_2, e_3) = 0, \quad \widetilde{\mathcal{S}}(e_2, e_3) = 0. \end{cases}$$

So, the scalar curvature $\widetilde{r}$ is given by:

$$\widetilde{r} = \sum_{i=1}^{3} g(e_i e_i)\widetilde{\mathcal{S}}(e_i, e_i) = 2.$$

If (g, ζ, λ) be a Ricci soliton on $\mathcal{M}^3$ with respect to quarter symmetric non-metric connection $\widetilde{\nabla}$, then from (4.3), we get

$$\widetilde{r} = -2n + \lambda(2n + 1),$$

which implies that

$$\lambda = \frac{2(1+n)}{2n+1}.$$

Thus the Ricci soliton (g, ζ, λ) on a Kenmotsu manifold $\mathcal{M}^3$ with respect to quarter symmetric non-metric connection $\widetilde{\nabla}$ is always expanding. So Theorem 4.1 is verified.

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