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Study on the degenerate higher-order new Fubini-type numbers and polynomials

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Abstract

In this paper, we first consider the new two-variables Fubini-type numbers and polynomials of the order r arising from the fermionic p-adic integral on $\underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r-times}$ and those degenerate version. We derive some interesting properties and recurrence relations on those numbers and polynomials. Second, we study the degenerate new two-variables Fubini-type numbers and polynomials of order r as one of the generalizations of these new two-variables Fubini-type numbers and polynomials by using the λ-Sheffer sequences. When $\lambda \to 0$, we obtain interesting identities for those degenerate new two-variables Fubini-type numbers and polynomials of order r. Some of them include the degenerate and other special polynomials and numbers such as the degenerate Bernoulli polynomials and numbers of order s, the degenerate Euler polynomials and numbers of order s, the degenerate Frobenius-Euler polynomials of order r, the degenerate Lah-Bell polynomials, and so on.

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1. Introduction

Many mathematicians have studied degenerate versions of some special polynomials and numbers considering a person's psychological burden or the surrounding environment, and they've discovered some interesting results (*cf.* [1, 10, 12, 14, 18]). Furthermore, one of the important tools for finding the combinatorial identities for the degenerate version of special numbers and polynomials is the umbral calculus (*cf.* [7, 10, 12, 21, 23]). The Fubini numbers and polynomials also play important role in connecting relationship between special numbers and polynomials. In particular, the generating function of two-variables Fubini numbers and polynomials can be represented by the fermionic p-adic integral on $\mathbb{Z}_p$ can be represented by the p-adic Volkenborn integral on $\mathbb{Z}_p$ (*cf.* [15, 16]). Various identities of two-variables Fubini numbers and polynomials have been studied in [8, 9, 13, 18, 27].

Recently, we considered the new two-variables Fubini-type numbers and polynomials and studied their interesting properties. In this paper, we first define the new two-variables Fubini-type numbers and polynomials of the order r arising from the fermionic p-adic integral on $\underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r-times}$ and those degenerate version. We explore recurrence

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relations on those numbers. Second, we study the representations of these new polynomials as linear combinations of well-known special polynomials (such as the degenerate Bernoulli polynomials and numbers of order s, the degenerate Euler polynomials and numbers of order s, the degenerate Frobenius-Euler polynomials of order r, the degenerate Lah-Bell polynomials, and so on) by using degenerate Sheffer sequences.

Let $\mathbb{R}$ denote the set of real numbers. For any $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_{\lambda}^{x}(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{1.1}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda))$, $(n \geq 1)$ (*cf.* [10, 12, 14, 17, 18]).

We note that

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \tag{1.2}$$

where $< x >_0 = 1$ and $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$, $(n \geq 1)$ (*cf.* [3]).

The degenerate Stirling numbers of the first kind are given by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \tag{1.3}$$

and the generating function

$$\frac{1}{k!}\big(\log_{\lambda}(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.4}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-(n-1))$ $(n \geq 1)$, and $\log_{\lambda}(e_{\lambda}(t)) = t = e_{\lambda}(\log_{\lambda}(t))$ (*cf.* [10, 12, 17, 18]).

The degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0) \tag{1.5}$$

and the generating function

$$\frac{1}{k!}(e_{\lambda}(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0) \tag{1.6}$$

(*cf.* [10, 12, 14, 17, 18]).

When $\lambda \to 0$, $S_2(n,k)$ are the Stirling numbers of the second kind.

By (1.6), we easily get

$$S_{2,\lambda}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j)_{n,\lambda}. \tag{1.7}$$

It is well known that the degenerate Bell polynomials are given by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k, \tag{1.8}$$

and the generating function of them

$$e^{x(e_{\lambda}(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x)\frac{t^n}{n!}$$

(*cf.* [17]).

When $\lambda \to 0$, $\phi_n(x)$ are the Bell polynomials.

Let $B_{n,\lambda}^{(s)}(x)$ $(n \geq 0)$ be the degenerate Bernoulli polynomials of the order s given by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^s e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(s)}(x)\frac{t^n}{n!} \tag{1.9}$$

(*cf.* [3, 12]).

When $s = 1$, $B_{n,\lambda}(x)$ $(n \geq 0)$ are called the degenerate Bernoulli polynomials.

The degenerate Frobenius-Euler polynomials of order s are defined by

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^s e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(s)}(x|u)\frac{t^n}{n!}, \quad (u \neq 1,\ u \in \mathbb{C},\ \ k \geq 0) \tag{1.10}$$

(*cf.* [3, 12]).

When $x = 0$, $h_{n,\lambda}^{(s)}(u) = h_{n,\lambda}^{(s)}(0|u)$ are called the degenerate Frobenius-Euler numbers of order s.

When $x = 0$ and $s = 1$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called the degenerate Frobenius-Euler numbers.

When $u = -1$, the degenerate Euler polynomials of order s, respectively are given by the generating function to be

$$\left(\frac{2}{e_\lambda(t)+1}\right)^s e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(s)}(x)\frac{t^n}{n!} \tag{1.11}$$

(*cf.* [3, 10]).

We note that $E_{n,\lambda}^{(s)} = E_{n,\lambda}^{(s)}(0)$ $(n \geq 0)$, are called the degenerate Euler numbers of order r.

When $x = 0$ and $s = 1$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are called the degenerate Euler numbers.

The unsigned Lah number $L(n,k)$ counts the number of ways of all distributions of n balls, labelled $1, 2, \cdots, n$, among k unlabelled, contents-ordered boxes, with no box left empty and have an explicit formula

$$\mathbf{L}(n,k) = \binom{n-1}{k-1}\frac{n!}{k!} \tag{1.12}$$

(*cf.* [3]).

From (1.12), the generating function of $L(n,k)$ is given by

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, \quad (k \geq 0) \tag{1.13}$$

(*cf.* [3]).

From (1.13), the Lah-Bell polynomials and the degenerate Lah-Bell polynomials respectively are given by

$$e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=0}^{\infty} BL_n(x)\frac{t^n}{n!}, \qquad (n,\ k \geq 0) \tag{1.14}$$

(*cf.* [12]), and

$$e_\lambda^x\left(\frac{1}{1-t}-1\right) = \sum_{n=0}^{\infty} BL_{n,\lambda}(x)\frac{t^n}{n!}, \quad (n,\ k \geq 0) \tag{1.15}$$

(*cf.* [12]).

When $x = 1$, $BL_n = BL_n(1)$ are called the Lah-Bell numbers.

When $x = 1$, $BL_{n,\lambda} := BL_{n,\lambda}(1)$ are called the n-th degenerate Lah-Bell numbers.

When $\lambda \to 0$, $\lim_{\lambda\to 0} BL_{n,\lambda} = BL_n$ are the n-th Lah-Bell numbers.

Cayley [2] considered the ordered Bell numbers (Fubini numbers) which used them to count certain plane trees with $n+1$ totally ordered leaves. The ordered Bell numbers are given by

$$\beta_n = \sum_{k=0}^{n} k! S_2(n,k), \qquad (n \geq 0). \tag{1.16}$$

We note that the generating function of β_n is given by

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{n!} \tag{1.17}$$

(*cf.* [3]).

The ordered Bell polynomials are defined by the generating function

$$\frac{1}{2-e^t} e^{xt} = \sum_{n=0}^{\infty} \beta_n(x) \frac{t^n}{n!} \tag{1.18}$$

(*cf.* [3, 27]).

The gamma function $\Gamma(s)$ belongs to the category of the special transcendental functions given by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt = (s-1)! \tag{1.19}$$

(*cf.* [3]).

The higher-order two variables Fubini polynomials are given by

$$\left(\frac{1}{1-y(e^t-1)}\right)^r e^{xt} = \sum_{n=0}^{\infty} F_n^{(r)}(y \mid x) \frac{t^n}{n!} \tag{1.20}$$

(*cf.* [11, 18, 27]).

When $x=0$, $F_n^{(r)}(y) = F_n^{(r)}(y \mid 0)$ are called the higher-order two variables Fubini numbers.

The degenerate higher-order two variables Fubini polynomials are given by

$$\left(\frac{1}{1-y(e_\lambda(t)-1)}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} F_{n,\lambda}^{(r)}(y \mid x) \frac{t^n}{n!} \tag{1.21}$$

(*cf.* [11, 18]).

When $x=0$, $F_{n,\lambda}^{(r)}(y) = F_{n,\lambda}^{(r)}(y \mid 0)$ are called the degenerate higher-order two variables Fubini numbers.

The p-adic analysis and their applications utilize p-adic distributions and p-adic measure, p-adic integrals, p-adic L-function, and other generalized functions. Among these, the p-adic integral and its applications are very important in finding solutions to special (differential) equations, real problems in both physics and engineering (*cf.* [9, 15, 16, 24, 26, 28]).

Let p be a prime number with $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let$|\cdot|_p$ be the p-adic norm with $|p|_p = \frac{1}{p}$.

For a $\mathbb{C}_p$-valued continuous function f on $\mathbb{Z}_p$, Kim [15, 16] introduced the p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^{N-1}} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \end{aligned} \tag{1.22}$$

(*cf.* [15, 16]).

Let $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. From (1.22), we observe that

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) \tag{1.23}$$

(*cf.* [15, 16]).

In (1.22), when $n = 1$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{1.24}$$

We naturally consider

$$\int_{\mathbb{Z}_p} (y(e^t+1))^z e^{xt} d\mu_{-1}(z) = \frac{2e^{xt}}{1+y(e^t+1)} \tag{1.25}$$

and

$$\int_{\mathbb{Z}_p} (y(e_\lambda(t)+1))^z e_\lambda^x(t) d\mu_{-1}(z) = \frac{2e_\lambda^x(t)}{1+y(e_\lambda(t)+1)}. \tag{1.26}$$

When $x = 0$ in (1.25) and (1.26) respectively, we have

$$\int_{\mathbb{Z}_p} (y(e^t+1))^z d\mu_{-1}(z) = \frac{2}{1+y(e^t+1)} \tag{1.27}$$

and

$$\int_{\mathbb{Z}_p} (y(e_\lambda(t)+1))^z d\mu_{-1}(z) = \frac{2}{1+y(e_\lambda(t)+1)} \tag{1.28}$$

(*cf.* [18]).

From (1.25), we consider the two variables Fubini-type polynomials of the first kind given by

$$\frac{1}{1+y(e^t+1)} e^{xt} = \sum_{n=0}^{\infty} J_n(y \mid x)\frac{t^n}{n!}. \tag{1.29}$$

When $x = 0$, $J_n(y) = J_n(y \mid 0)$ are called the Fubini-type numbers of the first kind.

From (1.26), we naturally the degenerate two variables Fubini-type polynomials of the first kind given by

$$\frac{1}{1+y(e_\lambda(t)+1)} e_\lambda^x(t) = \sum_{n=0}^{\infty} J_{n,\lambda}(y \mid x)\frac{t^n}{n!} = \frac{1}{1+2y}\sum_{k=0}^{n}\Big(-\frac{y}{1+2y}\Big)^k k! S_{2,\lambda}(n,k). \tag{1.30}$$

When $x = 0$, $J_{n,\lambda}(y) = J_{n,\lambda}(y \mid 0)$ are called the degenerate Fubini-type numbers of the first kind.

2. The new higher order Fubini-type numbers and polynomials

Now, we consider

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y(e^t+1))^{z_1+z_2+\cdots+z_r} e^{xt}\, d\mu_{-1}(z_1) d\mu_{-1}(z_2)\cdots d\mu_{-1}(z_r) = \Big(\frac{2}{1+y(e^t+1)}\Big)^r e^{xt} \tag{2.1}$$

and

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y(e_\lambda(t)+1))^{z_1+z_2+\cdots+z_r} e_\lambda^x(t)\, d\mu_{-1}(z_1) d\mu_{-1}(z_2)\cdots d\mu_{-1}(z_r) = \Big(\frac{2}{1+y(e_\lambda(t)+1)}\Big)^r e_\lambda^x(t). \tag{2.2}$$

For any $r \in \mathbb{N}$, from (2.1), the new two-variables Fubini-type polynomials of order r are defined by

$$\left(\frac{1}{1+y(e^t+1)}\right)^r e^{xt} = \sum_{n=0}^{\infty} J_n^{(r)}(y|x)\frac{t^n}{n!}. \tag{2.3}$$

When $x = 0$, $J_n^{(r)}(y) = J_n^{(r)}(y|0)$ are called the new Fubini-type numbers of order r.
In [8], Kilar-Simsek considered a new family of polynomials $a_n^{(l)}(x)$ by means of the generating function

$$F_a(x;t,l) = \frac{2^l}{(2-e^t)^{2l}}e^{xt} = \sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^n}{n!}. \tag{2.4}$$

From (2.5) and (2.4), we note that

$$J_n^{(2r)}\left(-\frac{1}{3} \mid x\right) = \left(\frac{9}{2}\right)^r a_n^r(x).$$

From (2.2), the degenerate new two-variables Fubini-type polynomials of order r are defind by

$$\left(\frac{1}{1+y(e_\lambda(t)+1)}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} J_{n,\lambda}^{(r)}(y|x)\frac{t^n}{n!}. \tag{2.5}$$

When $x = 0$, $J_{n,\lambda}^{(r)}(y) = J_{n,\lambda}^{(r)}(y|0)$ are called the degenerate new Fubini-type numbers of order r.
When $\lambda \to 0$, $J_{n,\lambda}^{(r)}(y|x) = J_n^{(r)}(y|x)$ are the new two-variables Fubini-type numbers of order r.
From (1.1) and (2.5), we note that

$$J_{n,\lambda}^{(r)}(y \mid x) = \sum_{l=0}^{n}\binom{n}{l}J_{l,\lambda}^{(r)}(y)(x)_{n-l,\lambda} = \sum_{l=0}^{n}\binom{n}{l}J_{n-l,\lambda}^{(r)}(y)(x)_{l,\lambda}. \tag{2.6}$$

By (2.5) for $x = 0$, we observe that

$$J_{n,\lambda}^{(r)}(y) = \sum_{j=0}^{n}\binom{n}{j}J_{n-j,\lambda}(y)J_{j,\lambda}^{(r-1)}(y).$$

The degenerate ordered Bell polynomials of the order r are given by the generating function

$$\left(\frac{1}{2-e_\lambda(t)}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \tag{2.7}$$

(*cf.* [3]).
When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the degenerate ordered Bell numbers.
When $r = 1$, $\beta_{n,\lambda}(x) = \beta_{n,\lambda}^{(1)}(x)$ are called the degenerate ordered Bell numbers.
When $y = -\frac{1}{3}$, for $r \in \mathbb{N}$, from (2.5) and (2.7), we have

$$J_{n,\lambda}^{(r)}(-\frac{1}{3}|x) = 3F_{n,\lambda}^{(r)}(1|x) = 3\beta_{n,\lambda}^{(r)}(x).$$

Theorem 2.1. *For $n \geq 0$, we have*

$$J_{n,\lambda}^{(r)}(y) = \sum_{l=0}^{n}(-1)^l l!\binom{l+r-1}{l}S_{2,\lambda}(n,l)y^l(1+2y)^{-(l+r)}.$$

When $\lambda \to 0$, we have

$$J_n^{(r)}(y) = \sum_{l=0}^{n}(-1)^l l!\binom{l+r-1}{l}S_2(n,l)y^l(1+2y)^{-(l+r)}.$$

Proof. By (1.2), (1.6) and (2.5), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} J_{n,\lambda}^{(r)}(y)\frac{t^n}{n!} &= (1+2y)^{-r}\Big(1+\frac{y}{1+2y}(e_\lambda(t)-1)\Big)^{-r}\\ &= (1+2y)^{-r}\sum_{l=0}^{\infty}(-1)^l <r>_l \left(\frac{y}{1+2y}\right)^l \frac{1}{l!}(e_\lambda(t)-1)^l\\ &= (1+2y)^{-r}\sum_{l=0}^{\infty}(-1)^l <r>_l \left(\frac{y}{1+2y}\right)^l \sum_{n=l}^{\infty} S_{2,\lambda}(n,l)\frac{t^n}{n!} \qquad (2.8)\\ &= (1+2y)^{-r}\sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^l\binom{l+r-1}{l} l! S_{2,\lambda}(n,l)\frac{y}{1+2y}^l\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of the both sides of (2.8), we have the desired identity. □

By (1.21) and (2.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} J_{n,\lambda}^{(r)}(y)\frac{t^n}{n!} &= \Big(\frac{1}{1+2y}\Big)^r\Big(\frac{1}{1+\frac{y}{1+2y}(e_\lambda(t)-1)}\Big)^r\\ &= (1+2y)^{-r}\sum_{n=0}^{\infty} F_{n,\lambda}^{(r)}\Big(\frac{y}{1+2y}\Big)\frac{t^n}{n!}. \qquad (2.9)\end{aligned}$$

By using (2.9), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$J_{n,\lambda}^{(r)}(y) = (1+2y)^{-r} F_{n,\lambda}^{(r)}\Big(\frac{y}{1+2y}\Big).$$

Theorem 2.3. *For $n \geq 0$ and $x \neq 1$, we have*

$$\sum_{r=0}^{\infty} J_{n,\lambda}^{(r)}(y)x^r = \frac{(1+2y)x}{1+2y-x} J_{n,\lambda}\Big(\frac{y}{1+2y}\Big).$$

When $\lambda \to 0$, we have

$$\sum_{r=0}^{\infty} J_{n}^{(r)}(y)x^r = \frac{(1+2y)x}{1+2y-x} J_{n}\Big(\frac{y}{1+2y}\Big).$$

Proof. By Theorem 2.1, we observe that

$$\begin{aligned}\sum_{r=0}^{\infty} J_{n,\lambda}^{(r)}(y)x^r &= \sum_{r=0}^{\infty}\sum_{l=0}^{n}(-1)^l\binom{l+r-1}{l} l! S_{2,\lambda}(n,l) y^l \Big(\frac{1}{1+2y}\Big)^{l+r} x^r\\ &= \sum_{l=0}^{n}(-1)^l l! S_{2,\lambda}(n,l)\Big(\frac{y}{1+2y}\Big)^l \sum_{r=0}^{\infty}\binom{l+r}{l}\Big(\frac{x}{1+2y}\Big)^{r+1}\\ &= \sum_{l=0}^{n}(-1)^l l! S_{2,\lambda}(n,l)\Big(\frac{y}{1+2y}\Big)^l\Big(\frac{x}{1+2y}\Big)\Big(\frac{1}{1-\frac{x}{1+2y}}\Big)\\ &= \frac{(1+2y)x}{1+2y-x} J_{n,\lambda}\Big(\frac{y}{1+2y}\Big). \qquad (2.10)\end{aligned}$$

□

Theorem 2.4. *For $r \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$J_{n,\lambda}^{(r)}(y) = \frac{1}{\Gamma(r)}\Big(\frac{1}{1+2y}\Big)^r \int_0^\infty z^{r-1}\phi_{n,\lambda}\Big(-\frac{yz}{1+2y}\Big)e^{-z}dz.$$

When $\lambda \to 0$, we have

$$J_n^{(r)}(y) = \frac{1}{\Gamma(r)}\Big(\frac{1}{1+2y}\Big)^r \int_0^\infty z^{r-1}\phi_n\Big(-\frac{yz}{1+2y}\Big)e^{-z}dz.$$

Proof. From (1.8) and Theorem 2.1, we have

$$\begin{aligned}\Big(\frac{1}{1+2y}\Big)^r \int_0^\infty z^{r-1}\phi_{n,\lambda}\Big(-\frac{yz}{1+2y}\Big)e^{-z}dz &= \Big(\frac{1}{1+2y}\Big)^r \sum_{l=0}^n S_{2,\lambda}(n,l)(-1)^l\Big(\frac{y}{1+2y}\Big)^l \int_0^\infty z^{r-1+l}e^{-z}dz \\ &= \sum_{l=0}^n (-1)^l S_{2,\lambda}(n,l)y^l\Big(\frac{1}{1+2y}\Big)^{r+l}(r+l-1)! \qquad (2.11) \\ &= \Gamma(r)\sum_{l=0}^n (-1)^l S_{2,\lambda}(n,l)y^l\Big(\frac{1}{1+2y}\Big)^{r+l} l!\binom{r+l-1}{l} = \Gamma(r)J_{n,\lambda}^{(r)}(y).\end{aligned}$$

From (2.11), we have the desired result. □

From Theorem 2.1, we observe that

$$\begin{aligned}\Big(\frac{1}{1+y(e_\lambda(t)+1)}\Big)^r &= \frac{1}{\Gamma(r)}\Big(\frac{1}{1+2y}\Big)^r \int_0^\infty z^{r-1}\phi_{n,\lambda}\Big(-\frac{yz}{1+2y}\Big)e^{-z}dz \\ &= \frac{1}{\Gamma(r)}\Big(\frac{1}{1+2y}\Big)^r \int_0^\infty z^{r-1}e^{-z\left(\frac{y}{1+2y}(e_\lambda(t)-1)+1\right)}dz. \qquad (2.12)\end{aligned}$$

Theorem 2.5. *For $n \geq 0$, we have*

$$r(1+y)J_{n,\lambda}^{(r+1)}(y) = J_{n+1,\lambda}^{(r)}(y) + (\lambda n - 1)J_{n,\lambda}^{(r)}(y).$$

When $\lambda \to 0$, we have

$$r(1+y)J_n^{(r+1)}(y) = J_{n+1}^{(r)}(y) - J_n^{(r)}(y).$$

Proof. Differentiating both sides of (2.5) with respect to t, we have

$$\frac{-rye_\lambda^{1-\lambda}(t)}{(1+y(e_\lambda(t)+1))^{r+1}} = \sum_{n=0}^\infty J_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!}. \qquad (2.13)$$

By using the left-hand side of (2.13), we observe that

$$\frac{-rye_\lambda^{1-\lambda}(t)}{(1+y(e_\lambda(t)+1))^{r+1}} = \frac{r}{1+\lambda t}\Big[(1+y)\sum_{n=0}^\infty J_{n,\lambda}^{(r+1)}(y)\frac{t^n}{n!} - \sum_{n=0}^\infty J_{n,\lambda}^{(r)}(y)\frac{t^n}{n!}\Big]. \qquad (2.14)$$

From (2.13) and (2.14), we have

$$(1+\lambda t)\sum_{n=0}^\infty J_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!} = r\sum_{n=0}^\infty \Big[(1+y)J_{n,\lambda}^{(r+1)}(y) - J_{n,\lambda}^{(r)}(y)\Big]\frac{t^n}{n!}. \qquad (2.15)$$

From (2.15), we obtain

$$\sum_{n=0}^\infty J_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!} + \lambda\sum_{n=1}^\infty nJ_{n,\lambda}^{(r)}(y)\frac{t^n}{n!} = r\sum_{n=0}^\infty \Big[(1+y)J_{n,\lambda}^{(r+1)}(y) - J_{n,\lambda}^{(r)}(y)\Big]\frac{t^n}{n!}. \qquad (2.16)$$

By (2.16), we get

$$\sum_{n=0}^{\infty} J_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!} + \lambda \sum_{n=0}^{\infty} nJ_{n,\lambda}^{(r)}(y)\frac{t^n}{n!} = r\sum_{n=0}^{\infty}\Big[(1+y)J_{n,\lambda}^{(r+1)}(y) - J_{n,\lambda}^{(r)}(y)\Big]\frac{t^n}{n!}. \tag{2.17}$$

By (2.17), we obtain the desired identity.

□

Theorem 2.6. *For $n \geq 0$ and $r > 0$, we have*

$$F_{n+1,\lambda}^{(r)}(y) = r(1+y)\sum_{l=0}^{n}\binom{n}{l}F_{n-l,\lambda}(y)F_{l,\lambda}^{(r)}(y) - (\lambda n + r)F_{n,\lambda}^{(r)}(y).$$

When $\lambda \to 0$, we have

$$F_{n+1}^{(r)}(y) = r(1+y)\sum_{l=0}^{n}\binom{n}{l}F_{n-l}(y)F_{l}^{(r)}(y) - rF_{n}^{(r)}(y).$$

Proof. From (2.13), we observe that

$$\begin{aligned}\frac{-rye_\lambda^{1-y}(t)}{(1+y\,(e_\lambda(t)+1))^{r+1}} &= \sum_{n=0}^{\infty} F_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!}\\ &= \frac{r}{1+\lambda t}\Big(\frac{1+y}{1+y\,(e_\lambda(t)+1)} - 1\Big)\Big(\frac{1}{1+y\,(e_\lambda(t)+1)}\Big)^r. \end{aligned}\tag{2.18}$$

Multiply $(1+\lambda t)$ in (2.18), we have

$$\begin{aligned}(1+\lambda t)\sum_{n=0}^{\infty} F_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!} &= r(1+y)\Big(\frac{1+y}{1+y\,(e_\lambda(t)+1)}\Big)\Big(\frac{1}{1+y\,(e_\lambda(t)+1)}\Big)^r - r\Big(\frac{1}{1+y\,(e_\lambda(t)+1)}\Big)^r\\ &= r(1+y)\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}F_{n-l,\lambda}(y)F_{l,\lambda}^{(r)}(y)\Big)\frac{t^n}{n!} - r\sum_{n=0}^{\infty}F_{n,\lambda}^{(r)}(y)\frac{t^n}{n!}. \end{aligned}\tag{2.19}$$

The left-side hand of (2.19) is

$$\begin{aligned}(1+\lambda t)\sum_{n=0}^{\infty} F_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} F_{n+1,\lambda}^{(r)}(y)\frac{t^n}{n!} + \lambda\sum_{n=1}^{\infty} nF_{n,\lambda}^{(r)}(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\Big\{F_{n+1,\lambda}^{(r)}(y) + \lambda nF_{n,\lambda}^{(r)}(y)\Big\}\frac{t^n}{n!}. \end{aligned}\tag{2.20}$$

By (2.19) and (2.20), we have the desired result. □

3. Some identities of the higher-order degenerate Fubini-type polynomials arising from λ-Sheffer sequences

The umbral calculus, based on the modern idea of linear functions, linear operators and adjoints, began to build a rigorous foundation by Rota in the 1970s, primarily as symbolic techniques for the manipulation of numerical and polynomial sequences (*cf.* [4, 5, 7, 10, 12], [19]-[23]). Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials (*cf.* [10, 12]). Recently, Kim-Kim [10] introduced the λ-Sheffer sequence and the degenerate Sheffer sequence.

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \Big\{f(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!}\ \Big|\ a_k \in \mathbb{C}\Big\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$.

$$\mathbb{P}_n = \{\, P(x) \in \mathbb{C}[x] \mid degP(x) \le n\}, \quad (n \ge 0).$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t) \mid \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\langle f(t) \mid (x)_{n,\lambda} \rangle_\lambda = a_n, \tag{3.1}$$

for all $n \ge 0$ (*cf.* [10]).

For $\lambda = 0$, we observe that the linear functional $\langle f(t) \mid \cdot \rangle_0$ agrees with the one in $\langle f(t) \mid x^n \rangle = a_k$, $(k \ge 0)$.

From $\langle f(t)g(t) \mid (x)_{n,\lambda} \rangle_\lambda = \langle f(t) \mid (g(t))_\lambda (x)_{n,\lambda} \rangle_\lambda$ and (1.16), we note that

$$\langle t^k \mid (x)_{n,\lambda} \rangle_\lambda = \langle 1 \mid (t^k)_\lambda (x)_{n,\lambda} \rangle_\lambda = \langle 1 \mid (n)_k (x)_{n-k,\lambda} \rangle_\lambda = n!\delta_{n,k}, \tag{3.2}$$

for all $n,\ k\ \ge 0$, where $\delta_{n,k}$ is the Kronecker's symbol.

From (3.2), for each $\lambda \in \mathbb{R}$, and each nonnegative integer k, the differential operator on $\mathbb{P}$ is given by

$$(t^k)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if} \quad k \le n \\ 0, & \text{if} \quad k \ge n \end{cases} \tag{3.3}$$

(*cf.* [10]), and for any power series $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$,

$$(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}, \quad (n \ge 0). \tag{3.4}$$

By (1.1) and (3.4), we note that

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{n-k,\lambda} (y)_{k,\lambda}. \tag{3.5}$$

From (1.1) and (3.5), it is easy to see that

$$(e_\lambda^y(t))_\lambda p(x) = p(x+y), \text{ for any } p(x) \in \mathbb{P}. \tag{3.6}$$

The order $o(f(t))$ of a power series $f(t)(\ne 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t)) = 0$. $f(t)$ is called a delta series if $o(f(t)) = 1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively, and $s_{n,\lambda}(x)$ be a degenerate polynomial of a degree n. Then there exists a unique sequences $s_{n,\lambda}(x)$ such that the orthogonality conditions

$$\langle g(t)(f(t))^k \mid s_{n,\lambda}(x) \rangle_\lambda = n!\delta_{n,k}, \quad (n,k \ge 0) \tag{3.7}$$

(*cf.* [10]).

The sequences $s_{n,\lambda}(x)$ are called the λ-Sheffer sequences for $(g(t), f(t))$, which are denoted by $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \ \sim\ (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g(\overline{f}(t))} e_\lambda^x(\overline{f}(t)) \ = \ \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k \quad (n,k \ge 0) \tag{3.8}$$

(*cf.* [10]).

Assume that for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t),\ f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0} \overline{f}_\lambda(t) = \overline{f}(t)$, where is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. Let $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$.

In this case, the family $\{s_{n,\lambda}(x)\}_{\lambda\in\mathcal{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ are called the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

Let $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t), g(t))_\lambda$, $(n \geq 0)$. Then

$$s_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} r_{k,\lambda}(x), \quad (n \geq 0), \tag{3.9}$$

where

$$\mu_{n,k} = \frac{1}{k!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}(l(\overline{f}(t)))^k \mid (x)_{n,\lambda}\right\rangle_\lambda, \quad (n,k \geq 0)$$

(*cf.* [10]).

Now, we will express the two variable higher-order degenerate Fubini-type polynomials $J^{(r)}_{n,\lambda}(y \mid x)$ as linear combinations of some well-known special polynomials by using the degenerate umbra calculus.

First, from (2.5), we have the λ-Sheffer sequence as follows:

$$J^{(r)}_{n,\lambda}(y \mid x) \sim ((1 + y\,(e_\lambda(t) + 1))^r,\ t)_\lambda. \tag{3.10}$$

Theorem 3.1. *For $n \geq 0$, we have*

$$J^{(r)}_{n+1,\lambda}(y \mid x) = -ryJ^{(r+1)}_{n,\lambda}(y \mid 1 - \lambda + x) + xJ^{(r)}_{n,\lambda}(y \mid x - \lambda).$$

When $\lambda \to 0$, we have

$$J^{(r)}_{n+1}(y \mid x) = -ryJ^{(r+1)}_{n}(y \mid 1 + x) + xJ^{(r)}_{n}(y \mid x).$$

Proof. For $n \in \mathbb{N}$, from (2.5) and (3.10), we get

$$\begin{aligned}
J^{(r)}_{n,\lambda}(y \mid x) &= \left\langle \frac{e^x_\lambda(t)}{(1 + y(e_\lambda(t) + 1))^r}\middle|(x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle \partial t\frac{1}{(1 + y\,(e_\lambda(t) + 1))^r}e^x_\lambda(t)\middle|(x)_{n-1,\lambda}\right\rangle_\lambda + \left\langle \frac{1}{(1 + y\,(e_\lambda(t) + 1))^r}\partial t(e^x_\lambda(t))\middle|(x)_{n-1,\lambda}\right\rangle_\lambda \\
&= -ryJ^{(r+1)}_{n-1,\lambda}(y \mid 1 - \lambda + x) + xJ^{(r)}_{n-1,\lambda}(y \mid x - \lambda).
\end{aligned} \tag{3.11}$$

From (3.11), we have the desired result. □

Theorem 3.2. *For $n \geq 0$, we have*

$$J^{(r)}_{n,\lambda}(y|x) = \sum_{j=0}^{n}\Big(\sum_{l=j}^{n}\binom{n}{l}S_{2,\lambda}(l,j)J^{(r)}_{n-l,\lambda}(y)\Big)(x)_n.$$

When $\lambda \to 0$, we have

$$J^{(r)}_{n}(y|x) = \sum_{j=0}^{n}\Big(\sum_{l=j}^{n}\binom{n}{l}S_{2}(l,j)J^{(r)}_{n-l}(y)\Big)(x)_n.$$

Proof. We consider the degenerate Sheffer sequence

$$(x)_n \sim (1, e_\lambda(t)-1)_\lambda, \tag{3.12}$$

since $(1+t)^n = \sum_{n=0}^{\infty}(x)_n \frac{t^n}{n!}$.

From (3.9) and (3.12),

$$J_{n,\lambda}^{(r)}(y|x) = \sum_{j=0}^{n} z_{n,j}(x)_n, \tag{3.13}$$

where, by (1.6),

$$\begin{aligned} z_{n,j} &= \frac{1}{j!}\left\langle \frac{1}{1+y(e_\lambda(t)+1)^r}(e_\lambda(t)-1)^j \Big| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{l=j}^{n}\binom{n}{l} S_{2,\lambda}(l,j) J_{n-l,\lambda}^{(r)}(y). \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), we have the desired representation. □

Theorem 3.3. *For* $1 \le r \le n$*, we have*

$$(x)_n = \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}\binom{r}{n-l} y^{n-l}(n-l)! S_{1,\lambda}(n,j)\right\} J_{j,\lambda}^{(r)}(y|x).$$

When $\lambda \to 0$*, we have*

$$(x)_n = \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}\binom{r}{n-l} y^{n-l}(n-l)! S_{1,\lambda}(n,j)\right\} J_{j}^{(r)}(y|x).$$

Proof. We consider two degenerate Sheffer sequences:

$$(x)_n \sim (1, e_\lambda(t)-1)_\lambda \quad \text{and} \quad J_{n,\lambda}^{(r)}(y|x) \sim ((1+y(e_\lambda(t)+1))^r, t). \tag{3.15}$$

From (3.9) and (3.15), we have

$$(x)_n = \sum_{j=0}^{n} \mu_{n,j} J_{j,\lambda}^{(r)}(y|x), \tag{3.16}$$

where, for $1 \le r \le n$, by (1.4), we observe that

$$\begin{aligned} \mu_{n,j} &= \frac{1}{j!}\langle (1+yt)^r (\log_\lambda(1+t)^j | (x)_{n,\lambda}\rangle_\lambda \\ &= \sum_{l=j}^{n}\binom{n}{l} S_{1,\lambda}(n,j) \sum_{m=0}^{r}\binom{r}{m} y^m \langle t^m | (x)_{n-l,\lambda}\rangle_\lambda \\ &= \sum_{l=j}^{n}\binom{n}{l} S_{1,\lambda}(n,j)\binom{r}{n-l} y^{n-l}(n-l)!. \end{aligned} \tag{3.17}$$

Combining (3.16) with (3.17), we obtain the desired identity. □

From (1.9) and (3.8), we get

$$B_{n,\lambda}^{(s)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^s, \ t\right)_\lambda. \tag{3.18}$$

Theorem 3.4. *For $n \geq 0$ and $r > 0$, we have*

$$J_{n,\lambda}^{(r)}(y \mid x) = \sum_{j=0}^{n} \Big(\sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} \frac{(1)_{l+1,\lambda}}{(l+1)} J_{n-j-l,\lambda}^{(r)}(y) \Big) B_{j,\lambda}(x),$$

where $B_{n,\lambda}(x)$ are the degenerate Bernoulli polynomials.
When $\lambda \to 0$, we have

$$J_{n}^{(r)}(y \mid x) = \sum_{j=0}^{n} \Big(\sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} \frac{(1)_{l+1}}{(l+1)} J_{n-j-l}^{(r)}(y) \Big) B_{j}(x),$$

where $B_n(x)$ are the Bernoulli polynomials.

Proof. From (3.10) and (3.18), we consider two degenerate Sheffer sequences as follows:

$$J_{n,\lambda}^{(r)}(y \mid x) \sim ((1 + y\,(e_\lambda(t)+1))^r, t)_\lambda \quad \text{and} \quad B_{n,\lambda}(x) \sim \Big(\frac{e_\lambda(t)-1}{t},\ t \Big)_\lambda. \tag{3.19}$$

From (3.9) and (3.19),

$$J_{n,\lambda}^{(r)}(y \mid x) = \sum_{j=0}^{n} z_{n,j} B_{j,\lambda}(x), \tag{3.20}$$

where

$$\begin{aligned} z_{n,j} &= \frac{1}{j!} \Big\langle \frac{1}{1+y(e_\lambda(t)+1)^r} \frac{e_\lambda(t)-1}{t} t^j \Big| (x)_{n,\lambda} \Big\rangle_\lambda \\ &= \binom{n}{j} \Big\langle \frac{1}{1+y(e_\lambda(t)+1)^r} \frac{e_\lambda(t)-1}{t} \Big| (x)_{n-j,\lambda} \Big\rangle_\lambda \\ &= \binom{n}{j} \sum_{l=0}^{n-j} \binom{n-j}{l} \frac{(1)_{l+1,\lambda}}{(l+1)} J_{n-j-l,\lambda}^{(r)}(y). \end{aligned} \tag{3.21}$$

Combining (3.20) with (3.21), we have the desired identity. □

To obtain the inverse formula of Theorem 3.4, we need the following lemma.

Lemma 3.5. *For $p(x) \in \mathbb{P}_n$, we have*

$$p(x) = \sum_{j=0}^{n} a_j J_{j,\lambda}^{(r)}(y \mid x),$$

where

$$a_j = \frac{1}{j!} \langle (1 + y(e_\lambda(t)+1))^r t^j | p(x) \rangle_\lambda.$$

Proof. For $p(x) \in \mathbb{P}_n$, we put

$$p(x) = \sum_{j=0}^{n} a_j J_{j,\lambda}^{(r)}(y \mid x). \tag{3.22}$$

By (3.7) and (3.10), for $0 \leq l \leq n$, we have

$$\begin{aligned} \langle (1 + y\,(e_\lambda(t)+1))^r t^l | p(x) \rangle_\lambda &= \sum_{j=0}^{n} a_j \langle (1 + y\,(e_\lambda(t)+1))^r t^l | J_{j,\lambda}^{(r)}(y \mid x) \rangle_\lambda \\ &= \sum_{j=0}^{n} a_j l! \delta_{j,l} = a_l l!. \end{aligned} \tag{3.23}$$

By (3.22) and (3.23), we obtain the desired result. □

By using Lemma 3.5, we obtain the inverse formula of Theorem 3.4.

Theorem 3.6. *Let* $p(x) = B^{(s)}_{n,\lambda}(x) \in \mathbb{P}_n$. *Then we have*

$$B^{(s)}_{n,\lambda}(x) = \sum_{j=0}^{n}\left[\binom{n}{j}\sum_{l=0}^{r}(1+y)^{r-l}y^l B^{(s)}_{n-j,\lambda}(l)\right]J^{(r)}_{j,\lambda}(y \mid x),$$

where $B^{(s)}_{n,\lambda}(x)$ *are the degenerate Bernoulli polynomials of order* s.

When $\lambda \to 0$, *we have*

$$B^{(s)}_{n}(x) = \sum_{j=0}^{n}\left[\binom{n}{j}\sum_{l=0}^{r}(1+y)^{r-l}y^l B^{(s)}_{n-j}(l)\right]J^{(r)}_{j}(y \mid x),$$

where $B^{(s)}_{n}(x)$ *are the Bernoulli polynomials of order* s.

Proof. Let $p(x) = B^{(s)}_{n,\lambda}(x)$. Then, by using Lemma 3.5, we have

$$B^{(s)}_{n,\lambda}(x) = \sum_{j=0}^{n} a_j J^{(r)}_{j,\lambda}(y \mid x), \tag{3.24}$$

where

$$\begin{aligned} a_j &= \frac{1}{j!}\langle (1+y\,(e_\lambda(t)-1))^r t^j | B^{(s)}_{n,\lambda}(x)\rangle_\lambda \\ &= \binom{n}{j}\sum_{l=0}^{r}(1+y)^{r-l}y^l B^{(s)}_{n-j,\lambda}(l). \end{aligned} \tag{3.25}$$

Combining (3.24) with (3.25), we get the desired result. □

Theorem 3.7. *For* $n \geq 0$, *we have*

$$J^{(r)}_{n,\lambda}(y \mid x) = \sum_{j=0}^{n}\binom{n}{j}\left[\sum_{m=0}^{n-j}\sum_{l=0}^{s}\binom{n-j}{m}\frac{1}{(1-u)^s}(-u)^{s-l}J^{(r)}_{m,\lambda}(y)(x+l)_{n-j-m,\lambda}\right]h^{(s)}_{j,\lambda}(x \mid u).$$

When $\lambda \to 0$, *we have*

$$J^{(r)}_{n}(y \mid x) = \sum_{j=0}^{n}\binom{n}{j}\left[\sum_{m=0}^{n-j}\sum_{l=0}^{s}\binom{n-j}{m}\frac{1}{(1-u)^s}(-u)^{s-l}J^{(r)}_{m}(y)(x+l)_{n-j-m}\right]h^{(s)}_{j}(x \mid u).$$

Proof. From (3.8) and (1.10) we have the Sheffer sequence for $h^{(s)}_{n,\lambda}(x \mid u)$ as follows:

$$\left(\left(\frac{e_\lambda(t)-u}{1-u}\right)^s,\ t\right)_\lambda \sim h^{(s)}_{n,\lambda}(x \mid u). \tag{3.26}$$

By (3.9), (3.10), and (3.26), we have

$$J^{(r)}_{n,\lambda}(y \mid x) = \sum_{j=0}^{n} z_{n,j}\, h^{(s)}_{j,\lambda}(x \mid u), \tag{3.27}$$

where

$$\begin{aligned} z_{n,j} &= \frac{1}{j!}\left\langle \frac{1}{(1+y(e_\lambda(t)+1))^r}\left(\frac{e_\lambda(t)-u}{1-u}\right)^s t^j \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \binom{n}{j}\left\langle \frac{1}{(1+y(e_\lambda(t)+1))^r}\left(\frac{e_\lambda(t)-u}{1-u}\right)^s \middle| (x)_{n-j,\lambda}\right\rangle_\lambda \\ &= \binom{n}{j}\sum_{m=0}^{n-j}\binom{n-j}{m}J_{m,\lambda}^{(r)}(y)\frac{1}{(1-u)^s}\sum_{l=0}^{s}(-u)^{s-l}\langle e_\lambda^l(t)|(x)_{n-j-m,\lambda}\rangle_\lambda \\ &= \binom{n}{j}\sum_{m=0}^{n-j}\binom{n-j}{m}J_{m,\lambda}^{(r)}(y)\frac{1}{(1-u)^s}\sum_{l=0}^{s}(-u)^{s-l}(x+l)_{n-j-m,\lambda}. \end{aligned} \tag{3.28}$$

Combining (3.27) with (3.28), we have the desired representation. □

Next theorem is the inverse formula of Theorem 3.7.

Theorem 3.8. *Let $p(x) = h_{n,\lambda}^{(s)}(x) \in \mathbb{P}_n$. Then we have*

$$h_{n,\lambda}^{(s)}(x) = \sum_{j=0}^{n}\left[\binom{n}{j}\sum_{l=0}^{r}\binom{r}{l}(1+y)^l y^l h_{n-j,\lambda}^{(s)}(l)\right] J_{j,\lambda}^{(r)}(y \mid x).$$

When $\lambda \to 0$, we have

$$h_{n}^{(s)}(x) = \sum_{j=0}^{n}\left[\binom{n}{j}\sum_{l=0}^{r}\binom{r}{l}(1+y)^l y^l h_{n-j}^{(s)}(l)\right] J_{j}^{(r)}(y \mid x).$$

Proof. Let $p(x) = h_{n,\lambda}^{(s)}(x)$. Then, by Lemma 3.5, we have

$$h_{n,\lambda}^{(s)}(x) = \sum_{j=0}^{n} a_j J_{j,\lambda}^{(r)}(y \mid x), \tag{3.29}$$

where

$$\begin{aligned} a_j &= \frac{1}{j!}\langle (1+y(e_\lambda(t)+1))^r t^j | h_{n,\lambda}^{(s)}(x)\rangle_\lambda \\ &= \binom{n}{j}\sum_{l=0}^{r}\binom{r}{l}(1+y)^{r-1} y^l h_{n-j,\lambda}^{(s)}(l). \end{aligned} \tag{3.30}$$

Combining (3.29) with (3.30), we have the desired identity. □

Corollary 3.9. *For $n \geq 0$, we have*

$$J_{n,\lambda}^{(r)}(y \mid x) = \frac{1}{2^s}\sum_{j=0}^{n}\sum_{l=0}^{s}\binom{n}{j}\binom{s}{l}J_{n-j,\lambda}^{(r)}(y \mid l)E_{j,\lambda}^{(s)}(x)$$

and

$$E_{n,\lambda}^{(s)}(x) = \sum_{j=0}^{n}\sum_{l=0}^{r}\binom{n}{j}(1+y)^{r-l}y^l E_{n-j,\lambda}^{(s)}(l)J_{j,\lambda}^{(r)}(y \mid x).$$

Now, we consider a connection to the degenerate Lah-Bell polynomials.

Theorem 3.10. *For $n \geq 0$, we have*

$$J_{n,\lambda}^{(r)}(y \mid x) = \sum_{j=0}^{n} \left[\sum_{l=j}^{n} \binom{n}{l} (-1) L(l, j) J_{n-l,\lambda}^{(r)}(y) \right] BL_{j,\lambda}(x).$$

When $\lambda \to 0$, we have

$$J_{n}^{(r)}(y \mid x) = \sum_{j=0}^{n} \left[\sum_{l=j}^{n} \binom{n}{l} (-1) L(l, j) J_{n-l}^{(r)}(y) \right] BL_{j}(x),$$

where, $L(n,k)$ are the unsigned Lah numbers.

Proof. From (3.9) and (1.15), we have the Sheffer sequence for $BL_{n,\lambda}(x)$ as follows:

$$BL_{n,\lambda}(x) \sim \left(1, \frac{t}{1+t}\right)_{\lambda}. \tag{3.31}$$

By using (3.9), (3.10) and (3.31), we have

$$J_{n,\lambda}^{(r)}(y \mid x) = \sum_{j=0}^{n} z_{n,j} BL_{j,\lambda}(x), \tag{3.32}$$

where, from (1.13), we have

$$\begin{aligned} z_{n,j} &= \frac{1}{j!} \left\langle \frac{1}{(1+y(e_{\lambda}(t)+1))^{r}} \left(\frac{t}{1+t}\right)^{j} \middle| (x)_{n,\lambda} \right\rangle_{\lambda} \\ &= \left\langle \frac{1}{(1+y(e_{\lambda}(t)+1))^{r}} \frac{1}{j!} (-1)^{j} \left(\frac{-t}{1-(-t)}\right)^{j} \middle| (x)_{n,\lambda} \right\rangle_{\lambda} \\ &= \sum_{l=j}^{n} (-1)^{j} L(l,j) \binom{n}{l} \left\langle \frac{1}{(1+y(e_{\lambda}(t)+1))^{r}} \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda} \\ &= \sum_{l=j}^{n} \binom{n}{l} (-1)^{j} L(l,j) J_{n-l,\lambda}^{(r)}(y). \end{aligned} \tag{3.33}$$

Combining (3.32) with (3.33), we have

$$J_{n,\lambda}^{(r)}(y \mid x) = \sum_{j=0}^{n} \left[\sum_{l=j}^{n} \binom{n}{l} (-1) L(l, j) J_{n-l,\lambda}^{(r)}(y) \right] BL_{j,\lambda}(x).$$

□

Next theorem is the inverse formula of Theorem 3.10 .

Theorem 3.11. *Let $p(x) = BL_{n,\lambda}(x) \in \mathbb{P}_n$. Then we have*

$$BL_{n,\lambda}(x) = \sum_{j=0}^{\infty} \left(\binom{n}{j} \sum_{l=0}^{r} \binom{r}{l} (1+y)^{l} y^{l} BL_{n-j,\lambda} \right) J_{j,\lambda}^{(r)}(y \mid x).$$

When $\lambda \to 0$, we have

$$BL_{n}(x) = \sum_{j=0}^{\infty} \left(\binom{n}{j} \sum_{l=0}^{r} \binom{r}{l} (1+y)^{l} y^{l} BL_{n-j} \right) J_{j}^{(r)}(y \mid x).$$

Proof. Let $p(x) = BL_{n,\lambda}(x)$. Then by using Lemma 3.5, we have

$$BL_{n,\lambda}(x) = \sum_{j=0}^{n} a_j J_{j,\lambda}^{(r)}(y \mid x), \tag{3.34}$$

where

$$\begin{aligned} a_j &= \frac{1}{j!}\langle (1 + y(e_\lambda(t) + 1))^r t^j | BL_{n,\lambda}(x)\rangle_\lambda \\ &= \binom{n}{j}\langle (1 + y(e_\lambda(t) + 1))^r | BL_{n-j,\lambda}(x)\rangle_\lambda \\ &= \binom{n}{j}\sum_{l=0}^{r}\binom{r}{l}(1 + y)^l y^l BL_{n-j,\lambda}(l). \end{aligned} \tag{3.35}$$

Combining (3.34) with (3.35), we have the inverse formula of Theorem 3.10. □

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