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Integral inequalities of Hermite–Hadamard type for products of s-logarithmically convex functions

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Abstract

In this paper, the authors establish several new integral inequalities of the Hermite–Hadamard type for the products of s-logarithmically convex functions and present simple applications to construct inequalities of the arithmetic and (generalized) logarithmic means.

Keywords: Hermite–Hadamard type, s-logarithmically convex function, product, mean, integral inequality

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1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $h : J \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$h(\theta u + (1-\theta)v) \le \theta f(u) + (1-\theta)h(v) \tag{1.1}$$

holds for $u, v \in J$ and $\theta \in [0, 1]$. If the inequality (1.1) reverses, then h is a concave function on J.

Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on J. Then the Hermite–Hadamard integral inequality reads that

$$h\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(x)\,\mathrm{d}\,x \le \frac{h(u)+h(v)}{2}, \quad u, v \in J.$$

Definition 1.2 (*cf.* [2])**.** If a positive function $h : J \subset \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ satisfies

$$h(\theta u + (1-\theta)v) \le h^{\theta}(u)h^{1-\theta}(v) \tag{1.2}$$

for all $u, v \in J$ and $\theta \in [0, 1]$, then h is called a logarithmically convex function on J. If the inequality (1.2) reverses, then we call h a logarithmically concave function on J.

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In [4, 5], the kind of s-logarithmically convex functions was generalized as follows.

Definition 1.3 (*cf.* [4, 5])**.** For some number $s \in (0, 1]$, we call $h : J \subseteq \mathbb{R} \to \mathbb{R}_+$ an s-logarithmically convex function if the inequality

$$h(\theta u + (1-\theta)v) \le [h(u)]^{\theta^s}[h(v)]^{(1-\theta)^s}$$

holds for all $u, v \in J$ and $\theta \in [0, 1]$.

Remark 1.4. Let $s \in (0, 1]$ and $h : J \subseteq \mathbb{R} \to \mathbb{R}_+$ be an s-logarithmically convex function.

1. The 1-logarithmically convex function is just a logarithmically convex function on $\mathbb{R}_+$.
2. If $s \in (0, 1)$, then $h(u) \ge 1$ for all $u \in J$.

In the articles [3, 6] and [7], some integral inequalities of the Hermite–Hadamard type were established for the products of strongly logarithmically convex functions, (α, m)-convex functions, and other convex functions.

In this paper, we will establish several new integral inequalities of the Hermite–Hadamard type for the products of s-logarithmically convex functions and present simple applications to derive inequalities of the arithmetic and (generalized) logarithmic means.

2. New integral inequalities of Hermite–Hadamard type

We are now in a position to establish several new integral inequalities of the Hermite–Hadamard type for the products of s-logarithmically convex functions.

Theorem 2.1. *Suppose $h : J \subseteq \mathbb{R} \to \mathbb{R}_+$ and $u, v \in J$ with $u < v$. If h is an s-logarithmically convex function on* $[u, v]$ *for some constant* $0 < s \le 1$*, then*

$$\begin{aligned} h^{1/2^{1-s}}\left(\frac{u+v}{2}\right) &\le \frac{1}{v-u}\int_u^v [h(x)h(u+v-x)]^{1/2}\,\mathrm{d}\,x \\ &\le \frac{1}{v-u}\int_u^v h(x)\,\mathrm{d}\,x \\ &\le [h(u)h(v)]^{1-s}L(h^s(u), h^s(v)) \\ &\le \frac{1}{2}[h(u)h(v)]^{1-s}[h^s(u)+h^s(v)], \end{aligned} \tag{2.1}$$

where $L(u, v)$ for $u, v > 0$ is the logarithmic mean

$$L(u,v) = \begin{cases} \dfrac{v-u}{\ln v - \ln u}, & u \ne v; \\ u, & u = v. \end{cases} \tag{2.2}$$

Proof. Since h is an s-logarithmically convex function, we have

$$\begin{aligned} h\left(\frac{u+v}{2}\right) &= h\left(\frac{\theta u + (1-\theta)v + (1-\theta)u + \theta v}{2}\right) \\ &\le [h(\theta u + (1-\theta)v)h((1-\theta)u + \theta v)]^{1/2^s} \\ &= [h(\theta u + (1-\theta)v)h(u+v-(\theta u + (1-\theta)v))]^{1/2^s} \end{aligned}$$

for all $\theta \in [0, 1]$. Hence, for all $\theta \in [0, 1]$, we obtain

$$\begin{aligned} h^{1/2^{1-s}}\left(\frac{u+v}{2}\right) &\le [h(\theta u + (1-\theta)v)h((1-\theta)u + \theta v)]^{1/2} \\ &= [h(\theta u + (1-\theta)v)h(u+v-(\theta u + (1-\theta)v))]^{1/2}. \end{aligned} \tag{2.3}$$

Integrating with respect to $\theta \in [0,1]$, making the transform $x = \theta u + (1-\theta)v$ for $\theta \in [0,1]$, and utilizing the geometric-arithmetic inequality, we obtain

$$\int_0^1 [h(\theta u + (1-\theta)v)h(u+v-(\theta u + (1-\theta)v))]^{1/2}\,\mathrm{d}\,\theta = \frac{1}{v-u}\int_u^v [h(x)h(u+v-x)]^{1/2}\,\mathrm{d}\,x \le \frac{1}{v-u}\int_u^v h(x)\,\mathrm{d}\,x.$$

The left-hand side inequality in (2.1) is thus proved.

Let $0 < \mu \le 1 \le \eta$ and $0 < s, \theta \le 1$. Then

$$\mu^{\theta^s} \le \mu^{s\theta} \quad \text{and} \quad \eta^{\theta^s} \le \eta^{s\theta+1-s}. \tag{2.4}$$

See [1, p. 4]. In view of the s-logarithmic convexity of h, we acquire

$$\frac{1}{v-u}\int_u^v h(x)\,\mathrm{d}\,x = \int_0^1 h(\theta u + (1-\theta)v)\,\mathrm{d}\,\theta \le \int_0^1 [h(u)]^{\theta^s}[h(v)]^{(1-\theta)^s}\,\mathrm{d}\,\theta.$$

By virtue of the second item in Remark 1.4 and the inequalities in (2.4), we write

$$[h(u)]^{\theta^s}[h(v)]^{(1-\theta)^s} \le [h(u)h(v)]^{1-s}[h(u)]^{s\theta}[h(v)]^{s(1-\theta)}$$

for $\theta \in [0,1]$. So, we obtain that

$$\begin{aligned}\int_0^1 [h(u)]^{\theta^s}[h(v)]^{(1-\theta)^s}\,\mathrm{d}\,\theta &\le [h(u)h(v)]^{1-s}\int_0^1 [h(u)]^{s\theta}[h(v)]^{s(1-\theta)}\,\mathrm{d}\,\theta\\ &= [h(u)h(v)]^{1-s}L(h^s(u),h^s(v))\\ &\le \frac{1}{2}[h(u)h(v)]^{1-s}[h^s(u)+h^s(v)].\end{aligned}$$

Theorem 2.1 is thus proved. □

If taking $s = 1$ in Theorem 2.1, we can derive the following corollary.

Corollary 2.2. *Under the conditions of Theorem* 2.1*, when* $s = 1$*, we have*

$$h\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v [h(x)h(u+v-x)]^{1/2}\,\mathrm{d}\,x \le \frac{1}{v-u}\int_u^v h(x)\,\mathrm{d}\,x \le L\,(h(u),h(v)) \le \frac{h(u)+h(v)}{2},$$

where $L(u,v)$ *is the logarithmic mean defined by* (2.2).

Theorem 2.3. *Let* $h, g : J \subseteq \mathbb{R} \to \mathbb{R}_+$ *and* $u, v \in J$ *with* $u < v$. *If* h *is an* s_1*-logarithmically convex function on* $[u,v]$ *and* g *is an* s_2*-logarithmically convex function on* $[u,v]$ *for some numbers* $0 < s_1, s_2 \le 1$*, then*

$$\begin{aligned}h^{1/2^{1-s_1}}\left(\frac{u+v}{2}\right)g^{1/2^{1-s_2}}\left(\frac{u+v}{2}\right) &\le \frac{1}{v-u}\int_u^v [h(x)g(x)h(u+v-x)g(u+v-x)]^{1/2}\,\mathrm{d}\,x\\ &\le \frac{1}{v-u}\int_u^v h(x)g(x)\,\mathrm{d}\,x\\ &\le [h(u)h(v)]^{1-s_1}[g(u)g(v)]^{1-s_2}L([h(u)]^{s_1}[g(u)]^{s_2},[h(v)]^{s_1}[g(v)]^{s_2})\\ &\le \frac{1}{2}\,[h(u)h(v)]^{1-s_1}\,[g(u)g(v)]^{1-s_2}\Big\{[h(u)]^{s_1}[g(u)]^{s_2}+[h(v)]^{s_1}[g(v)]^{s_2}\Big\}.\end{aligned} \tag{2.5}$$

Proof. Using the s-logarithmic convexity of h and g, employing the inequality (2.3), we have

$$h^{1/2^{1-s_1}}\left(\frac{u+v}{2}\right)g^{1/2^{1-s_2}}\left(\frac{u+v}{2}\right) \le [h(\theta u+(1-\theta)v)h(u+v-(\theta u+(1-\theta)v))g(\theta u+(1-\theta)v)g(u+v-(\theta u+(1-\theta)v))]^{1/2}$$

for $\theta \in [0,1]$. Similarly, we have

$$\begin{aligned}
&\int_0^1 \left[h(\theta u+(1-\theta)v)h(u+v-(\theta u+(1-\theta)v))g(\theta u+(1-\theta)v)g(u+v-(\theta u+(1-\theta)v))\right]^{1/2}\,\mathrm{d}\,\theta \\
&\qquad = \frac{1}{v-u}\int_u^v [h(x)g(x)h(u+v-x)g(u+v-x)]^{1/2}\,\mathrm{d}\,x \le \frac{1}{v-u}\int_u^v h(x)g(x)\,\mathrm{d}\,x.
\end{aligned}$$

The left-hand side inequality in (2.5) is thus proved.

On the other hand,

$$\begin{aligned}
\frac{1}{v-u}\int_u^v h(x)g(x)\,\mathrm{d}\,x &= \int_0^1 h(\theta u+(1-\theta)v)g(\theta u+(1-\theta)v)\,\mathrm{d}\,\theta \\
&\le \int_0^1 [h(u)]^{\theta^{s_1}}[g(u)]^{\theta^{s_2}}[h(v)]^{(1-\theta)^{s_1}}[g(v)]^{(1-\theta)^{s_2}}\,\mathrm{d}\,\theta \\
&\le [h(u)h(v)]^{1-s_1}[g(u)g(v)]^{1-s_2}\int_0^1 \{[h(u)]^{s_1}[g(u)]^{s_2}\}^{\theta}\{[h(v)]^{s_1}[g(v)]^{s_2}\}^{1-\theta}\,\mathrm{d}\,\theta \\
&= [h(u)h(v)]^{1-s_1}[g(u)g(v)]^{1-s_2}L([h(u)]^{s_1}[g(u)]^{s_2},[h(v)]^{s_1}[g(v)]^{s_2}) \\
&\le \frac{1}{2}[h(u)h(v)]^{1-s_1}[g(u)g(v)]^{1-s_2}\{[h(u)]^{s_1}[g(u)]^{s_2}+[h(v)]^{s_1}[g(v)]^{s_2}\}.
\end{aligned}$$

Theorem 2.3 is thus proved. □

If putting $s_1 = s_2 = 1$ in Theorem 2.3, we deduce the following corollary.

Corollary 2.4. *Under the conditions of Theorem 2.3, if* $s_1 = s_2 = 1$, *then*

$$\begin{aligned}
h\left(\frac{u+v}{2}\right)g\left(\frac{u+v}{2}\right) &\le \frac{1}{v-u}\int_u^v [h(x)g(x)h(u+v-x)g(u+v-x)]^{1/2}\,\mathrm{d}\,x \\
&\le \frac{1}{v-u}\int_u^v h(x)g(x)\,\mathrm{d}\,x \\
&\le L\,(h(u)g(u),h(v)g(v)) \\
&\le \frac{h(u)g(u)+h(v)g(v)}{2},
\end{aligned}$$

where $L(u,v)$ *is the logarithmic mean defined in* (2.2).

3. Simple applications to simple means

For $u,v \in \mathbb{R}_+$ and $r \in \mathbb{R}$ with $r \ne 0$, define $A(u,v) = \frac{u+v}{2}$ and

$$L_r(u,v) = \begin{cases} \left[\dfrac{v^{r+1}-u^{r+1}}{(r+1)(v-u)}\right]^{1/r}, & u \ne v, r \ne -1; \\ \dfrac{v-u}{\ln v - \ln u}, & u \ne v, r = -1; \\ u, & u = v. \end{cases}$$

These means are respectively called the arithmetic mean and the generalized logarithmic means of two positive number $u > 0$ and $v > 0$.

We consider the function $h(u) = \frac{1}{rx^r}$ for $u \in \mathbb{R}_+$ and $r \in \mathbb{R}_+$. Since

$$h(tu+(1-t)v) \le \frac{1}{r}(u^t v^{1-t})^r = [h(u)]^t[h(v)]^{1-t}$$

holds for all $u,v \in \mathbb{R}_+$ and $t \in [0,1]$, then $h(u)$ is logarithmically convex on $\mathbb{R}_+$.

Theorem 3.1. *Let $v > u > 0$ and $r \in \mathbb{R}_+$. Then*

$$\frac{1}{[A(u,v)]^r} \le \frac{1}{[L_{-r}(u,v)]^r} \le L\left(\frac{1}{u^r}, \frac{1}{v^r}\right) \le A\left(\frac{1}{u^r}, \frac{1}{v^r}\right), \tag{3.1}$$

where $A(u,v)$, $L(u,v)$, and $L_r(u,v)$ are the arithmetic mean, the logarithmic mean, and the generalized logarithmic mean, respectively.

Proof. From the relations

$$h\left(\frac{u+v}{2}\right) = \frac{1}{r[A(u,v)]^r} \quad \text{and} \quad \frac{1}{v-u}\int_u^v h(x)\,\mathrm{d}\,x = \frac{1}{r[L_{-r}(u,v)]^r},$$

by virtue of Corollary 2.2, we conclude that the inequality (3.1) holds. Theorem 3.1 is thus proved. □

Similarly, in view of Corollary 2.4, we have

Theorem 3.2. *Let $v > u > 0$ and $r_1, r_2 \in \mathbb{R}_+$. Then*

$$\frac{1}{[A(u,v)]^{r_1+r_2}} \le \frac{1}{[L_{-(r_1+r_2)}(u,v)]^{r_1+r_2}} \le L\left(\frac{1}{u^{r_1+r_2}}, \frac{1}{v^{r_1+r_2}}\right) \le A\left(\frac{1}{u^{r_1+r_2}}, \frac{1}{v^{r_1+r_2}}\right).$$

Specially, if $r_1 = r_2 = r$, then

$$\frac{1}{[A(u,v)]^{2r}} \le \frac{1}{[L_{-2r}(u,v)]^{2r}} \le L\left(\frac{1}{u^{2r}}, \frac{1}{v^{2r}}\right) \le A\left(\frac{1}{u^{2r}}, \frac{1}{v^{2r}}\right).$$

4. Conclusions

The main results in this paper are those integral inequalities in (2.1) and (2.5) of the Hermite–Hadamard type for the products of s-logarithmically convex functions and those inequalities on the arithmetic mean, logarithmic mean, and the generalized logarithmic mean.

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