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On the 2-dissection and 4-dissection for Ramanujan-Selberg continued fraction and its reciprocal

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Abstract

In this paper, we establish the 2-dissection and 4-dissection for Ramanujan-Selberg continued fraction and its reciprocal. We show an eight-terms expansion of the reciprocal of Ramanujan-Selberg continued fraction. In the end we give some remarks and suggest two open problems and conjectures on the dissections of the Ramanujan-Selberg continued fraction.

Keywords: Ramanujan-Selberg continued fraction, dissection, reciprocal, Jacobi triple product identity, conjecture

2020 MSC: 11F27, 11P84, 11A55, 30B70

1. Introduction

For the two indeterminates q and z with $|q| < 1$, the q-shifted factorial of infinite order is defined by

$$(z;q)_\infty = \prod_{k=0}^{\infty}\left(1-zq^k\right) = (1-z)(1-zq)\cdots(1-zq^n)\cdots.$$

We have

$$(z;q)_\infty = \left(z, zq; q^2\right)_\infty, \tag{1.1}$$

$$\left(z^2;q^2\right)_\infty = (z;q)_\infty(-z;q)_\infty. \tag{1.2}$$

We adopt the following compact notations for the multiple q-shifted factorial:

$$(z_1, z_2, \ldots, z_m; q)_\infty = (z_1;q)_\infty (z_2;q)_\infty \cdots (z_m;q)_\infty.$$

Jacobi triple product identity is given by [7, Theorem 1.3.3, p. 10]

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty}\left(1+zq^{2n-1}\right)\left(1+q^{2n-1}/z\right)\left(1-q^{2n}\right) = \left(-zq, -q/z, q^2; q^2\right)_\infty, \quad z \neq 0. \tag{1.3}$$

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Definition 1.1. The m-dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the presentation of as $P = P_0 + P_1 + \cdots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k},\ k = 0, 1, \ldots, m-1$.

The celebrated Rogers-Ramanujan continued fraction is defined as [3, p. 153]

$$R(q) := \frac{1}{1+}\ \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+} \cdots = \frac{\left(q, q^4; q^5\right)_\infty}{(q^2, q^3; q^5)_\infty}. \tag{1.4}$$

On page 365 of his Lost Notebook [22], Ramanujan wrote five identities which shows the relation between $R(q)$ and five continued fractions $R(-q)$, $R\left(q^2\right), R\left(q^3\right), R\left(q^4\right)$, and $R\left(q^5\right)$. Rogers [25] found the identity relating $R(q)$ with $R\left(q^n\right)$ for $n = 2, 3, 5$ and 11. The identity (1.4) was first established by Rogers [24]. Ramanujan [22] gave 2-dissections of $R(q)$ and its reciprocal, and these were first proved by Andrews [2]. Ramanujan [22] gave 5-dissections of $R(q)$ and its reciprocal, and these results were improved upon and proved by Hirschhorn [15], Hirschhorn conjectured formulas for 4-dissections of $R(q)$ and its reciprocal, and these were first proved by Lewis and Liu [18]. Hirschhorn [14] also gave an elementary proof of his conjecture. Recently, Hirschhorn [13] gave the 2- and 4-dissections of the Rogers-Ramanujan functions which the 2-dissections lead to Gordon's formulas.

On page 366 of his Lost Notebook [22], Ramanujan investigated the continued fraction (now call Ramanujan cubic continued fraction) [3, p. 154]

$$G(q) := \frac{1}{1+}\ \frac{q+q^2}{1+}\ \frac{q^2+q^4}{1+}\ \frac{q^3+q^6}{1+} \cdots = \frac{\left(q, q^5; q^6\right)_\infty}{(q^3, q^3; q^6)_\infty}, \tag{1.5}$$

and claimed that there are many results of $G(q)$. This identity was first established by Ramanujan [22]. The 2- and 4-dissections of $1/G(q)$ were first given by Srivastava [30]. Hirschhorn and Roselin [16] also obtained the 2-, 3-, 4- and 6-dissections of Ramanujan cubic continued fraction and its reciprocal.

On page 299 of his Second Notebook [21], Ramanujan recorded another continued fraction (now call Ramanujan-Göllnitz-Gordon continued fraction)

$$H(q) := \frac{1}{1+q+}\ \frac{q^2}{1+q^3+}\ \frac{q^4}{1+q^5+}\ \frac{q^6}{1+q^7+} \cdots = \frac{\left(q, q^7; q^8\right)_\infty}{(q^3, q^5; q^8)_\infty}. \tag{1.6}$$

This identity was first established by Ramanujan [21]. Without any knowledge of Ramanujan's work, Göllnitz [10] and Gordon [11] rediscovered and proved (1.6) independently. On page 44 of his Lost Notebook [22], Ramanujan also stated

$$H(q) = \frac{1}{1+}\ \frac{q+q^2}{1+}\ \frac{q^4}{1+}\ \frac{q^3+q^6}{1+}\ \frac{q^8}{1+} \cdots = \frac{\left(q, q^7; q^8\right)_\infty}{(q^3, q^5; q^8)_\infty}, \tag{1.7}$$

which first proved by Selberg [28, Eq. (53)], [29, p. 18-19]. Hirschhorn [12] established 8-dissections of $H(q)$ and its reciprocal, thereby demonstrating the periodicity of the sign of the coefficients in expansions of $H(q)$ and its reciprocal, and in particular that certain coefficients are zero, a phenomenon first observed and shown by Richmond and Szekeres [23]. Alladi and Gordon [1], Andrews and Bressoud [5] and Chan and Yesilyurt [9] generalized these themes. Vasuki and Srivasta Kumar [31] obtained certain identities for Ramanujan-Göllnitz-Gordan continued fraction. Xia and Yao [32, 34] proved the 8-dissection of the Ramanujan-Göllnitz-Gordon continued fraction by an iterative method.

The Ramanujan-Selberg continued fraction is defined by [3, p. 150], [29]

$$T(q) := \frac{1}{1+}\ \frac{q}{1+}\ \frac{q+q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^2+q^4}{1+} \cdots. \tag{1.8}$$

Independently, Ramanujan and Selberg studied this interesting continued fraction. Ramanujan asserted in his notebook [21, p. 290] that

$$T(q) = \frac{\left(-q^2; q^2\right)_\infty}{(-q; q^2)_\infty} = \frac{\left(q; q^2\right)_\infty}{(q^2, q^2; q^4)_\infty}. \tag{1.9}$$

It was A. Selberg, unaware of Ramanujan's assertion at the time of writing, who gave the first proof of (1.9) in print [28]. Other proofs were discovered by Ramanathan [20], Andrews, Berndt, Jacobsen and Lamphere [4], and Zhang [36]. See also the paper by Baruah and Saikia [6] for a penetrating study of $T(q)$ (and related continued fractions). Zhang [35, 37, 38] investigated the Ramanujan-Selberg continued fractions and given some explicit evaluations of two Ramanujan-Selberg continued fractions. Chan [8] obtained a modular equation of the Ramanujan-Selberg continued fraction to give a proof of the celebrated Jacobian identity. Saikia [26] showed two theta-function identities for the Ramanujan-Selberg continued fraction lead to some modular equations of $T(q)$. Lee and Park [17] studied the modularity of the Ramanujan-Selberg continued fraction. Saikia and Boruah [27] gave some new explicit values of Ramanujan-Selberg continued fraction. Nevertheless it seem not to consider by people on the dissections of Ramanujan-Selberg continued fraction.

Motivated by the above works, especially by the wonderful ideas and methods of Hirschhorn ([12] and [13]), we will investigate the 2-dissection and 4-dissection for Ramanujan-Selberg continued fraction and its reciprocal based on Jacobi triple product identity and a result of Hirschhorn.

In the Section 2 of this paper, we establish the 2-dissection and 4-dissection for $1/T(q)$ with Jacobi triple product identity. We also show an eight-terms expansion of $1/T(q)$ based on Hirschhorn's result [13, p. 231]. In the Section 3, we establish the 2-dissection of $T(q)$ and 4-dissection for $(q^2;q^4)_\infty^3 T(q)$. In the last section, we give some remarks, and suggest two open problems and conjectures on the dissections of Ramanujan-Selberg continued fraction.

2. 2-dissection and 4-dissection of $1/T(q)$

In this section, we will establish the 2-dissection and 4-dissection of the reciprocal of Ramanujan-Selberg continued fraction.

Theorem 2.1. *The following* 2*-dissection of* $1/T(q)$ *holds*

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\left(q,q^7,q^8;q^8\right)_\infty \left(q^6,q^{10};q^{16}\right)_\infty}{(q^2;q^2)_\infty}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} a_{2n+1}q^n = \frac{\left(q^3,q^5,q^8;q^8\right)_\infty \left(q^2,q^{14};q^{16}\right)_\infty}{(q^2;q^2)_\infty}. \tag{2.2}$$

Proof. Let $1/T(q) = \sum_{n=0}^{\infty} a_n q^n$. Write

$$\sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_{2n}q^{2n} + \sum_{n=0}^{\infty} a_{2n+1}q^{2n+1}. \tag{2.3}$$

On the other hand, we have, from (1.9)

$$\begin{aligned} 1/T(q) =& \frac{\left(q^2,q^2;q^4\right)_\infty}{(q;q^2)_\infty} = \frac{\left(q^2,q^2;q^4\right)_\infty}{(q,q^3;q^4)_\infty} = \frac{\left(q^2,q^2;q^4\right)_\infty}{\prod_{n=1}^{\infty}(1-q^{4n-3})(1-q^{4n-1})} \\ =& \frac{\left(q^2,q^2;q^4\right)_\infty \prod_{n=1}^{\infty}\left(1+q^{4n-3}\right)\left(1+q^{4n-1}\right)\left(1-q^{4n}\right)}{\prod_{n=1}^{\infty}(1-q^{8n-6})(1-q^{8n-2})(1-q^{4n})} \\ =& \frac{\left(q^2;q^4\right)_\infty \prod_{n=1}^{\infty}\left(1+q^{4n-3}\right)\left(1+q^{4n-1}\right)\left(1-q^{4n}\right)}{(q^4;q^4)_\infty}. \end{aligned} \tag{2.4}$$

Let $q \longmapsto q^2$ and then set $z = q$ in (1.3), we have

$$\prod_{n=1}^{\infty}\left(1+q^{4n-1}\right)\left(1+q^{4n-3}\right)\left(1-q^{4n}\right) = \sum_{n=-\infty}^{\infty} q^{2n^2+n}. \tag{2.5}$$

Replace (2.5) into (2.4), becomes

$$\begin{aligned}1/T(q) =&\frac{\left(q^2;q^4\right)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty} q^{2n^2+n}\\ =&\frac{\left(q^2;q^4\right)_\infty}{(q^4;q^4)_\infty}\left\{\sum_{n=-\infty}^{\infty} q^{8n^2+2n}+q\sum_{n=-\infty}^{\infty} q^{8n^2-6n}\right\}.\end{aligned}\tag{2.6}$$

Let $q\longmapsto q^8$ and then take $z=q^2$ and $z=q^{-6}$ in (1.3), respectively, we get

$$\sum_{n=-\infty}^{\infty} q^{8n^2+2n}=\left(-q^6,-q^{10},q^{16};q^{16}\right)_\infty,\tag{2.7}$$

$$\sum_{n=-\infty}^{\infty} q^{8n^2-6n}=\left(-q^2,-q^{14},q^{16};q^{16}\right)_\infty.\tag{2.8}$$

Substitute (2.7) and (2.8) into (2.6) and use (1.1) and (1.2), we obtain

$$\begin{aligned}1/T(q) =&\frac{\left(q^2;q^4\right)_\infty}{(q^4;q^4)_\infty}\left\{\left(-q^6,-q^{10},q^{16};q^{16}\right)_\infty+q\left(-q^2,-q^{14},q^{16};q^{16}\right)_\infty\right\}\\ =&\frac{\left(q^2;q^4\right)_\infty}{(q^4;q^4)_\infty}\left\{\frac{\left(q^{12},q^{20};q^{32}\right)_\infty\left(q^{16};q^{16}\right)_\infty}{(q^6,q^{10};q^{16})_\infty}+q\frac{\left(q^4,q^{28};q^{32}\right)_\infty\left(q^{16};q^{16}\right)_\infty}{(q^2,q^{14};q^{16})_\infty}\right\}.\end{aligned}\tag{2.9}$$

Hence, by (2.3) and (2.9), we find that

$$\sum_{n=0}^{\infty} a_{2n}q^{2n}=\frac{\left(q^2;q^4\right)_\infty\left(q^{12},q^{20};q^{32}\right)_\infty\left(q^{16};q^{16}\right)_\infty}{(q^4;q^4)_\infty(q^6,q^{10};q^{16})_\infty},$$

$$\sum_{n=0}^{\infty} a_{2n+1}q^{2n}=\frac{\left(q^2;q^4\right)_\infty\left(q^{4},q^{28};q^{32}\right)_\infty\left(q^{16};q^{16}\right)_\infty}{(q^4;q^4)_\infty(q^2,q^{14};q^{16})_\infty}.$$

Replace q by $q^{\frac{1}{2}}$ in the above identities, we obtain Theorem 2.1. This proof is completed. □

Theorem 2.2. *The following* 4*-dissection of* $1/T(q)$ *holds*

$$\sum_{n=0}^{\infty} a_{4n}q^{n}=\frac{\left(q^3,q^{13},q^{16};q^{16}\right)_\infty\left(q^{10},q^{22};q^{32}\right)_\infty}{(q;q)_\infty},\tag{2.10}$$

$$\sum_{n=0}^{\infty} a_{4n+1}q^{n}=\frac{\left(q,q^{15},q^{16};q^{16}\right)_\infty\left(q^{14},q^{18};q^{32}\right)_\infty}{(q;q)_\infty},\tag{2.11}$$

$$\sum_{n=0}^{\infty} a_{4n+2}q^{n}=-\frac{\left(q^5,q^{11},q^{16};q^{16}\right)_\infty\left(q^{6},q^{26};q^{32}\right)_\infty}{(q;q)_\infty},\tag{2.12}$$

$$\sum_{n=0}^{\infty} a_{4n+3}q^{n}=-q\frac{\left(q^7,q^{9},q^{16};q^{16}\right)_\infty\left(q^{2},q^{30};q^{32}\right)_\infty}{(q;q)_\infty}.\tag{2.13}$$

Proof. From (2.3) write

$$\sum_{n=0}^{\infty} a_{2n}q^{2n}=\sum_{n=0}^{\infty} a_{4n}q^{4n}+\sum_{n=0}^{\infty} a_{4n+2}q^{4n+2},$$

$$\sum_{n=0}^{\infty} a_{2n+1}q^{2n+1}=\sum_{n=0}^{\infty} a_{4n+1}q^{4n+1}+\sum_{n=0}^{\infty} a_{4n+3}q^{4n+3}.$$

By (2.1), we have

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\left(q, q^7, q^8; q^8\right)_\infty \left(q^6, q^{10}; q^{16}\right)_\infty}{(q^2; q^2)_\infty}$$
$$= \frac{\left(q^6, q^{10}; q^{16}\right)_\infty \prod_{n=1}^{\infty} \left(1 - q^{8n-7}\right)\left(1 - q^{8n-1}\right)\left(1 - q^{8n}\right)}{(q^2; q^2)_\infty}. \tag{2.14}$$

Let $q \longmapsto -q^4$ and then set $z = q^3$ in (1.3), we have

$$\prod_{n=1}^{\infty} \left(1 - q^{8n-7}\right)\left(1 - q^{8n-1}\right)\left(1 - q^{8n}\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2+3n}. \tag{2.15}$$

Then (2.14) becomes

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\left(q^6, q^{10}; q^{16}\right)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2+3n}$$
$$= \frac{\left(q^6, q^{10}; q^{16}\right)_\infty}{(q^2; q^2)_\infty} \left\{ \sum_{n=-\infty}^{\infty} q^{16n^2+6n} - q \sum_{n=-\infty}^{\infty} q^{16n^2-10n} \right\}. \tag{2.16}$$

Let $q \longmapsto q^{16}$ and then take $z = q^6$ and $z = q^{-10}$ in (1.3), respectively, we deduce that

$$\sum_{n=-\infty}^{\infty} q^{16n^2+6n} = \left(-q^{10}, -q^{22}, q^{32}; q^{32}\right)_\infty, \tag{2.17}$$
$$\sum_{n=-\infty}^{\infty} q^{16n^2-10n} = \left(-q^6, -q^{26}, q^{32}; q^{32}\right)_\infty. \tag{2.18}$$

Therefore, by (1.2) we obtain

$$\sum_{n=0}^{\infty} a_{4n}q^{2n} = \frac{\left(q^6, q^{10}; q^{16}\right)_\infty \left(q^{20}, q^{32}, q^{44}, q^{64}; q^{64}\right)_\infty}{(q^2; q^2)_\infty (q^{10}, q^{22}; q^{32})_\infty},$$
$$\sum_{n=0}^{\infty} a_{4n+2}q^{2n} = -\frac{\left(q^6, q^{10}; q^{16}\right)_\infty \left(q^{12}, q^{32}, q^{52}, q^{64}; q^{64}\right)_\infty}{(q^2; q^2)_\infty (q^6, q^{26}; q^{32})_\infty}.$$

Replace q by $q^{\frac{1}{2}}$ in the above identities, we get

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{\left(q^3, q^{13}, q^{16}; q^{16}\right)_\infty \left(q^{10}, q^{22}; q^{32}\right)_\infty}{(q; q)_\infty},$$
$$\sum_{n=0}^{\infty} a_{4n+2}q^n = -\frac{\left(q^5, q^{11}, q^{16}; q^{16}\right)_\infty \left(q^6, q^{26}; q^{32}\right)_\infty}{(q; q)_\infty}.$$

In the same manner, we can obtain

$$\sum_{n=0}^{\infty} a_{4n+1}q^n = \frac{\left(q, q^{15}, q^{16}; q^{16}\right)_\infty \left(q^{14}, q^{18}; q^{32}\right)_\infty}{(q; q)_\infty},$$
$$\sum_{n=0}^{\infty} a_{4n+3}q^n = -q\frac{\left(q^7, q^9, q^{16}; q^{16}\right)_\infty \left(q^2, q^{30}; q^{32}\right)_\infty}{(q; q)_\infty}.$$

This completes our proof. □

We below give an eight-terms expansion of $1/T(q)$. We need the following lemma.

Lemma 2.3 (*cf.* Hirschhorn [13, p. 231])**.**

$$\frac{1}{(q;q)_\infty}=\frac{1}{(q^2;q^2)_\infty^2}\left(\frac{\left(q^{12},q^{16},q^{20},q^{32};q^{32}\right)_\infty}{(q^6,q^{10};q^{16})_\infty}+q\frac{\left(q^4,q^{16},q^{28},q^{32};q^{32}\right)_\infty}{(q^2,q^{14};q^{16})_\infty}\right). \tag{2.19}$$

Theorem 2.4. *We have*

$$\begin{aligned}1/T(q)=&\frac{\left(q^{12},q^{52},q^{64};q^{64}\right)_\infty\left(q^{48},q^{40},q^{64},q^{80},q^{88},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{24},q^{40};q^{64})_\infty}\\
&+q\frac{\left(q^{4},q^{60},q^{64};q^{64}\right)_\infty\left(q^{48},q^{56},q^{64},q^{72},q^{80},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{24},q^{40};q^{64})_\infty}\\
&-q^2\frac{\left(q^{20},q^{44},q^{64};q^{64}\right)_\infty\left(q^{24},q^{48},q^{64},q^{80},q^{104},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{24},q^{40};q^{64})_\infty}\\
&+q^4\frac{\left(q^{12},q^{52},q^{64};q^{64}\right)_\infty\left(q^{16},q^{40}q^{64},q^{88},q^{112},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{8},q^{56};q^{64})_\infty}\\
&+q^5\frac{\left(q^{4},q^{60},q^{64};q^{64}\right)_\infty\left(q^{16},q^{56},q^{64},q^{72},q^{112},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{8},q^{56};q^{64})_\infty}\\
&-q^6\frac{\left(q^{20},q^{44},q^{64};q^{64}\right)_\infty\left(q^{16},q^{24},q^{64},q^{108},q^{112},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{8},q^{56};q^{64})_\infty}\\
&-q^7\frac{\left(q^{28},q^{36},q^{64};q^{64}\right)_\infty\left(q^{8},q^{48},q^{64},q^{80},q^{120},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{24},q^{40};q^{64})_\infty}\\
&-q^{11}\frac{\left(q^{28},q^{36},q^{64};q^{64}\right)_\infty\left(q^{8},q^{16},q^{64},q^{112},q^{120},q^{128};q^{128}\right)_\infty}{(q^8;q^8)_\infty^2\,(q^{8},q^{56};q^{64})_\infty}.\end{aligned}$$

Proof. Applying (2.19) to Theorem 2.2, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}a_{4n}q^n=&\frac{\left(q^{3},q^{13},q^{16};q^{16}\right)_\infty\left(q^{12},q^{10},q^{16},q^{20},q^{22},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2\,(q^6,q^{10};q^{16})_\infty}\\
&+q\frac{\left(q^{3},q^{13},q^{16};q^{16}\right)_\infty\left(q^{4},q^{10}q^{16},q^{22},q^{28},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2\,(q^2,q^{14};q^{16})_\infty},\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}a_{4n+1}q^n=&\frac{\left(q,q^{15},q^{16};q^{16}\right)_\infty\left(q^{12},q^{14},q^{16},q^{18},q^{20},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2\,(q^6,q^{10};q^{16})_\infty}\\
&+q\frac{\left(q,q^{15},q^{16};q^{16}\right)_\infty\left(q^{4},q^{14},q^{16},q^{18},q^{28},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2\,(q^2,q^{14};q^{16})_\infty},\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}a_{4n+2}q^n=&-\frac{\left(q^{5},q^{11},q^{16};q^{16}\right)_\infty\left(q^{6},q^{12},q^{16},q^{20},q^{26},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2\,(q^6,q^{10};q^{16})_\infty}\\
&-q\frac{\left(q^{5},q^{11},q^{16};q^{16}\right)_\infty\left(q^{4},q^{6}q^{16},q^{26},q^{28},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2\,(q^2,q^{14};q^{16})_\infty},\end{aligned}$$

$$\sum_{n=0}^{\infty} a_{4n+3}q^n = -q\frac{\left(q^7,q^9,q^{16};q^{16}\right)_\infty \left(q^2,q^{12},q^{16},q^{20},q^{30},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2 (q^6,q^{10};q^{16})_\infty}$$
$$-q^2\frac{\left(q^7,q^9,q^{16};q^{16}\right)_\infty \left(q^2,q^4,q^{16},q^{28},q^{30},q^{32};q^{32}\right)_\infty}{(q^2;q^2)_\infty^2 (q^2,q^{14};q^{16})_\infty}.$$

It follows that Theorem 2.4. The proof is completed. □

3. 2-dissection of $T(q)$ and 4-dissection of $\left(q^2;q^4\right)_\infty^3 T(q)$

In this section, we establish the 2-dissection of Ramanujan-Selberg continued fraction and 4-dissection of $\left(q^2;q^4\right)_\infty^3 T(q)$.

Theorem 3.1. *The following* 2*-dissection of* $T(q)$ *holds*

$$\sum_{n=0}^{\infty} b_{2n}q^n = \frac{\left(q,q^7,q^8;q^8\right)_\infty \left(q^6,q^{10};q^{16}\right)_\infty}{(q^2;q^2)_\infty (q;q^2)_\infty^3}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^n = -\frac{\left(q^3,q^5,q^8;q^8\right)_\infty \left(q^2,q^{14};q^{16}\right)_\infty}{(q^2;q^2)_\infty (q;q^2)_\infty^3}. \tag{3.2}$$

Proof. Let $T(q) = \sum_{n=0}^{\infty} b_n q^n$. Write

$$\sum_{n=0}^{\infty} b_n q^n = \sum_{n=0}^{\infty} b_{2n}q^{2n} + \sum_{n=0}^{\infty} b_{2n+1}q^{2n+1}. \tag{3.3}$$

On the other hand, we have, from (1.9)

$$T(q) = \frac{\left(q;q^2\right)_\infty}{(q^2,q^2;q^4)_\infty} = \frac{\left(q,q^3;q^4\right)_\infty}{(q^2,q^2;q^4)_\infty} = \frac{\prod_{n=1}^{\infty}\left(1-q^{4n-3}\right)\left(1-q^{4n-1}\right)\left(1-q^{4n}\right)}{(q^2,q^2,q^4;q^4)_\infty}. \tag{3.4}$$

Let $q \longmapsto q^2$ and then set $z = -q$ in (1.3), we have

$$\prod_{n=1}^{\infty}\left(1-q^{4n-1}\right)\left(1-q^{4n-3}\right)\left(1-q^{4n}\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n}. \tag{3.5}$$

Replace (3.5) into (3.4), becomes

$$T(q) = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}}{(q^2,q^2,q^4;q^4)_\infty} = \frac{1}{(q^2,q^2,q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty} q^{8n^2+2n} - q\sum_{n=-\infty}^{\infty} q^{8n^2-6n}\right]. \tag{3.6}$$

Let $q \longmapsto q^8$ and then take $z = q^2$ and $z = q^{-6}$ in (1.3), respectively, we get

$$\sum_{n=-\infty}^{\infty} q^{8n^2+2n} = \left(-q^6,-q^{10},q^{16};q^{16}\right)_\infty, \tag{3.7}$$

$$\sum_{n=-\infty}^{\infty} q^{8n^2-6n} = \left(-q^2,-q^{14},q^{16};q^{16}\right)_\infty. \tag{3.8}$$

Substitute (3.7) and (3.8) into (3.6) and use (1.1) and (1.2), we obtain

$$\begin{aligned} T(q) =& \frac{1}{(q^2,q^2,q^4;q^4)_\infty}\left[\left(-q^6,-q^{10},q^{16};q^{16}\right)_\infty - q\left(-q^2,-q^{14},q^{16};q^{16}\right)_\infty\right] \\ =& \frac{1}{(q^2,q^2,q^4;q^4)_\infty}\left[\frac{\left(q^{12},q^{20};q^{32}\right)_\infty\left(q^{16};q^{16}\right)_\infty}{(q^6,q^{10};q^{16})_\infty} - q\frac{\left(q^4,q^{28};q^{32}\right)_\infty\left(q^{16};q^{16}\right)_\infty}{(q^2,q^{14};q^{16})_\infty}\right]. \end{aligned} \tag{3.9}$$

Hence, by (3.3) and (3.9), we find that

$$\sum_{n=0}^{\infty} b_{2n}q^{2n} = \frac{\left(q^{16};q^{16}\right)_\infty\left(q^{12},q^{20};q^{32}\right)_\infty}{(q^2,q^2,q^4;q^4)_\infty(q^6,q^{10};q^{16})_\infty},$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^{2n} = -\frac{\left(q^{16};q^{16}\right)_\infty\left(q^4,q^{28};q^{32}\right)_\infty}{(q^2,q^2,q^4;q^4)_\infty(q^2,q^{14};q^{16})_\infty}.$$

Replace q by $q^{\frac{1}{2}}$ in the above identities after using $\left(q^2;q^4\right)_\infty = \left(q^2,q^6,q^{10},q^{14};q^{16}\right)_\infty$, we obtain Theorem 3.1. This proof is completed. □

Theorem 3.2. *The following* 4*-dissection of* $\left(q^2;q^4\right)_\infty^3 T(q)$ *holds*

$$\sum_{n=0}^{\infty} c_{4n}q^n = \frac{\left(q^3,q^{13},q^{16};q^{16}\right)_\infty\left(q^{10},q^{22};q^{32}\right)_\infty}{(q;q)_\infty}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} c_{4n+1}q^n = -\frac{\left(q,q^{15},q^{16};q^{16}\right)_\infty\left(q^{14},q^{18};q^{32}\right)_\infty}{(q;q)_\infty}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} c_{4n+2}q^n = -\frac{\left(q^5,q^{11},q^{16};q^{16}\right)_\infty\left(q^6,q^{26};q^{32}\right)_\infty}{(q;q)_\infty}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} c_{4n+3}q^n = q\frac{\left(q^7,q^9,q^{16};q^{16}\right)_\infty\left(q^2,q^{30};q^{32}\right)_\infty}{(q;q)_\infty}. \tag{3.13}$$

Proof. Let $\left(q^2;q^4\right)_\infty^3 T(q) = \sum_{n=0}^{\infty} c_nq^n$. Write

$$\sum_{n=0}^{\infty} c_nq^n = \sum_{n=0}^{\infty} c_{4n}q^{4n} + \sum_{n=0}^{\infty} c_{4n+1}q^{4n+1} + \sum_{n=0}^{\infty} c_{4n+2}q^{4n+2} + \sum_{n=0}^{\infty} c_{4n+3}q^{4n+3}. \tag{3.14}$$

On the other hand, we write, from (3.3)

$$\sum_{n=0}^{\infty} b_{2n}q^{2n} = \sum_{n=0}^{\infty} b_{4n}q^{4n} + \sum_{n=0}^{\infty} b_{4n+2}q^{4n+2}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^{2n+1} = \sum_{n=0}^{\infty} b_{4n+1}q^{4n+1} + \sum_{n=0}^{\infty} b_{4n+3}q^{4n+3}. \tag{3.16}$$

It is easily seen that, from Theorem 2.1 and Theorem 3.1

$$\sum_{n=0}^{\infty} b_{2n}q^n = \frac{1}{(q;q^2)_\infty^3}\sum_{n=0}^{\infty} a_{2n}q^n, \tag{3.17}$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^n = -\frac{1}{(q;q^2)_\infty^3}\sum_{n=0}^{\infty} a_{2n+1}q^n. \tag{3.18}$$

Applying Theorem 2.2 and combining (3.15) and (3.16), we obtain

$$\sum_{n=0}^{\infty} b_{4n}q^n = \frac{\left(q^3, q^{13}, q^{16}; q^{16}\right)_\infty \left(q^{10}, q^{22}; q^{32}\right)_\infty}{\left(q^{\frac{1}{2}}; q\right)_\infty^3 (q;q)_\infty}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} b_{4n+1}q^n = -\frac{\left(q, q^{15}, q^{16}; q^{16}\right)_\infty \left(q^{14}, q^{18}; q^{32}\right)_\infty}{\left(q^{\frac{1}{2}}; q\right)_\infty^3 (q;q)_\infty}, \tag{3.20}$$

$$\sum_{n=0}^{\infty} b_{4n+2}q^n = -\frac{\left(q^5, q^{11}, q^{16}; q^{16}\right)_\infty \left(q^{6}, q^{26}; q^{32}\right)_\infty}{\left(q^{\frac{1}{2}}; q\right)_\infty^3 (q;q)_\infty}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} b_{4n+3}q^n = q\frac{\left(q^7, q^{9}, q^{16}; q^{16}\right)_\infty \left(q^{2}, q^{30}; q^{32}\right)_\infty}{\left(q^{\frac{1}{2}}; q\right)_\infty^3 (q;q)_\infty}. \tag{3.22}$$

Multiplying both sides of the above identities by $\left(q^{\frac{1}{2}}; q\right)_\infty$, noting that (3.14), we complete the proof of Theorem 3.2. □

By Theorem 2.2 and Theorem 3.2, we easily get the following relations between their coefficients a_n and c_n.

Conjecture 3.3. We have, for $n \geq 0$

$$a_{4n} = c_{4n}, \quad a_{4n+1} = -c_{4n+1},$$
$$a_{4n+2} = c_{4n+2}, \quad a_{4n+3} = -c_{4n+3}.$$

4. Remarks, two open problems and conjectures

In this section, we give some remarks and suggest two open problems and conjectures open the dissections of the Ramanujan-Selberg continued fraction.

Remark 4.1. We can give the second proof the 2-dissection of $1/T(q)$ using Hirschhorn's result. If we rewrite the representation of $1/T(q)$ as

$$1/T(q) = \frac{\left(q^2, q^2; q^4\right)_\infty \left(q^2; q^2\right)_\infty}{(q;q)_\infty}.$$

Applying (2.19), Theorem 2.1 follows.

Remark 4.2. In fact, our Theorem 3.2 give the 4-dissection of $\frac{(q;q)_\infty^2}{(q^4;q^4)_\infty}$:

$$\begin{aligned}\frac{(q;q)_\infty^2}{(q^4;q^4)_\infty} =& \left(q^3, q^{13}, q^{16}; q^{16}\right)_\infty \left(q^{10}, q^{22}; q^{32}\right)_\infty - q\left(q, q^{15}, q^{16}; q^{16}\right)_\infty \left(q^{14}, q^{18}; q^{32}\right)_\infty \\ &- q^2\left(q^5, q^{11}, q^{16}; q^{16}\right)_\infty \left(q^{6}, q^{26}; q^{32}\right)_\infty + q^7\left(q^7, q^{9}, q^{16}; q^{16}\right)_\infty \left(q^{2}, q^{30}; q^{32}\right)_\infty.\end{aligned}$$

Hirschhorn [16] gave more dissections of q-infinite products, the following are two analogue forms $\frac{(q^2;q^2)_\infty}{(q;q)_\infty}$ and $\frac{(q;q)_\infty}{(q^2;q^2)_\infty}$ on the 3-dissection as:

$$\frac{\left(q^2;q^2\right)_\infty}{(q;q)_\infty} = \frac{\left(q^6;q^6\right)_\infty}{(q^3;q^3)_\infty^2}\left[\left(-q^{12}, -q^{15}, q^{27}; q^{27}\right)_\infty + q\left(-q^{6}, -q^{21}, q^{27}; q^{27}\right)_\infty + q^2\left(-q^{3}, -q^{24}, q^{27}; q^{27}\right)_\infty\right],$$

$$\frac{(q;q)_\infty}{(q^2;q^2)_\infty} = \frac{\left(q^3;q^3\right)_\infty}{(q^6;q^6)_\infty^2}\left[\left(q^{21}, q^{33}, q^{54}; q^{54}\right)_\infty - q\left(q^{15}, q^{39}, q^{54}; q^{54}\right)_\infty - q^5\left(q^{3}, q^{51}, q^{54}; q^{54}\right)_\infty\right].$$

Remark 4.3. We can prove that Theorem 2.2 is equivalent to the result of Xia and Yao [33, Eq. (1.7), p. 2036] which has a typing error in the last term.

Open Problem 4.4. How do we establish the 4-dissection of Ramanujan-Selberg continued fraction $T(q)$?

Conjecture 4.5. Let $T(q) = \sum_{n=0}^{\infty} b_n q^n$. If the 4-dissection of $T(q)$ exists and can be found, then the sign of each of b_n is periodic with period 4, for $n \geq 0$

$$b_{4n} > 0,\ b_{4n+1} < 0,\ b_{4n+2} < 0,\ b_{4n+3} > 0.$$

Open Problem 4.6. How do we find the 8-dissection of $1/T(q)$?

Conjecture 4.7. Let $1/T(q) = \sum_{n=0}^{\infty} a_n q^n$. If the 8-dissection of $1/T(q)$ exists and can be found, then the sign of each of a_n is periodic with period 8, for $n \geq 0$

$$a_{8n} > 0,\ a_{8n+1} > 0,\ a_{8n+2} < 0,\ a_{8n+3} < 0,$$
$$a_{8n+4} > 0,\ a_{8n+5} > 0,\ a_{8n+6} < 0,\ a_{8n+7} < 0.$$

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References

[1] K. Alladi and B. Gordon, *Vanishing coefficients in the expansion of products of Rogers-Ramanujan type, The Rademacher legacy to mathematics*, Contemp. Math. **166**, 129–139, 1994.
[2] G. E. Andrews, *Ramanujan's lost notebook III The Rogers-Ramanujan continued fraction*, Adv. Math. **41**, 186–208, 1981.
[3] G. E. Andrews and B. C. Berndt, *Ramanujan's lost notebook part I*, Springer-Verlag, New York, 2012.
[4] G. E. Andrews, B. C. Berndt, L. Jacobsen and R. L. Lamphere, *The continued fractions found in the unorganized portions of Ramanujan notebooks*, Mem. Amer. Math. Soc. **99 (477)**, 1992.
[5] G. E. Andrews and D. Bressoud, *Vanishing coefficients in infinite product expansions*, J. Aust. Math. Soc. **27**, 199–202, 1979.
[6] N. D. Baruah and N. Saikia, *Modular relations and explicit values of Ramanujan-Selberg continued fractions*, Int. J. Math. Math. Sci. **2006**, 2016; Article ID: 54901.
[7] B. C. Berndt, *Number theory in the spirit of Ramanujan*, American Mathematical Society, 2006.
[8] H.-C. Chan, *From a Ramanujan-Selberg continued fraction to a Jacobian identity*, Proc. Amer. Math. Soc. **137**, 2849–2856, 2009.
[9] S. H. Chan and H. Yesilyurt, *The periodicity of the signs of the coefficients of certain infinite products*, Pac. J. Math. **225**, 13–32, 2006.
[10] H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225**, 154–190, 1967.
[11] B. Gordon, *Some continued fractions of the Rogers-Ramanujan type*, Duke Math. J. **32**, 741–748, 1965.
[12] M. D. Hirschhorn, *On the expansion of a continued fraction of Gordon*, Ramanujan J. **5**, 369–375, 2001.
[13] M. D. Hirschhorn, *On the 2- and 4-dissections of the Rogers-Ramanujan functions*, Ramanujan J. **40**, 227–235, 2016.
[14] M. D. Hirschhorn, *On the 2- and 4-dissections of Ramanujan's continued fraction and its reciprocal*, Ramanujan J. **24**, 85–92, 2011.
[15] M. D. Hirschhorn, *On the expansion of Ramanujan's continued fraction*, Ramanujan J. **2**, 521–527, 1998.
[16] M. D. Hirschhorn and Roselin, *On the 2-, 3-, 4- and 6-dissections of Ramanujan's cubic continued fraction and its recipricol*, In: Proc. of Ramanujan Rediscovered, Bangalore, India, June 1-5, 2009, RMS Lecture Note Series, **14**, 125–138, 2009.
[17] Y. Lee and Y. K. Park, *Modularity of a Ramanujan-Selberg continued fraction*, J. Math. Anal. Appl. **438**, 373–394, 2016.
[18] R. P. Lewis and Z. G. Liu, *A conjecture of Hirschhorn on the 4-dissection of Ramanujan's continued fraction*, Ramanujan J. **4**, 347–352, 1991.
[19] K. G. Ramanathan, *Hypergeometric series and continued fraction*, Proc. Indian Acad. Sci. Math. Sci. **97**, 227–296, 1987.
[20] K. G. Ramanathan, *Ramanujan's continued fraction*, Indian J. Pure Appl. Math. **16**, 695–724, 1985.
[21] S. Ramanujan, *Notebooks* (2 Volumes), Tata Institute of Fundamental Research, Bombay, 1957.
[22] S. Ramanujan, *The "lost" notebook and other unpublished papers*, Narosa, New Delhi, 1988.

[23] B. Richmond and G. Szehers, *The Taylor coefficients of certain infinite products*, Acta Sci. Math. **40**, 347–369, 1978.
[24] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. Lond. Math. Soc. **25**, 318–343, 1894.
[25] L. J. Rogers, *On a type of modular relation*, Proc. Lond. Math. Soc. **19**, 387–397, 1921.
[26] N. Saikia, *Two theta-function identities for the Ramanujan-Selberg continued fraction and applications*, J. Number Theory **151**, 46–53, 2015.
[27] N. Saikia and C. Boruah, *Some new explicit values of Ramanujan-Selberg continued fraction*, Afr. Mat. **28**, 1131–1141, 2017.
[28] A. Selberg, *Über einige arithmetische identitäten*, Avh. Norske Vid.-Akad. Oslo I. Mat.-Naturv. KI. **8**, 3–23, 1936.
[29] A. Selberg, *Collected papers* (Volume I), Springer-Verlag, Berlin, 1989.
[30] B. Srivastava, *On 2-dissection and 4-dissection of Ramanujan's cubic continued fraction and identities*, Tamsui Oxf. J. Math. Sci. **23**, 305–315, 2007.
[31] K. R. Vasuki and B. R. Srivasta Kumar, *Certian identities for Ramanujan-Göllnitz-Gordan continued fraction*, J. Comput. Appl. Math. **187**, 87–95, 2006.
[32] E. X. W. Xia and X. M. Yao, *The 8-dissection of the Ramanujan-Gollnitz-Gordon continued fraction by an iterative method*, Int. J. Number Theory **7**, 1589–1593, 2011.
[33] E. X. W. Xia and X. M. Yao, *On 2- and 4-dissections for some infinite products*, Rocky Mountain J. Math. **43**, 2033–2047, 2013.
[34] E. X. W. Xia and X. M. Yao, *4-dissections and 8-dissections for some infinite products*, Rocky Mountain J. Math. **44**, 1685–1696, 2014.
[35] L.-C. Zhang, *Some important continued fractions of Ramanujan and Selberg*, University of Illinois at Urbana-Champaign, Ph.D. Thesis, 1990.
[36] L.-C. Zhang, *q-difference equations and Ramanujan-Selberg continued fractions*, Acta Arith. **57**, 307–355, 1991.
[37] L.-C. Zhang, *Explicit evaluations of a Ramanujan-Selberg continued fraction*, Proc. Amer. Math. Soc. **130**, 9–14, 2002.
[38] L.-C. Zhang, *Explicit evaluations of two Ramanujan-Selberg continued fractions*, Int. J. Number Theory **1**, 593–601, 2005.